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Existence of solutions for a shear thickening fluid-particle system with non-Newtonian potential

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Abstract: This paper is concerned with a compressible shear thickening fluid-particle interaction model for the evolution of particles dispersed in a viscous non-Newtonian fluid. Taking the influence of non-Newtonian gravitational potential into consideration, the existence and uniqueness of strong solutions are established.

Keywords: Existence, Strong solutions, Compressible, Non-Newtonian fluid

MSC: 76A05, 76A10

1 Introduction

We consider a compressible non-Newtonian fluid-particle interaction model which reads as follows

\[
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + (\rho u^2)_x + \rho \Psi_x - \lambda \left( (u^2 + \mu_1) \frac{\rho u}{\rho} - u_x \right)_x + (P + \eta) x &= -\eta \Psi_x, \\
(\Psi_{\beta})_{x} &= \frac{4 \pi g (\rho - \frac{1}{2} \int_{\Omega} \rho d\Omega)}{\frac{1}{\beta}}, \\
\eta_{x} + [\eta (u - \Phi_x)]_{x} &= \eta_{xx}
\end{align*}
\]

(1)

with the initial and boundary conditions

\[
\begin{align*}
(\rho, u, \eta)|_{t=0} &= (\rho_0, u_0, \eta_0), \\
(\rho x, u x)|_{\partial \Omega} &= 0, \\
(\eta x + \eta \Phi x)|_{\partial \Omega} &= 0,
\end{align*}
\]

(2)

and the no-flux condition for the density of particles

\[(\eta_x + \eta \Phi_x)|_{\partial \Omega} = 0, \quad t \in [0, T].
\]

(3)

where \(\rho, u, \eta, P(\rho) = a \rho^\gamma\) denote the fluid density, velocity, the density of particle in the mixture and pressure respectively. \(\Psi\) denotes the non-Newtonian gravitational potential and the given function \(\Phi(x)\) denotes the external potential. \(\alpha > 0, \gamma > 1, \mu_1 > 0, p > 2, 1 < q < 2, \lambda > 0\) is the viscosity coefficient and \(\beta \neq 0\) is a constant. \(\Omega\) is a one-dimensional bounded interval, for simplicity we only consider \(\Omega = (0, 1), \Omega_T = \Omega \times [0, T].\)

In fact, there are extensive studies concerning the theory of strong and weak solutions for the multidimensional fluid-particle interaction models for the Newtonian case. In [1], Carrillo et al. discussed the...

The non-Newtonian fluid is an important type of fluid because of its immense applications in many fields of engineering fluid mechanics such as inks, paints, jet fuels etc., and biological fluids such as blood (see [7]). Many researchers turned to the study of this type of fluid under different conditions both theoretically and experimentally. For details, we refer to the readers to [8-12] and the references therein. To our knowledge, there seems to be a very few mathematical results for the case of the fluid-interaction model systems with non-Newtonian gravitational potential. There are still no existence results to problem (1)-(3) when \( p > 2, 1 < q < 2 \) which describes that the motion of the compressible viscous isentropic gas flow is driven by a non-Newtonian gravitational force.

We are interested in the existence and uniqueness of strong solutions on a one dimensional bounded domain. The strong nonlinearity of (1) bring us new difficulties in getting the upper bound of \( \rho \) and the method used in [2] is not suitable for us. Motivated by the work of Cho et al. [13, 14] on Navier-Stokes equations, we establish local existence and uniqueness of strong solutions by the iteration techniques.

Throughout the paper we assume that \( a = \lambda = 1 \). In the following sections, we will use simplified notations for standard Sobolev spaces and Bochner spaces, such as
\[
\text{for all } t, \quad L^p = L^p(\Omega), \quad H^1 = H^1(\Omega), \quad C(0, T ; H^1) = C(0, T ; H^1(\Omega)).
\]

We state the definition of strong solution as follows:

**Definition 1.1.** The \((\rho, u, \Phi, \eta)\) is called a strong solution to the initial boundary value problem (1)-(3), if the following conditions are satisfied:

(i) \( \rho \in L^\infty(0, T_\ast; H^1(\Omega)), u \in L^\infty(0, T_\ast; W^{1,p}_0(\Omega) \cap H^2(\Omega)), \)

(ii) For all \( \varphi \in L^\infty(0, T_\ast; H^1(\Omega)), \varphi_t \in L^\infty(0, T_\ast; L^2(\Omega)), \) for a.e. \( t \in (0, T) \), we have

\[
\int_{\Omega} \rho \varphi(x, t) \, dx - \int_{0}^{t} \int_{\Omega} (\rho \varphi_t + \rho u \varphi_x)(x, s) \, dx \, ds = \int_{\Omega} \rho_0 \varphi(x, 0) \, dx
\]

(iii) For all \( \phi \in L^\infty(0, T_\ast; W^{1,p}_0(\Omega) \cap H^2(\Omega)), \phi_t \in L^2(0, T_\ast; H^1_0(\Omega)), \) for a.e. \( t \in (0, T) \), we have

\[
\int_{\Omega} \rho \phi(x, t) \, dx - \int_{0}^{t} \int_{\Omega} \left\{ \rho u^2 \phi_x - \rho \psi_x \phi - \lambda (u_x^2 + \mu_1) \frac{u_x^2}{2} u_x \phi_x 
+ (P + \eta) \phi_x + \eta \psi_x \phi \right\} \, dx \, ds = \int_{\Omega} \rho_0 u_0 \phi(x, 0) \, dx
\]

(iv) For all \( \psi \in L^\infty(0, T_\ast; H^2(\Omega)), \psi_t \in L^\infty(0, T_\ast; H^1(\Omega)), \) for a.e. \( t \in (0, T) \), we have

\[
- \int_{0}^{t} \int_{\Omega} |\psi_x|^{-2} \psi \phi x(x, s) \, dx \, ds = \int_{0}^{t} \int_{\Omega} 4 \pi g(\rho - \frac{1}{|\xi|}) \int_{\Omega} \rho \phi(x, s) \, dx \, ds
\]

(v) For all \( \psi \in L^\infty(0, T_\ast; H^2(\Omega)), \psi_t \in L^\infty(0, T_\ast; L^2(\Omega)), \) for a.e. \( t \in (0, T) \), we have

\[
\int_{\Omega} \eta \psi(x, t) \, dx - \int_{0}^{t} \int_{\Omega} [\eta(u - \Phi_x) - \eta \psi_x](x, s) \, dx \, ds = \int_{\Omega} \eta_0 \psi(x, 0) \, dx
\]
1.1 Main results

**Theorem 1.2.** Let \( \mu_1 > 0 \) be a positive constant and \( \Phi \in C^2(\Omega) \), and assume that the initial data \((\rho_0, u_0, \eta_0)\) satisfy the following conditions

\[
0 \leq \rho_0 \in H^3(\Omega), u_0 \in \dot{H}_0^1(\Omega) \cap H^2(\Omega), \eta_0 \in H^4(\Omega)
\]

and the compatibility condition

\[
-[(u_x^2 + \mu_1) \frac{\mu_0}{p} u_0]_x + (P(\rho_0) + \eta_0)_x + \eta_0 \Phi_x = \rho_0^\frac{1}{p} (g + \beta \Phi_x),
\]

for some \( g \in L^2(\Omega) \). Then there exist a \( T_* \in (0, +\infty) \) and a unique strong solution \((\rho, u, \eta)\) to (1)-(3) such that

\[
\begin{align*}
\rho &\in L^\infty(0, T_*; H^4(\Omega)), \rho_t \in L^\infty(0, T_*; L^2(\Omega)), \\
u &\in L^\infty(0, T_*; W_0^{1,p}(\Omega) \cap H^2(\Omega)),\ u_t \in L^2(0, T_*; \dot{H}_0^1(\Omega)), \\
\eta &\in L^\infty(0, T_*; H^2(\Omega)), \eta_t \in L^\infty(0, T_*; L^2(\Omega)), \\
\Phi &\in L^\infty(0, T_*; H^2(\Omega)),\ \Phi_t \in L^\infty(0, T_*; \dot{H}_0^1(\Omega)), \\
\sqrt{\Phi} u_t &\in L^\infty(0, T_*; L^2(\Omega)), ((u_x^2 + \mu_1) \frac{\mu_0}{p} u_x)_x \in L^2(0, T_*; L^2(\Omega)).
\end{align*}
\]

2 A priori Estimates for Smooth Solutions

In this section, we will prove the local existence of strong solutions. By virtue of the continuity equation (1)_1, we deduce the conservation of mass

\[
\int_{\Omega} \rho(t) dx = \int_{\Omega} \rho_0 dx = m_0, \quad (t > 0, m_0 > 0).
\]

Provided that \((\rho, u, \eta)\) is a smooth solution of (1)-(3) and \(\rho_0 \geq \delta\), where \(0 < \delta \ll 1\) is a positive number. We denote by \(M_0 = 1 + \mu_1 + \mu_1^{-1} + |\rho_0|_{H^1} + |g|_{L^2}\), and introduce an auxiliary function

\[
Z(t) = \sup_{0 \leq s \leq t} \left( 1 + |u(s)|_{W_0^{1,p}} + |\rho(s)|_{H^1} + |\eta(s)|_{L^2} + |\eta(t)|_{H^1} + |\sqrt{\rho} u_t(t)|_{L^2} \right).
\]

Then we estimate each term of \(Z(t)\) in terms of some integrals of \(Z(t)\), apply arguments of Gronwall-type and thus prove that \(Z(t)\) is locally bounded.

2.1 Estimate for \(|u|_{W_0^{1,p}}\)

By using (1)_1, we rewrite the (1)_2 as

\[
\rho u_t + \rho u u_x + \rho \Phi_x - [(u_x^2 + \mu_1) \frac{\mu_0}{p} u_x]_x + (P + \eta)_x = -\eta \Phi_x.
\]

Multiplying (10) by \(u_t\), integrating (by parts) over \(\Omega_T\), we have

\[
\iint_{\Omega_T} \rho |u|^2 dx ds + \int_{\Omega_T} (u_x^2 + \mu_1) \frac{\mu_0}{p} u_x u_t dx ds = - \int_{\Omega_T} \left( \rho u u_x + \rho \Phi_x + P_x + \eta_x + \eta \Phi_x \right) u_t dx ds.
\]

We deal with each term as follows:

\[
\int_{\Omega} (u_x^2 + \mu_1) \frac{\mu_0}{p} u_x u_t dx = \frac{1}{2} \int_{\Omega} (u_x^2 + \mu_1) \frac{\mu_0}{p} (u_t^2)_t dx = \frac{1}{2} \frac{d}{dt} \int_{\Omega} (s + \mu_1) \frac{\mu_0}{p} u_t^2 ds dx
\]

\[
\int_{0}^{t} (s + \mu_1) \frac{\mu_0}{p} ds = \int_{\mu_1}^{u_t^2 + \mu_1} t \frac{\mu_0}{p} dt = \frac{2}{p} \left[ (u_x^2 + \mu_1 \frac{\mu_0}{p}) - \mu_1 \frac{\mu_0}{p} \right] \geq \frac{2}{p} |u_x|^p - \frac{2}{p} \mu_1 \frac{\mu_0}{p}.
\]
Substituting the above into (11), we obtain

$$\begin{align*}
- \iint_{\Omega_T} P_s u_t \, dx \, ds &= \iint_{\Omega_T} P u_t \, dx \, ds - \frac{d}{dt} \iint_{\Omega_T} P u_s \, dx \, ds - \iint_{\Omega_T} P_t u_s \, dx \, ds.
\end{align*}$$

Since from (11), we get

$$P_t = -\gamma P u_x - P_s u$$

$$- \iint_{\Omega_T} (\eta_x + \eta \Phi_x) u_t \, dx \, ds - \frac{d}{dt} \iint_{\Omega_T} (\eta_x + \eta \Phi_x) u \, dx \, ds - \iint_{\Omega_T} (\eta_x + \eta \Phi_x)_t u \, dx \, ds$$

Substituting the above into (11), we obtain

$$\begin{align*}
& \iint_0^t \left| \sqrt{\rho} u_t(s) \right|^2 ds + \frac{1}{p} \int_{\Omega} \left| u_s(t) \right|^p dx \\
& \leq C + \int_{\Omega} |P u_x| dx + \iint_{\Omega_T} (|\rho u_s u_t| + |\rho \Phi u_t| + |\gamma P u_s| + |P_s u u_s|) \, dx \, ds \\
& \quad + \iint_{\Omega_T} (|\eta u_{x,x} + |\eta \Phi u_{x,x}| + |\eta \Phi u_{x} + |\eta u_{x} + |\eta \Phi_{x,x} u_{x} + |\eta \Phi_{x} u_{x}| + |\eta \Phi_{x} u_{x}| + |\eta \Phi_{x} u_{x}|) \, dx \, ds.
\end{align*}$$

Using Young’s inequality, we obtain

$$\begin{align*}
& \iint_0^t \left| \sqrt{\rho} u_t(s) \right|^2 ds + \left| u_s(t) \right|^p dx \\
& \leq C + C \left( |\rho|_{L^\infty} |u_s|_{L^2}^2 + |\rho|_{L^\infty} |\Phi_{x,x}|_{L^2}^2 + |P|_{L^\infty} u_s_{x}^2 + |P_s|_{L^1} |u_s|_{L^2} + |\rho|_{L^\infty} |\Phi_{x,x}|_{L^2}^2 + |\rho|_{L^\infty} |u_s|_{L^2} + |\eta|_{L^\infty} |u_{x,x}|_{L^2} + |\eta|_{L^\infty} |u_{x}|_{L^2} \\
& \quad + |\eta|_{L^\infty} |u_{x,x}|_{L^2} + |\eta|_{L^\infty} |u_{x}|_{L^2} + |\eta|_{L^\infty} |u_{x}|_{L^2} + |\eta|_{L^\infty} |u_{x,x}|_{L^2} + |\eta|_{L^\infty} |u_{x}|_{L^2}^2 \\
& \quad + |\eta|_{L^\infty} |u_{x,x}|_{L^2} + |\eta|_{L^\infty} |u_{x}|_{L^2} + |\eta|_{L^\infty} |u_{x,x}|_{L^2} + |\eta|_{L^\infty} |u_{x}|_{L^2} + |\eta|_{L^\infty} |u_{x,x}|_{L^2} \\
& \right) \, dx \, ds + C |P(t)|_{L^2}^p.
\end{align*}$$

On the other hand, multiplying (13) by \(\psi\) and integrating over \(\Omega\), we get

$$\iint_{\Omega} |\psi_s|^q dx = - \int_{\Omega} (|\psi_s|^{q-2} \psi_s)_x \psi \, dx = -4\pi g \int_{\Omega} \rho \psi \, dx - m_0 \int_{\Omega} \psi \, dx$$

$$\leq 8\pi g m_0 |\psi|_{L^\infty} \leq 8\pi g m_0 |\psi|_{L^p} \leq \frac{1}{q} |\psi|_{L^q}^q + \frac{1}{p} (8\pi g m_0)^p$$

Then we have

$$\int_{\Omega} |\psi_s|^q dx \leq C(m_0), \quad 1 < q < 2.$$
Hence, we deduce that
\[ C \int_{\Omega} |\psi_x|^{q-2} \psi_{xx}^2 \, dx \geq C |\psi_x|^{q-2} |\psi_{xx}|_{L^2}^2, \]
\[ \geq C |\psi_{xx}|_{L^2}^{q-2} |\psi_{xx}|_{L^2}^2 \geq C |\psi_{xx}|_L^q \]
and
\[ -4\pi g \int_{\Omega} \rho_x \psi_x \, dx \leq C \int_{\Omega} \rho_x ||\psi_x|| \, dx \leq C |\rho_x|_{L^p}^p + C |\psi_x|_{L^q}^q \leq C |\rho_x|_{L^p}^p + C(x) |\psi_{xx}|_{L^q}^q. \]

Therefore,
\[ |\psi_{xx}|_{L^2} \leq CZ^2(t). \quad (14) \]

We deal with the term of \(|u_{xx}|_{L^2}\). Notice that
\[ \left| \left( u_x^2 + \mu \right) \frac{\partial}{\partial x} u_x \right| \geq \mu \frac{\partial}{\partial x} |u_x|. \]
Then
\[ |u_{xx}| \leq C |\rho u_t + \rho uu_x + \rho \psi_x + (P + \eta) x + \eta \psi_x|. \]

Taking the above inequality by \(L^2\) norm, we get
\[ |u_{xx}|_{L^2} \leq C |\rho u_t + \rho uu_x + \rho \psi_x + (P + \eta) x + \eta \psi_x|_{L^2} \]
\[ \leq C \left( |\rho|_{L^\infty}^{\frac{1}{2}} |\sqrt{\rho} u_t|_{L^2} + |\rho|_{L^\infty} |u|_{L^2} |u_x|_{L^2} + |\rho|_{L^2} |\psi_x|_{L^2} + |P_x|_{L^2} + |\eta|_{L^2} + |\eta|_{L^2} |\psi_x|_{L^2} \right). \]

Hence, we deduce that
\[ |u_{xx}|_{L^2} \leq CZ^{\max\left(\frac{\xi}{4}, 1, 3\right)}(t). \quad (15) \]

Moreover, using (1), we have
\[ |P(t)|_{L^2}^p \leq \int_{\Omega} |P(t)|^2 \, dx = \int_{\Omega} |P(0)|^2 \, dx + \int_0^t \frac{d}{ds} \left( \int_{\Omega} P(s)^2 \, dx \right) \, ds \]
\[ \leq \int_{\Omega} |P(0)|^2 \, dx + 2 \int_0^t \int_{\Omega} P(s) \gamma \rho^{-1} (-\rho x u - \rho u_x) \, dx \, ds \]
\[ \leq C + C \int_0^t |\rho|_{L^\infty} |\rho|_{L^\infty}^{-1} |\rho|_{L^p} |u_x|_{L^2} \, ds \]
\[ \leq C + \int_0^t Z^{2\gamma+1}(s) \, ds. \quad (16) \]

Combining (13)-(16), yields
\[ \int_0^t \left( |\sqrt{\rho} u_t(s)|_{L^2}^2(s) + |u_x(t)|_{L^2}^p \right) \, ds \leq C(1 + \int_0^t Z^{\max\left(\frac{\xi}{4}, 2\gamma+1\right)}(s) \, ds), \quad (17) \]
where \(C\) is a positive constant, depending only on \(M_0\).

### 2.2 Estimate for \(|\rho|_{H^1}\)

From (1), taking it by \(L^2\) norm, we get
\[ |\eta_{xx}|_{L^2} \leq |\eta_x + (\eta(u - \phi_x))|_{L^2}. \]
Multiplying (1) by $\rho$, integrating over $\Omega$, we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\rho|^2 dx + \int_{\Omega} (\rho u_x) \rho dx = 0.
\]
Integrating by parts, using Sobolev inequality, we deduce that
\[
\frac{d}{dt} \|\rho(t)\|_{L^2}^2 \leq \int \|u_x\| \|\rho\|_{L^2}^2 dx \leq \|u_{xx}\|_{L^2} \|\rho\|_{L^2}^2.
\]
(19)

Differentiating (1) with respect to $x$, and multiplying by $\rho_x$, integrating over $\Omega$, using Sobolev inequality, we have
\[
\frac{d}{dt} \int_{\Omega} |\rho_x|^2 dx = - \int_{\Omega} \left( \frac{3}{2} u_x(\rho_x)^2 + \rho \rho_x u_{xx} \right)(t) dx 
\leq C \left[ \|u_x\|_{L^2} \|\rho_x\|_{L^2} + \|\rho\|_{L^2} \|\rho_x\|_{L^2} \|u_{xx}\|_{L^2} \right] 
\leq C \|\rho\|_{H^1}^2 \|u_{xx}\|_{L^2}.
\]
(20)

From (19) and (20), by Gronwall’s inequality, it follows that
\[
\sup_{0 \leq s \leq T} \|\rho(t)\|_{H^1}^2 \leq \|\rho_0\|_{H^1}^2 \exp \left( C t \int_0^t \|u_{xx}\|_{L^2} ds \right) \leq C \exp \left( C t \int_0^t Z_{\text{max}}(s) ds \right),
\]
(21)

Besides, by using (1), we can also get the following estimates:
\[
|\rho_x(t)|_{L^2} \leq \|\rho_x(t)\|_{L^2} = |u(t)|_{L^\infty} + \|\rho(t)\|_{L^\infty} \|u_x(t)\|_{L^2} \leq C \|\rho\|_{H^1}^2(t).
\]
(22)

### 2.3 Estimate for $|\eta|_{L^2}$ and $|\eta|_{H^1}$

Multiplying (1) by $\eta$, integrating the resulting equation over $\Omega_T$, using the boundary conditions (3), Young’s inequality, we have
\[
\int_0^t \|\eta(s)\|_{L^2}^2 ds + \frac{1}{2} \|\eta(t)\|_{L^2}^2 \leq \int_{\Omega_T} (|\eta u| + |\eta \phi_x \eta_x|) dxds 
\leq \frac{1}{4} \int_0^t \|\eta(s)\|_{L^2}^2 ds + C \int_0^t \|u\|_{L^\infty} |\eta|_{H^1}^2 ds + C \int_0^t |\eta|_{H^1}^2 + C 
\leq \frac{1}{4} \int_0^t \|\eta(s)\|_{L^2}^2 ds + C(1 + \int_0^t Z(t) ds).
\]
(23)

Multiplying (1) by $\eta$, integrating (by parts) over $\Omega_T$, using the boundary conditions (3), Young’s inequality, we have
\[
\int_0^t \|\eta(s)\|_{L^2}^2 ds + \frac{1}{2} \|\eta(t)\|_{L^2}^2 \leq \int_{\Omega_T} |\eta (u - \phi_x) \eta_x| dxds 
\leq \frac{1}{4} \int_0^t \|\eta_x(s)\|_{L^2}^2 ds + C \int_0^t \|\eta|_{H^1}^2 \|u_x\|_{L^2} ds + C \int_0^t |\eta|_{H^1}^2 ds + C
\]
Combining (12), (27) can be rewritten into
\[
\frac{1}{4} \int_0^t |\eta_{\alpha}(s)|^2 \, ds + C(1 + \int_0^t Z^{\alpha}(t) \, ds).
\] (24)

Differentiating (1) with respect to \( t \), multiplying the resulting equation by \( \eta \), integrating (by parts) over \( \Omega \), we get
\[
\int_0^t |\eta_{\alpha}(s)|^2 \, ds + \frac{1}{2} |\eta(t)|^2 = \int_{\Omega} (\eta(u - \Phi_x)) \, dx \, ds
\]
\[
\leq C + \int_{\Omega} (|\eta u\eta_{\alpha}| + |\eta\Phi_x \eta_{\alpha}| + |\eta u \eta_{\alpha}| + |\eta u \eta|) \, dx \, ds
\]
\[
\leq C(1 + \int_0^t (|\eta_{\alpha}|^2 + |\eta|^2 \alpha + |\eta_{\alpha}|^2 + |\eta|^2 \alpha + |\eta_{\alpha}|^2 + |\eta|^2 \alpha) \, dx)
\]
\[
+ \frac{1}{2} \int_0^t |\eta_{\alpha}|^2 + \frac{1}{2} \int_0^t |u_{\alpha}|^2 \, dx
\]
\[
\leq C(1 + \int_0^t Z^{2\gamma + \delta}(s) \, ds).
\] (25)

Combining (23)-(25), we get
\[
|\eta|^2 + |\eta_{\alpha}|^2 + \int_0^t (|\eta_{\alpha}|^2 + |\eta|^2 \alpha) \, ds \leq C(1 + \int_0^t Z^{2\gamma + \delta}(s) \, ds).
\] (26)

### 2.4 Estimate for \( |\sqrt{\rho}u_t|_{L^2} \)

Differentiating equation (10) with respect to \( t \), multiplying the result equation by \( u_t \), and integrating it over \( \Omega \) with respect to \( x \), we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho |u_t|^2 \, dx + \int_{\Omega} \left[ (u^2 + \mu_1) \Phi_x \right] u_{xt} \, dx
\]
\[
= \int_{\Omega} \left[ (\rho u_t)^2 + uu_t u_t + \psi_x u_t - \rho u_t u_{\alpha} - \rho \psi_{\alpha} u_{\alpha} - (P + \eta) u_{xt} - \eta \phi_{\alpha} u_{\alpha} \right] \, dx.
\] (27)

Note that
\[
\left[ (u^2 + \mu_1) \Phi_x \right] u_{xt} = \left( u^2 + \mu_1 \right) \Phi_x \frac{p - 1}{p} \left( u^2 + \mu_1 \right) u_{\alpha} \geq \frac{p - 1}{p} \left( u^2 + \mu_1 \right) u_{\alpha}.
\]

Combining (12), (27) can be rewritten into
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho |u_t|^2 \, dx + \int_{\Omega} |u_{xt}|^2 \, dx
\]
\[
\leq 2 \int_{\Omega} \rho |u|^2 |u_t| |u_{xt}| \, dx + \int_{\Omega} \rho |u|^2 |u_{\alpha}| |u_{xt}| \, dx + \int_{\Omega} \rho |u|^2 |u_{xt}| |u_t| \, dx
\]
\[
+ \int_{\Omega} \rho |u|^2 |u_{\alpha}| |u_{xt}| \, dx + \int_{\Omega} \rho |u| |\psi_{xt}| |u_t| \, dx + \int_{\Omega} \rho |u| |\psi_{\alpha}| |u_{xt}| \, dx
\]
\[
+ \int_{\Omega} \rho |u_{\alpha}| u_{xt} \, dx + \int_{\Omega} |P| |u_{\alpha}| |u_{xt}| \, dx + \int_{\Omega} |P_{xt}| |u_{xt}| \, dx
\]
\[
+ \int_{\Omega} |\eta| |u_{xt}| \, dx + \int_{\Omega} |\eta| |\psi_{xt}| |u_t| \, dx + \int_{\Omega} \rho |\psi_{\alpha}| |u_t| \, dx = \sum_{j=1}^{12} I_j.
\] (28)
By using Sobolev inequality, Hölder inequality and Young’s inequality, (14),(15), we estimate each term of \( I_j \) as follows

\[
I_1 = 2 \int_\Omega \rho |u| |u_x| |u| dx \leq 2 |\rho|^{1/2} |u|^{1/2} |u|_{L^\infty} |u_x|_{L^2} \leq CZ^5(t) + \frac{1}{7} |u_x|_{L^2}^2,
\]

\[
I_2 = \int_\Omega |\rho| |u|^2 |u| dx \leq |\rho|^{1/2} |u|^{1/2} |u|_{L^2}^{2} |\sqrt{\rho} u|_{L^2}^{1/2} \leq CZ^5(t),
\]

\[
I_3 = \int_\Omega |\rho| |u|^2 |u_x| dx \leq |\rho|^{1/2} |u|^{1/2} |u_x|_{L^2}^{2} |\sqrt{\rho} u|_{L^2} \leq CZ^{\text{max}(\frac{5}{2}+\gamma,7)}(t),
\]

\[
I_4 = \int_\Omega |\rho| |u|^2 |\Psi| dx \leq |\rho|^{1/2} |u|^{1/2} |\Psi|_{L^2}^{2} |\sqrt{\rho} u|_{L^2} \leq CZ^{\frac{5}{2}+\gamma}(t) + \frac{1}{7} |\Psi|_{L^2}^2,
\]

\[
I_5 = \int_\Omega |\rho| |u|^2 |u| dx \leq |\rho|^{1/2} |u|^{1/2} |\Psi|_{L^2}^{2} |\sqrt{\rho} u|_{L^2} \leq CZ^{\frac{5}{2}+\gamma}(t),
\]

\[
I_6 = \int_\Omega |\rho| |u|^2 |\Psi| dx \leq |\rho|^{1/2} |u|^{1/2} |\Psi|_{L^2}^{2} |\sqrt{\rho} u|_{L^2} \leq CZ^{\frac{5}{2}+\gamma}(t) + \frac{1}{7} |\Psi|_{L^2}^2,
\]

\[
I_7 = \int_\Omega |\rho| |u|^2 |u| dx \leq |\rho|^{1/2} |u|^{1/2} |\Psi|_{L^2}^{2} |\sqrt{\rho} u|_{L^2} \leq CZ^{\text{max}(\frac{5}{2}+\gamma,5)}(t),
\]

\[
I_8 = \int_\Omega |\rho| |u|^2 |u| dx \leq |\rho|^{1/2} |u|^{1/2} |\Psi|_{L^2}^{2} |\sqrt{\rho} u|_{L^2} \leq CZ^{2\gamma+2}(t) + \frac{1}{7} |\Psi|_{L^2}^2,
\]

\[
I_9 = \int_\Omega |\rho| |u|^2 |u| dx \leq |\rho|^{1/2} |u|^{1/2} |\Psi|_{L^2}^{2} |\sqrt{\rho} u|_{L^2} \leq CZ^{2\gamma+2}(t) + \frac{1}{7} |\Psi|_{L^2}^2,
\]

\[
I_{10} = \int_\Omega |\rho| |u|^2 |u| dx \leq |\rho|^{1/2} |u|^{1/2} |\Psi|_{L^2}^{2} |\sqrt{\rho} u|_{L^2} \leq CZ^{2}(t) + \frac{1}{7} |\Psi|_{L^2}^2,
\]

\[
I_{11} = \int_\Omega |\rho| |u|^2 |u| dx \leq |\rho|^{1/2} |u|^{1/2} |\Psi|_{L^2}^{2} |\sqrt{\rho} u|_{L^2} \leq CZ^{2}(t) + \frac{1}{7} |\Psi|_{L^2}^2,
\]

\[
I_{12} = \int_\Omega |\rho| |u|^2 |u| dx \leq |\rho|^{1/2} |u|^{1/2} |\Psi|_{L^2}^{2} |\sqrt{\rho} u|_{L^2},
\]

where \( C \) is a positive constant, depending only on \( M_0 \).

Next, we deal with the term \( |\Psi|_{L^2}^2 \) of \( I_{12} \). Differentiating (13) with respect to \( t \), multiplying it by \( \Psi \), integrating over \( \Omega \) and using Young’s inequality, we obtain

\[
\int_\Omega (|\Psi|^{q-2} \Psi) \psi_x dx = -4\pi g \int_\Omega \rho_0 \psi_t dx.
\]

By virtue of

\[
\int_\Omega (|\Psi|^{q-2} \Psi) \psi_x dx = \int_\Omega (|\Psi|^{q-2} \psi_x^2 + \psi_x^2) \psi_x dx
\]

\[
\geq C \int_\Omega |\Psi|^{q-2} \psi_x^2 dx \geq C |\Psi|^{q-2} |\psi_x|_{L^2}^2,
\]

and

\[
-4\pi g \int_\Omega \rho_0 \psi_t dx \leq C \int_\Omega |\rho_1||\psi_t| dx \leq C |\rho_1|_{L^2}^2 + C |\psi_t|_{L^2}^2 \leq C |\rho_1|_{L^2}^2 + C(\varepsilon)|\Psi|_{L^2}^2,
\]

then \( |\Psi|_{L^2}^2 \leq CZ^{\frac{2\gamma+6-\gamma}{4}}(t) \). Therefore,

\[
I_{12} = \int_\Omega |\rho| |u|^2 |u| dx \leq |\rho|^{1/2} |u|^{1/2} |\Psi|_{L^2}^{2} |\sqrt{\rho} u|_{L^2} \leq CZ^{2\gamma+6-\gamma}(t).
\]
Substituting $I_j (j = 1, 2, \ldots, 12)$ into (28), and integrating over $(\tau, t) \subset (0, T)$ on the time variable, we have

$$\left| \sqrt{\rho} u_t (t) \right| \leq C \int_{\tau}^{t} \left| u_x \right|^2 \, ds \leq \left| \sqrt{\rho} u_t (\tau) \right| + C \int_{\tau}^{t} Z^{\max (\frac{2}{3} + \alpha, \beta)}(s) \, ds. \quad (29)$$

To obtain the estimate of $\left| \sqrt{\rho} u_t (t) \right|$, we need to estimate $\lim_{\tau \to 0} \left| \sqrt{\rho} u_t (\tau) \right|$. Multiplying (10) by $u_t$ and integrating over $\Omega$, we have

$$\int_{\Omega} \rho |u_t|^2 \, dx \leq 2 \int_{\Omega} \rho |u_x|^2 + \rho |\Psi|^2 + | \left[ (u_x^2 + \mu_1) \frac{\rho}{\bar{\rho}} u_x \right]_x + (P + \eta) x + \eta \phi_x |^2 \, dx.$$

According to the smoothness of $(\rho, u, \eta)$, we obtain

$$\lim_{\tau \to 0} \int_{\Omega} \left( \rho |u|^2 |u_x|^2 + \rho |\Psi|^2 + \rho^{-1} \right) - \left[ (u_x^2 + \mu_1) \frac{\rho}{\bar{\rho}} u_x \right]_x + (P + \eta) x + \eta \phi_x |^2 \, dx$$

$$= \int_{\Omega} \left( \rho |u_0|^2 |u_{0x}|^2 + \rho_0 |\Psi|^2 + \rho_0^{-1} \right) - \left[ (u_{0x}^2 + \mu_1) \frac{\rho}{\bar{\rho}} u_{0x} \right]_x + (P_0 + \eta_0) x + \eta_0 \phi_x |^2 \, dx$$

$$\leq \rho_0 |u_0|^2 |u_{0x}|^2 + \rho_0 |\Psi|^2 + \rho_0^{-1} + \omega \phi_x |^2 \leq C.$$

Therefore, taking a limit on $\tau$ in (29), as $\tau \to 0$, we conclude that

$$\left| \sqrt{\rho} u_t (t) \right| + \int_{0}^{t} \left| u_x \right|^2 \, ds \leq C (1 + \int_{0}^{t} Z^{\max (\frac{2}{3} + \alpha, \beta)}(s) \, ds) \quad (30)$$

where $C$ is a positive constant, depending only on $M_0$.

Combining the estimates of (15), (18), (21), (22), (17), (26), (30) and the definition of $Z(t)$, we conclude that

$$Z(t) \leq \tilde{C} \exp \frac{\kappa}{\bar{C}} \int_{0}^{t} Z^{\max (\frac{2}{3} + \alpha, \beta)}(s) \, ds \quad (31)$$

where $\tilde{C}$ and $\bar{C}$ are positive constants, depending only on $M_0$. This means that there exist a time $T_1 > 0$ and a constant $C > 0$, such that

$$\text{ess sup}_{0 \leq \tau \leq T_1} (|\rho|_{H^1} + |u|_{H^\alpha, H^\gamma}, |\eta|_{H^\beta} + |\eta|_{L^2} + |\sqrt{\rho} u|_{L^2} + |\rho|_{L^2})$$

$$+ \int_{0}^{T_1} (|\sqrt{\rho} u|^2 + |u_x|^2 + |\eta|^2 + |\eta|_{L^2} + |\eta|_{H^\beta}^2) \, ds \leq C. \quad (32)$$

### 3 Proof of the Main Theorem

In this section, our proof will be based on the usual iteration argument and some ideas developed in [13, 14]. Precisely, we construct the approximate solutions, by using the iterative scheme, inductively, as follows: first define $u^0 = 0$ and assuming that $u^{k-1}$ was defined for $k \geq 1$, let $\rho^k, u^k, \eta^k$ be the unique smooth solution to the following problems:

$$\begin{align*}
\rho^k_t + \rho^k u^k_{x}^{k-1} + \rho^k u^k_{x}^{k-1} &= 0 \\
\rho^k u^k_t + \rho^k u^k_{x}^{k-1} + \rho^k \Psi^k - \left[ \left( (u^k_x)^2 + \mu_1 \right) \frac{\rho}{\bar{\rho}} u^k_x \right]_x + P^k + \eta^k &= \eta^k \phi^k \\
\left( |\Psi^k|^2 - 2 \bar{\rho} \right) &= 4 \pi g \rho^k = m_0 \\
\eta^k + \left( \eta^k (u^k_{x}^{k-1} - \phi^k) \right)_x &= \eta^k \phi^k
\end{align*}$$

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with the initial and boundary conditions

\[
(\rho^k, u^k, \eta^k)|_{t=0} = (\rho_0, u_0, \eta_0)
\]

\[
u^k|_{\partial \Omega} = (\eta^k + \eta^k \phi_x)|_{\partial \Omega} = 0
\]

with the process, the nonlinear coupled system has been deduced into a sequence of decoupled problems and each problem admits a smooth solution. And the following estimates hold

\[
\text{ess sup}_{0 \leq t \leq T_1} \left( |\rho^k|_{H^1} + |u^k|_{W^{1,p} \cap H^{1,p}} + |\eta^k|_{H^1} + |\sqrt{\rho^k} u^k|_{L^2} + |\nu^k|_{L^2} \right)
\]

\[
+ \int_0^{T_1} \left( |\sqrt{\rho^k} u^k_{x}^2|_{L^2} + |u^k_{x x}^2|_{L^2} + |\eta^k_{x}^2|_{L^2} + |\eta^k|_{H^{1,p}}^2 \right) \, dt \leq C,
\]

(33)

where \(C\) is a generic constant depending only on \(M_0\), but independent of \(k\).

In addition, we first find \(\rho^k\) from the initial problem

\[
\rho^k + u^k - \rho^k \rho_x + u^k \rho_k = 0, \quad \text{and} \quad \rho^k|_{t=0} = \rho_0
\]

with smooth function \(u^{k-1}\), obviously, there is a unique solution \(\rho^k\) to the above problem and also by a standard argument, we could obtain that

\[
\rho^k(x, t) \geq \delta \exp \left[ - \int_0^T |u^k_{x}(-, s)|_{L^2} \, ds \right] > 0, \text{ for all } t \in (0, T_1).
\]

Next, we have to prove that the approximate solution \((\rho^k, u^k, \eta^k)\) converges to a solution to the original problem (1) in a strong sense. To this end, let us define

\[
\rho^{k+1} = \rho^{k+1} - \rho^k, \quad u^{k+1} = u^{k+1} - u^k, \quad \eta^{k+1} = \eta^{k+1} - \eta^k,
\]

then we can verify that the functions \(\rho^{k+1}, u^{k+1}, \eta^{k+1}\) satisfy the system of equations

\[
\rho^{k+1} + (\rho^k u^k)_{x} + (\rho^k u^k)_{x} = 0
\]

\[
n\rho^{k+1} u^k_{x} + \rho^{k+1} u^k u^{k+1} - \left( \left( \left( u^k_{x} \right)^2 + \mu_1 \right) \frac{\nabla \cdot u^k}{\nabla \cdot u^k} \right)_{x} - \left( \left( u^k \right)^2 + \mu_1 \right) \frac{\nabla \cdot u^k}{\nabla \cdot u^k} \right)_{x} = -\rho^{k+1} \left( u^k + u^k u^k + \phi_x \right) - \left( P^{k+1} - P^k \right) \rho^k (\rho^k u^k - \eta^{k+1}) - \eta^{k+1} \eta^k \eta^k
\]

\[
\left( \frac{\eta^{k+1} - \eta^k}{\rho^k} \right)_{x} - \left( \frac{\eta^{k+1} - \eta^k}{\rho^k} \right)_{x} = 4 \pi \eta^k\eta^k\eta^k
\]

\[
\eta^{k+1} + (\eta^k u^k)_{x} + (\eta^{k+1} (u^k - \phi_x))_{x} = \eta^k
\]

Multiplying (34) by \(\rho^{k+1}\), integrating over \(\Omega\) and using Young’s inequality, we obtain

\[
\frac{d}{dt} |\rho^{k+1}|_{L^2} \leq C |\rho^{k+1}|_{L^2} |u^k|_{L^\infty} + |\rho^k|_{H^1} |u^k|_{L^2} |\rho^{k+1}|_{L^2}
\]

\[
\leq C |u^k|_{x x L^2} |\rho^{k+1}|_{L^2} + C |\rho^k|_{H^1} |\rho^{k+1}|_{L^2} + |u^k|_{x x L^2}
\]

\[
\leq C |\rho^{k+1}|_{L^2} + |u^k|_{x x L^2}.
\]

(38)

where \(C\) is a positive constant, depending on \(M_0\) and \(C\) for all \(t \leq T_1\) and \(k \geq 1\).

Multiplying (35) by \(u^{k+1}\), integrating over \(\Omega\) and using Young’s inequality, we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |\rho^{k+1}|_{L^2} \, dx + \int_\Omega \left( \left( \left( u^k_{x} \right)^2 + \mu_1 \right) \frac{\nabla \cdot u^k}{\nabla \cdot u^k} \right)_{x} - \left( \left( u^k \right)^2 + \mu_1 \right) \frac{\nabla \cdot u^k}{\nabla \cdot u^k} \right)_{x} \rho^{k+1} \, dx
\]

\[
\leq C \int_\Omega \left( |\rho^k|_{L^2} |\rho^{k+1}|_{L^2} + |u^k|_{x x L^2} |\rho^{k+1}|_{L^2} + |P^{k+1} - P^k| |u^{k+1}|_{L^2} + |\rho^k| |\rho^k| |u^k|_{L^2} |\rho^{k+1}|_{L^2} + |\rho^k| |\rho^k| |u^k|_{L^2} |\rho^{k+1}|_{L^2} + |\rho^k| |\rho^k| |u^k|_{L^2} |\rho^{k+1}|_{L^2} + |\rho^k| |\rho^k| |u^k|_{L^2} |\rho^{k+1}|_{L^2} + |\rho^k| |\rho^k| |u^k|_{L^2} |\rho^{k+1}|_{L^2} \right) \, dx.
\]

(39)
Let
\[ \sigma(s) = (s^2 + \mu_1)^{\frac{p}{2}} s, \]
then
\[ \sigma'(s) = \left((s^2 + \mu_1)^{\frac{p}{2}} s\right)' = (s^2 + \mu_1)^{\frac{p}{2}} \left( (p - 1)s^2 + \mu_1 \right) \geq \mu_1 s^2. \]

We estimate the second term of (39) as follows
\[
\int_{\Omega} \left( (u_x^{k+1})^2 + \mu_1 u_x^{k+1} \right)_x - \left( (u_x^k)^2 + \mu_1 u_x^k \right)_x \hat{u}^{k+1} \, dx
= \int_{\Omega} \int_0^1 \sigma'((1 - \theta)u_x^k + \theta u_x^k) \, d\theta |\hat{u}^{k+1}| \, dx \geq \mu_1 \int_{\Omega} |\hat{u}^{k+1}|^2 \, dx. \tag{40}
\]

Similarly, multiplying (36) by \( \hat{\psi}^{k+1} \), integrating over \( \Omega \), we get
\[
\int_{\Omega} \left( (\psi_x^{k+1})^{q-2} \psi_x^{k+1} \right)_x - \left( (\psi_x^k)^{q-2} \psi_x^k \right)_x \hat{\psi}^{k+1} \, dx = 4\pi g \int_{\Omega} \rho \hat{\psi}^{k+1} \, dx, \tag{41}
\]
since
\[
\int_{\Omega} \left[ (\psi_x^{k+1})^{q-2} \psi_x^{k+1} - (\psi_x^k)^{q-2} \psi_x^k \right] \hat{\psi}^{k+1} \, dx
= (q - 1) \int_{\Omega} \left( \int_0^1 \left( |\hat{\psi}_x^{k+1}| + (1 - \theta) |\psi_x^k|^{q-2} \right) \, d\theta \right) (\hat{\psi}^{k+1})^2 \, dx
\]
and
\[
\int_0^1 |\hat{\psi}_x^{k+1}| + (1 - \theta) |\psi_x^k|^{q-2} \, d\theta = \int_0^1 \frac{1}{1 - \theta |\psi_x^k|^{2-q} + \theta} \, d\theta
\geq \int_0^1 \frac{1}{(|\psi_x^{k+1}| + |\psi_x^k|)^{2-q}} \, d\theta = \frac{1}{(|\psi_x^{k+1}| + |\psi_x^k|)^{2-q}}.
\]

Then
\[
\int_{\Omega} \left[ (\psi_x^{k+1})^{q-2} \psi_x^{k+1} - (\psi_x^k)^{q-2} \psi_x^k \right] \hat{\psi}^{k+1} \, dx \geq \frac{1}{(|\psi_x^{k+1}| + |\psi_x^k|)^{2-q}} \int_{\Omega} (\hat{\psi}^{k+1})^2 \, dx.
\]
That means (41) turns into
\[
\int_{\Omega} (\hat{\psi}^{k+1})^2 \, dx \leq C_1 \rho^{k+1}_{L^2}. \tag{42}
\]

Substituting (40) and (42) into (39), using Young’s inequality, yields
\[
\frac{d}{dt} \int_{\Omega} \rho^{k+1} |u_x^{k+1}|^2 \, dx + \int_{\Omega} |\hat{u}_x^{k+1}|^2 \, dx
\leq C (\rho^{k+1}_{L^2} |u_x^{k+1}|_{L^2} |u_x^{k+1}|_{L^2} + \rho^{k+1}_{L^2} |u_x^k|_{L^2} |u_x^k|_{L^2} |\hat{u}_x^{k+1}|_{L^2} + \rho^{k+1}_{L^2} |\psi_x^{k+1}|_{L^2} |\hat{u}_x^{k+1}|_{L^2} + \rho^{k+1}_{L^2} |\psi_x^k|_{L^2} |\hat{u}_x^{k+1}|_{L^2}
+ |\hat{\psi}_x^{k+1}|_{L^2} |\hat{u}_x^{k+1}|_{L^2} + |\hat{\psi}_x^{k+1}|_{L^2} |\hat{u}_x^{k+1}|_{L^2} + \rho^{k+1}_{L^2} |\psi_x^{k+1}|_{L^2} |\hat{u}_x^{k+1}|_{L^2} + \rho^{k+1}_{L^2} |\psi_x^k|_{L^2} |\hat{u}_x^{k+1}|_{L^2} + \rho^{k+1}_{L^2} |\psi_x^{k+1}|_{L^2} |\hat{u}_x^{k+1}|_{L^2})
\leq B_\zeta(t) \rho^{k+1}_{L^2} |\hat{u}_x^{k+1}|_{L^2} + C (|\sqrt{\rho^{k+1}_{L^2}} \hat{u}^{k+1}_{L^2} |\hat{u}^{k+1}_{L^2} |\hat{u}^{k+1}_{L^2} + \zeta |\hat{u}^{k+1}_{L^2}|^2), \tag{43}
\]
where \( B_\zeta(t) = C(1 + |u_x^{k+1}(t)|_{L^2}^2) \), for all \( t \leq T_1 \) and \( k \geq 1 \). Using (33) we derive
\[
\int_0^t B_\zeta(s) \, ds \leq C + C t.
Collecting (38), (43) and (44), we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\bar{\eta}^{k+1}|^2 \, dx + \int_{\Omega} \sum_{x} (|\eta| |\bar{u}^k| |\bar{\eta}^{k+1}|) \, dx \\
\leq \int_{\Omega} |\eta| \bar{u}^k - \varphi_x |\bar{\eta}^{k+1}| \, dx + \int_{\Omega} (|\eta| |\bar{u}^k| |\bar{\eta}^{k+1}|) \, dx \\
\leq |\eta| |\bar{u}^k| |\bar{\eta}^{k+1}| + |\eta| |\bar{u}^k| |\bar{\eta}^{k+1}| + |\eta| |\bar{u}^k| |\bar{\eta}^{k+1}| + |\eta| |\bar{u}^k| |\bar{\eta}^{k+1}| + |\eta| |\bar{u}^k| |\bar{\eta}^{k+1}| \\
\leq C \bar{\eta}^{k+1} + \zeta \bar{\eta}^{k+1} + c \bar{u}_x^{k+1}. 
\]

(44)

Collecting (38), (43) and (44), we obtain

\[
\frac{d}{dt} (|\bar{\rho}^{k+1}(t)|^2 + |\sqrt{\bar{\rho}^{k+1}} \bar{u}^{k+1}(t)|^2 + |\bar{\eta}^{k+1}(t)|^2) + |\bar{u}^k(t)|^2 + |\bar{\eta}^{k+1}|^2 \\
\leq E_C(t) |\bar{\rho}^{k+1}(t)|^2 + C \sqrt{\bar{\rho}^{k+1}} |\bar{u}^{k+1}(t)|^2 + C \bar{\eta}^{k+1} + c \bar{u}_x^k, 
\]

(45)

where \( E_C(t) \) depends only on \( B_C(t) \) and \( C_C \), for all \( t \leq T_1 \) and \( k \geq 1 \). Using (33), we have

\[
\int_0^t E_C(s) \, ds \leq C + C_C t. 
\]

Integrating (45) over \((0, t) \subset (0, T_1)\) with respect to \( t \), using Gronwall’s inequality, we have

\[
|\bar{\rho}^{k+1}(t)|^2 + |\sqrt{\bar{\rho}^{k+1}} \bar{u}^{k+1}(t)|^2 + |\bar{\eta}^{k+1}(t)|^2 + \int_0^t |\bar{u}^k(t)|^2 + \int_0^t |\bar{\eta}^{k+1}|^2 \\
\leq C \exp(C_C t) \int_0^t (|\sqrt{\bar{\rho}^{k+1}} \bar{u}^k(s)|^2 + |\bar{u}_x^k(s)|^2) \, ds. 
\]

(46)

From the above recursive relation, choose \( \zeta > 0 \) and \( 0 < T_* < T_1 \) such that \( C \exp(C_C T_*) < \frac{1}{2} \), using Gronwall’s inequality, we deduce that

\[
\sum_{k=1}^{T_*} \left[ \sup_{0 \leq t \leq T_*} (|\bar{\rho}^{k+1}(t)|^2 + |\sqrt{\bar{\rho}^{k+1}} \bar{u}^{k+1}(t)|^2 + |\bar{\eta}^{k+1}(t)|^2) \right] dt \\
+ \int_0^{T_*} |\bar{u}^k(t)|^2 + \int_0^{T_*} |\bar{\eta}^{k+1}|^2 \, dt < C. 
\]

(47)

Since all of the constants do not depend on \( \delta \), as \( k \to \infty \), we conclude that sequence \( (\rho^k, u^k, \eta^k) \) converges to a limit \( (\rho^\delta, u^\delta, \eta^\delta) \) in the following convergence

\[
\rho \to \rho^\delta \quad \text{in} \quad L^\infty(0, T_*; L^2(\Omega)), \\
u \to u^\delta \quad \text{in} \quad L^\infty(0, T_*; L^2(\Omega)) \cap L^2(0, T_*; H^3(\Omega)), \\
\eta \to \eta^\delta \quad \text{in} \quad L^\infty(0, T_*; L^2(\Omega)) \cap L^2(0, T_*; H^3(\Omega)), 
\]

and there also holds

\[
\text{ess sup}_{0 \leq t \leq T_1} (|\rho^\delta|_{H^s} + |u^\delta|_{W^{1,p}_{\rho,\delta} \cap H^s} + |\eta^\delta|_{H^s} + |\eta^\delta|_{L^2} + |\sqrt{\rho^\delta} u^\delta_{\delta} |_{L^2} + |\rho^\delta|_{L^2}) \\
+ \int_0^{T_*} (|\sqrt{\rho^\delta} u^\delta_{\delta} |_{L^2} + |\eta^\delta|_{L^2} + |\eta^\delta|_{L^2} + |\eta^\delta|_{L^2}) \, ds \leq C. 
\]

(51)

For each \( \delta > 0 \), let \( \rho^\delta_0 = J_{\delta} \ast \rho_0 + \delta, J_{\delta} \) is a mollifier on \( \Omega \), and \( u^\delta_0 \in H^3_0(\Omega) \cap H^2(\Omega) \) is a smooth solution of the boundary value problem

\[
\begin{cases}
- \left[ (|u^{\delta}_0|^2 + \mu_1) \frac{\partial}{\partial x} \right]^{\lambda_2} u^{\delta}_0 \right]_x + \left( P(\rho^\delta_0) + \eta^\delta_0 \right) \right)_x + \eta^\delta_0 \phi_x = (\rho^\delta_0)^{3/2}(g^\delta + \beta \phi_x), \\
u^\delta_0(0) = u^\delta_0(1) = 0,
\end{cases}
\]

(52)
where \( g^\delta \in C^\infty_0 \) and satisfies \( |g^\delta|_{L^2} \leq |g|_{L^2} \), \( \lim_{\delta \to 0^+} |g^\delta - g|_{L^2} = 0 \).

We deduce that \( (\rho^\delta, u^\delta, \eta^\delta) \) is a solution of the following initial boundary problem

\[
\begin{aligned}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + (\rho u u)_x + \rho \psi_x - \lambda\left[ (u_x^2 + \mu_1) \frac{\partial u}{\partial x} \right]_x + (P + \eta)_x &= -\eta \psi_x, \\
(\eta u^q)_x - 4\pi g(\rho - \frac{1}{|\Omega|}) \int_{\Omega} \rho \, dx) &= 0, \\
\eta \eta + (u (u - \Phi_x))_x &= \eta_{xx}, \\
(\rho, u, \eta)|_{t=0} &= (\rho_0^\delta, u_0^\delta, \eta_0^\delta), \\
u|_{\partial \Omega} = \psi|_{\partial \Omega} &= (\eta_\kappa + \eta \Phi_x)|_{\partial \Omega} = 0,
\end{aligned}
\]

where \( \rho_0^\delta \geq \delta, p > 2, 1 < q < 2 \).

By the proof of Lemma 2.3 in [11], there exists a subsequence \( \{u_0^\delta\} \) of \( \{u_0^\delta\} \), as \( \delta_j \to 0^- \), \( u_0^\delta \to u_0 \) in \( H^1_0(\Omega) \cap H^2(\Omega) \), \( -u_0^\delta \to -u_0 \) in \( L^2(\Omega) \). Hence, \( u_0 \) satisfies the compatibility condition (8) of Theorem 1.2. By virtue of the lower semi-continuity of various norms, we deduce that \( (\rho, u, \eta) \) satisfies the following uniform estimate

\[
\text{ess sup}_{0 \leq t \leq T_1} \left( |\rho|_{H^1} + |u|_{W^{1,p} \cap H^1} + |\eta|_{H^2} + |\eta|_{L^2} + |\sqrt{\rho} u_t|_{L^2} + |\rho|_{L^2} \right) \\
+ \int_0^r \left( |\sqrt{\rho} u_t|_{L^2}^2 + |u_x|_{L^2}^2 + |\eta_x|_{L^2}^2 + |\eta_x|_{L^2}^2 + |\eta_{xx}|_{L^2}^2 \right) ds \leq C,
\]

where \( C \) is a positive constant, depending only on \( M_0 \).

The uniqueness of solution can be obtained by the same method as the above proof of convergence, we omit the details here. This completes the proof.

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