Open Mathematics
Research Article

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On the different kinds of separability of the space of Borel functions

https://doi.org/10.1515/math-2018-0070
Received June 18, 2017; accepted May 24, 2018.
Abstract: In paper we prove that:
- a space of Borel functions $B(X)$ on a set of reals $X$, with pointwise topology, to be countably selective sequentially separable if and only if $X$ has the property $S_1(B_Γ, B_Γ)$;
- there exists a consistent example of sequentially separable selectively separable space which is not selective sequentially separable. This is an answer to the question of A. Bella, M. Bonanzinga and M. Matveev;
- there is a consistent example of a compact $T_2$ sequentially separable space which is not selective sequentially separable. This is an answer to the question of A. Bella and C. Costantini;
- $\min\{b, q\} = \{κ : 2^κ \text{ is not selective sequentially separable}\}$. This is a partial answer to the question of A. Bella, M. Bonanzinga and M. Matveev.

Keywords: $S_1(D, D)$, $S_1(S, S)$, $S_{fin}(S, S)$, Function spaces, Selection principles, Borel function, $σ$-set, $S_1(B_Ω, B_Ω)$, $S_1(B_Γ, B_Γ)$, $S_1(B_Ω, B_Γ)$, Sequentially separable, Selectively separable, Selective sequentially separable, Countably selective sequentially separable

MSC: 54C35, 54C05, 54C65, 54A20

1 Introduction

In [12], Osipov and Pytkeev gave necessary and sufficient conditions for the space $B_1(X)$ of the Baire class 1 functions on a Tychonoff space $X$, with pointwise topology, to be (strongly) sequentially separable. In this paper, we consider some properties of a space $B(X)$ of Borel functions on a set of reals $X$, with pointwise topology, that are stronger than (sequential) separability.

2 Main definitions and notation

Many topological properties are defined or characterized in terms of the following classical selection principles. Let $A$ and $B$ be sets consisting of families of subsets of an infinite set $X$. Then:

$S_1(A, B)$ is the selection hypothesis: for each sequence $(A_n : n ∈ N)$ of elements of $A$ there is a sequence $(b_n : n ∈ N)$ such that for each $n$, $b_n ∈ A_n$, and $(b_n : n ∈ N)$ is an element of $B$.

$S_{fin}(A, B)$ is the selection hypothesis: for each sequence $(A_n : n ∈ N)$ of elements of $A$ there is a sequence $(B_n : n ∈ N)$ of finite sets such that for each $n$, $B_n ⊆ A_n$, and $\bigcup_{n ∈ N} B_n ∈ B$.

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$U_{fin}(A, B)$ is the selection hypothesis: whenever $U_1, U_2, \ldots \in A$ and none contains a finite subcover, there are finite sets $F_n \subseteq U_n, n \in \mathbb{N}$, such that $\{\bigcup F_n : n \in \mathbb{N}\} \in B$.

An open cover $\mathcal{U}$ of a space $X$ is:
- an $\omega$-cover if $X$ does not belong to $\mathcal{U}$ and every finite subset of $X$ is contained in a member of $\mathcal{U}$;
- a $\gamma$-cover if it is infinite and each $x \in X$ belongs to all but finitely many elements of $\mathcal{U}$.

For a topological space $X$ we denote:
- $\Omega$ — the family of all countable open $\omega$-covers of $X$;
- $\Gamma$ — the family of all countable open $\gamma$-covers of $X$;
- $B_\omega$ — the family of all countable Borel $\omega$-covers of $X$;
- $B_\gamma$ — the family of all countable Borel $\gamma$-covers of $X$;
- $F_\Gamma$ — the family of all countable closed $\gamma$-covers of $X$;
- $D$ — the family of all countable dense subsets of $X$;
- $S$ — the family of all countable sequentially dense subsets of $X$.

A $\gamma$-cover $\mathcal{U}$ of co-zero sets of $X$ is $\gamma_F$-shrinkable if there exists a $\gamma$-cover $\{F(U) : U \in \mathcal{U}\}$ of zero-sets of $X$ with $F(U) \subseteq U$ for every $U \in \mathcal{U}$.

For a topological space $X$ we denote $\Gamma_F$, the family of all countable $\gamma_F$-shrinkable $\gamma$-covers of $X$.

We will use the following notations.
- $C_p(X)$ is the set of all real-valued continuous functions $C(X)$ defined on a space $X$, with pointwise topology.
- $B_1(X)$ is the set of all first Baire class functions $B_1(X)$ i.e., pointwise limits of continuous functions, defined on a space $X$, with pointwise topology.
- $B(X)$ is the set of all Borel functions, defined on a space $X$, with pointwise topology.

If $X$ is a space and $A \subseteq X$, then the sequential closure of $A$, denoted by $[A]_\text{seq}$, is the set of all limits of sequences from $A$. A set $D \subseteq X$ is said to be sequentially dense if $X = [D]_\text{seq}$. If $D$ is a countable, sequentially dense subset of $X$ then $X$ call sequentially separable space.

Call a space $X$ strongly sequentially separable if $X$ is separable and every countable dense subset of $X$ is sequentially dense.

A space $X$ is (countably) selectively separable (or M-separable, [3]) if for every sequence $(D_n : n \in \mathbb{N})$ of (countable) dense subsets of $X$ one can pick finite $F_n \subseteq D_n, n \in \mathbb{N}$, so that $\bigcup\{F_n : n \in \mathbb{N}\}$ is dense in $X$.

In [3], the authors started to investigate a selective version of sequential separability.

A space $X$ is (countably) selectively sequentially separable (or M-sequentially separable, [3]) if for every sequence $(D_n : n \in \mathbb{N})$ of (countable) sequentially dense subsets of $X$, one can pick finite $F_n \subseteq D_n, n \in \mathbb{N}$, so that $\bigcup\{F_n : n \in \mathbb{N}\}$ is sequentially dense in $X$.

In Scheepers’ terminology [16], countably selectively separability equivalently to the selection principle $S_{fin}(D, D)$, and countably selective sequentially separability equivalently to the $S_{fin}(S, S)$.

Recall that the cardinal $p$ is the smallest cardinal so that there is a collection of $p$ many subsets of the natural numbers with the strong finite intersection property but no infinite pseudo-intersection. Note that $\omega \leq p \leq c$.

For $f, g \in \mathbb{N}^\omega$, let $f \preceq g$ if $f(n) \leq g(n)$ for all but finitely many $n$. $b$ is the minimal cardinality of a $\preceq^*$-unbounded subset of $\mathbb{N}^\omega$. A set $B \subseteq \mathbb{N}^\omega$ is unbounded if the set of all increasing enumerations of elements of $B$ is unbounded in $\mathbb{N}^\omega$, with respect to $\preceq^*$. It follows that $|B| \geq b$. A subset $S$ of the real line is called a $Q$-set if each one of its subsets is a $G_\delta$. The cardinal $q$ is the smallest cardinal so that for any $\kappa < q$ there is a $Q$-set of size $\kappa$. (See [7] for more on small cardinals including $p$).
3 Properties of a space of Borel functions

Theorem 3.1. For a set of reals $X$, the following statements are equivalent:
1. $B(X)$ satisfies $S_1(S, S)$ and $B(X)$ is sequentially separable;
2. $X$ satisfies $S_1(B_r, B_r)$;
3. $B(X) \in S_{\text{fin}}(S, S)$ and $B(X)$ is sequentially separable;
4. $X$ satisfies $S_{\text{fin}}(B_r, B_r)$;
5. $B_1(X)$ satisfies $S_1(S, S)$;
6. $X$ satisfies $S_1(F_r, F_r)$;
7. $B_1(X)$ satisfies $S_{\text{fin}}(S, S)$.

Proof. It is obvious that $(1) \Rightarrow (3)$.

$(2) \Rightarrow (4)$. By Theorem 1 in [15], $U_{\text{fin}}(B_r, B_r) = S_1(B_r, B_r) = S_{\text{fin}}(B_r, B_r)$.

$(3) \Rightarrow (2)$. Let $\{\mathcal{F}_i\} \subset B_r$ and $\mathcal{S} = \{h_m\}_{m \in \mathbb{N}}$ be a countable sequentially dense subset of $B(X)$. For each $i \in \mathbb{N}$ we consider a countable sequentially dense subset $\mathcal{S}_i$ of $B(X)$ and $\mathcal{F}_i = \{F_i^n\}_{m \in \mathbb{N}}$ where

$$S_i \equiv \{F_i^n\}_{m \in \mathbb{N}} = \{f_i^n \in B(X) : f_i^n \upharpoonright \mathcal{S} = h_m \text{ and } f_i^n \upharpoonright (X \setminus F_i^n) = 1 \text{ for } m \in \mathbb{N} \}.$$

Since $\mathcal{F}_i = \{F_i^n\}_{m \in \mathbb{N}}$ is a Borel $\gamma$-cover of $X$ and $\mathcal{S}$ is a countable sequentially dense subset of $B(X)$, we have that $S_i$ is a countable sequentially dense subset of $B(X)$ for each $i \in \mathbb{N}$. Indeed, let $h \in B(X)$, there is a sequence $\{h_s\}_{s \in \mathbb{N}} \subset \mathcal{S}$ such that $\{h_s\}_{s \in \mathbb{N}}$ converges to $h$. We claim that $\{f_i^n\}_{s \in \mathbb{N}}$ converges to $h$. Let $K = \{x_1, \ldots, x_k\}$ be a finite subset of $X$, $\epsilon > 0$ and let $W = (h, K, \epsilon) = \{ g \in B(X) : |g(x_j) - h(x_j)| < \epsilon \text{ for } j = 1, \ldots, k \}$ be a base neighborhood of $h$, then there is $m_0 \in \mathbb{N}$ such that $K \subset F_i^{m_0}$ for each $m > m_0$ and $h \in W$ for each $s > m_0$. Since $f_i^n \upharpoonright K = h_s \upharpoonright K$ for every $s > m_0$, $f_i^n \upharpoonright W$ for every $s > m_0$. It follows that $\{f_i^n\}_{s \in \mathbb{N}}$ converges to $h$.

Since $B(X)$ satisfies $S_{\text{fin}}(S, S)$, there is a sequence $(\mathcal{F}_i = \{F_i^{m_i} : i \in \mathbb{N}\})$ such that for each $i$, $F_i \subset S_i$, and $\bigcup_{j \in \mathbb{N}} F_i$ is a countable sequentially dense subset of $B(X)$.

For $0 \in B(X)$ there is a sequence $\{f_i^{m_i}(n)\}_{j \in \mathbb{N}} \subset \bigcup_{j \in \mathbb{N}} F_i$ such that $\{f_i^{m_i}(n)\}_{j \in \mathbb{N}}$ converges to 0. Consider a sequence $(F_i^{m_i}(n) : j \in \mathbb{N})$.

$(1)$ $F_i^{m_i}(j) \in \mathcal{F}_i$;

$(2)$ $\{F_i^{m_i}(j) : j \in \mathbb{N}\}$ is a $\gamma$-cover of $X$.

Indeed, let $K$ be a finite subset of $X$ and $U = (0, K, \frac{1}{j})$ be a base neighborhood of 0, then there is $j_0 \in \mathbb{N}$ such that $f_i^{m_i}(j) \in U$ for every $j > j_0$. It follows that $K \subset F_i^{m_i}$ for every $j > j_0$. We thus get that $X$ satisfies $U_{\text{fin}}(B_r, B_r)$, and, hence, by Theorem 1 in [15], $X$ satisfies $S_1(B_r, B_r)$.

$(2) \Rightarrow (1)$. Let $\{S_i\} \subset S$ and $S = \{d_n : n \in \mathbb{N}\} \subset S$. Consider the topology $\tau$ generated by the family $\mathcal{P} = \{f_i^{-1}(G) : G \text{ is an open set of } \mathbb{R} \text{ and } f \in S \cup \{S_i\}\}$. Since $P = S \cup \{S_i\}$ is a countable dense subset of $B(X)$ and $X$ is Tychonoff, we have that the space $Y = (X, \tau)$ is a separable metrizable space. Note that a function $f \in P$, considered as mapping from $Y$ to $\mathbb{R}$, is a continuous function i.e. $f \in C(Y)$ for each $f \in P$. Note also that an identity map $\varphi$ from $X$ on $Y$, is a Borel bijection. By Corollary 12 in [6], $Y$ is a $QN$-space and, hence, by Corollary 20 in [17], $Y$ has the property $S_1(B_r, B_r)$. By Corollary 21 in [17], $B(Y)$ is an $\alpha_2$ space.

Let $q : \mathbb{N} \to \mathbb{N}$ be a bijection. Then we enumerate $\{S_i\}_{i \in \mathbb{N}}$ as $\{S_{(i)}(j) : j \in \mathbb{N}\}_{i \in \mathbb{N}}$. For each $d_n \in S$ there are sequences $s_{n,m} \subset S_{n,m}$ such that $s_{n,m}$ converges to $d_n$ for each $m \in \mathbb{N}$. Since $B(Y)$ is an $\alpha_2$ space, there is $\{b_{n,m} : m \in \mathbb{N}\}$ such that for each $m, b_{n,m} \in S_{n,m}$, and, $b_{n,m} \to d_n (m \to \infty)$. Let $B = \{b_{n,m} : n, m \in \mathbb{N}\}$. Note that $S \subset [B]_{\text{seq}}$.

Since $X$ is a $\sigma$-set (that is, each Borel subset of $X$ is $F_\sigma$)(see [17]), $B_1(X) = B(X)$ and $\varphi(B(Y)) = \varphi(B_1(Y)) \subseteq B(X)$ where $\varphi(B_1) := \{p \circ \varphi : p \in B(Y)\}$ and $\varphi(B_1(Y)) := \{p \circ \varphi : p \in B_1(Y)\}$.

Since $S$ is a countable, sequentially dense subset of $B(X)$, for any $g \in B(X)$ there is a sequence $\{g_n\}_{n \in \mathbb{N}} \subset S$ such that $\{g_n\}_{n \in \mathbb{N}}$ converges to $g$. But $g$ we can consider as a mapping from $Y$ into $\mathbb{R}$ and a set $\{g_n : n \in \mathbb{N}\}$ as subset of $C(Y)$. It follows that $g \in B_1(Y)$. We get that $\varphi(B(Y)) = B(X)$.

We claim that $B \in S$, i.e. that $B_{\text{seq}} = B(X)$. Let $f \in B(Y)$ and $\{f_k : k \in \mathbb{N}\} \subset S$ such that $f_k \to f (k \to \infty)$. For each $k \in \mathbb{N}$ there is $\{f_k^n : n \in \mathbb{N}\} \subset B$ such that $f_k^n \to f_k (n \to \infty)$. Since $Y$ is a $QN$-space (Theorem 16 in
[6], there exists an unbounded \( \beta \in \mathbb{N}^\mathbb{N} \) such that \( \{ f_k^{(j)} \} \) converges to \( f \) on \( Y \). It follows that \( \{ f_k^{(j)} : k \in \mathbb{N} \} \) converge to \( f \) on \( X \) and \( [B]_{seq} = B(X) \).

\[(5) \Rightarrow (6) \] By Velichko's Theorem ([18]), a space \( B_1(X) \) is sequentially separable for any separable metric space \( X \).

Let \( \{ F_i \} \subset F_T \) and \( S = \{ h_n \}_{n \in \mathbb{N}} \) be a countable sequentially dense subset of \( B_1(X) \).

Similarly implication \( (3) \Rightarrow (2) \) we get \( X \) satisfies \( U_{fin}(F_T, F_T) \), and, hence, by Lemma 13 in [17], \( X \) satisfies \( S_1(F_T, F_T) \).

\[(6) \Rightarrow (5) \] By Corollary 20 in [17], \( X \) satisfies \( S_1(B_T, B_T) \). Since \( X \) is a \( \sigma \)-set (see [17]), \( B_1(X) = B(X) \) and, by implication \( (2) \Rightarrow (1) \), we get \( B_1(X) \) satisfies \( S_1(S, S) \).

In [16], Theorem 13 M. Scheepers proved the following result.

**Theorem 3.2** (Scheepers). For \( X \) a separable metric space, the following are equivalent:

1. \( C_p(X) \) satisfies \( S_1(\mathcal{D}, \mathcal{D}) \);
2. \( X \) satisfies \( S_1(\mathcal{O}, \mathcal{O}) \).

We claim the theorem for a space \( B(X) \) of Borel functions.

**Theorem 3.3.** For a set of reals \( X \), the following are equivalent:

1. \( B(X) \) satisfies \( S_1(\mathcal{D}, \mathcal{D}) \);
2. \( X \) satisfies \( S_1(\mathcal{B}_T, \mathcal{B}_T) \).

**Proof.** \((1) \Rightarrow (2) \). Let \( X \) be a set of reals satisfying the hypotheses and \( \beta \) be a countable base of \( X \). Consider a sequence \( \{ B_i \}_{i \in \mathbb{N}} \) of countable Borel \( \omega \)-covers of \( X \) where \( B_i = \{ W^k_j \}_{j \in \mathbb{N}} \) for each \( i \in \mathbb{N} \).

Consider a topology \( \tau \) generated by the family \( \mathcal{P} = \{ W^k_i \cap A : i, j \in \mathbb{N} \} \cup \{ (X \setminus W^k_i) \cap A : i, j \in \mathbb{N} \} \) and \( \rho \in \beta \).

Note that if \( \chi_{\rho} \) is a characteristic function of \( P \) for each \( P \in \mathcal{P} \), then a diagonal mapping \( \varphi = \Delta_{P \in \mathcal{P}} \chi_{\rho} : X \to 2^\omega \) is a Borel bijection. Let \( Z = \varphi(X) \).

Note that \( \{ B_i \} \) is countable open \( \omega \)-cover of \( Z \) for each \( i \in \mathbb{N} \). Since \( B(Z) \) is a dense subset of \( B(X) \), then \( B(Z) \) also has the property \( S_1(\mathcal{D}, \mathcal{D}) \). Since \( C_p(Z) \) is a dense subset of \( B(Z) \), \( C_p(Z) \) has the property \( S_1(\mathcal{D}, \mathcal{D}) \), too.

By Theorem 3.2, the space \( Z \) has the property \( S_1(\mathcal{O}, \mathcal{O}) \). It follows that there is a sequence \( \{ W^k_i(\mathcal{O}) \}_{i \in \mathbb{N}} \) such that \( W^k_i(\mathcal{O}) \in B_1 \) and \( \{ W^k_i(\mathcal{O}) : i \in \mathbb{N} \} \) is an open \( \omega \)-cover of \( Z \). It follows that \( \{ W^k_i(\mathcal{O}) : i \in \mathbb{N} \} \) is Borel \( \omega \)-cover of \( X \).

\((2) \Rightarrow (1) \). Assume that \( X \) has the property \( S_1(\mathcal{B}_T, \mathcal{B}_T) \). Let \( \{ D_k \}_{k \in \mathbb{N}} \) be a sequence countable dense subsets of \( B(X) \) and \( D_k = \{ f^k_i : i \in \mathbb{N} \} \) for each \( k \in \mathbb{N} \). We claim that for any \( f \in B(X) \) there is a sequence \( \{ f_k \} \subset B(X) \) such that \( f_k \in D_k \) for each \( k \in \mathbb{N} \) and \( f \in \{ f_k : k \in \mathbb{N} \} \). Without loss of generality we can assume \( f = 0 \). For each \( f^k_i \in D_k \) let \( W^k_i = \{ x \in X : -\frac{1}{k} < f^k_i(x) < \frac{1}{k} \} \).

If for each \( j \in \mathbb{N} \) there is \( k(j) \) such that \( W^k_1^{(j)}(\mathcal{O}) = X \), then a sequence \( f_k^{(j)} = f_k^{(j)}(\mathcal{O}) \) uniformly converges to \( f \) and, hence, \( f \in \{ f_k^{(j)} : j, i \in \mathbb{N} \} \).

We can assume that \( W^k_i \neq X \) for any \( k, i \in \mathbb{N} \).

(a) \( \{ W^k_i \}_{i \in \mathbb{N}} \) a sequence of Borel sets of \( X \).

(b) For each \( k \in \mathbb{N} \), \( \{ W^k_i : i \in \mathbb{N} \} \) is a \( \omega \)-cover of \( X \).

By (2), \( X \) has the property \( S_1(\mathcal{B}_T, \mathcal{B}_T) \), hence, there is a sequence \( \{ W^k_i(\mathcal{B}_T) \}_{i \in \mathbb{N}} \) such that \( W^k_i(\mathcal{B}_T) \in \{ W^k_i \}_{i \in \mathbb{N}} \) for each \( k \in \mathbb{N} \) and \( \{ W^k_i(\mathcal{B}_T) \}_{i \in \mathbb{N}} \) is a \( \omega \)-cover of \( X \).

Consider \( \{ f_k^{(j)}(\mathcal{B}_T) \} \). We claim that \( f \in \{ f_k^{(j)}(\mathcal{B}_T) : k \in \mathbb{N} \} \). Let \( K \) be a finite subset of \( X \), \( \epsilon > 0 \) and \( U = \{ f, K, \epsilon \} \) be a base neighborhood of \( f \), then there is \( k_0 \in \mathbb{N} \) such that \( 0 < k_0 \epsilon < \epsilon \) and \( K \subset W^k_{i(k)} \). It follows that \( f_k^{(j)}(\mathcal{B}_T) \in U \).

Let \( D = \{ d_n : n \in \mathbb{N} \} \) be a dense subspace of \( B(X) \). Given a sequence \( \{ D_i \}_{i \in \mathbb{N}} \) of dense subspace of \( B(X) \), enumerate it as \( \{ D_{n,m} : n, m \in \mathbb{N} \} \). For each \( n \in \mathbb{N} \), pick \( d_{n,m} \in D_{n,m} \) so that \( d_n \in \{ d_{n,m} : m \in \mathbb{N} \} \). Then \( \{ d_{n,m} : m, n \in \mathbb{N} \} \) is dense in \( B(X) \).

In [16], (Theorem 35) and [4] (Corollary 2.10) proved the following result.
Theorem 3.4 (Scheepers). For a separable metric space, the following are equivalent:
1. $C_p(X)$ satisfies $S_{fn}(D, D)$;
2. $X$ satisfies $S_{fn}(\Omega, \Omega)$.

Then for the space $B(X)$ we have an analogous result.

Theorem 3.5. For a set of reals $X$, the following are equivalent:
1. $B(X)$ satisfies $S_{fn}(D, D)$;
2. $X$ satisfies $S_{fn}(B_{\Omega}, B_{\Omega})$.

Proof. It is proved similarly to the proof of Theorem 3.3.

4 Question of A. Bella, M. Bonanzinga and M. Matveev

In [3], Question 4.3, it is asked to find a sequentially separable selectively separable space which is not selective sequentially separable.

The following theorem answers this question.

Theorem 4.1 (CH). There is a consistent example of a space $Z$, such that $Z$ is sequentially separable, selectively separable, not selectively sequentially separable.

Proof. By Theorem 40 and Corollary 41 in [15], there is a c-Lusin set $X$ which has the property $S_1(B_{\Omega}, B_{\Omega})$, but $X$ does not have the property $U_{fn}(\Gamma, \Gamma)$.

Consider a space $Z = C_p(X)$. By Velichko’s Theorem ([18]), a space $C_p(X)$ is sequentially separable for any separable metric space $X$.

(a). $Z$ is sequentially separable. Since $X$ is Lindelöf and $X$ satisfies $S_1(B_{\Omega}, B_{\Omega})$, $X$ has the property $S_1(\Omega, \Omega)$.

By Theorem 3.2, $C_p(X)$ satisfies $S_1(D, D)$, and, hence, $C_p(X)$ satisfies $S_{fn}(D, D)$.

(b). $Z$ is selectively separable. By Theorem 4.1 in [11], $U_{fn}(\Gamma, \Gamma) = U_{fn}(\Gamma \Gamma, \Gamma)$ for Lindelöf spaces. Since $X$ does not have the property $U_{fn}(\Gamma, \Gamma)$, $X$ does not have the property $S_{fn}(\Gamma \Gamma, \Gamma)$. By Theorem 8.11 in [9], $C_p(X)$ does not have the property $S_{fn}(S, S)$.

(c). $Z$ is not selective sequentially separable.

Theorem 4.2 (CH). There is a consistent example of a space $Z$, such that $Z$ is sequentially separable, countably selectively separable, countably selectively separable, not countably selective sequentially separable.

Proof. Consider the c-Lusin set $X$ (see Theorem 40 and Corollary 41 in [15]), then $X$ has the property $S_1(B_{\Omega}, B_{\Omega})$, but $X$ does not have the property $U_{fn}(\Gamma, \Gamma)$ and, hence, $X$ does not have the property $S_{fn}(B_{\Gamma}, B_{\Gamma})$.

Consider a space $Z = B_1(X)$. By Velichko’s Theorem in [18], a space $B_1(X)$ is sequentially separable for any separable metric space $X$.

(a). $Z$ is sequentially separable. By Theorem 3.3, $B(X)$ satisfies $S_1(D, D)$. Since $Z$ is dense subset of $B(X)$ we have that $Z$ satisfies $S_1(D, D)$ and, hence, $Z$ satisfies $S_{fn}(D, D)$.

(b). $Z$ is countably selectively separable. Since $X$ does not have the property $S_{fn}(B_{\Gamma}, B_{\Gamma})$, by Theorem 3.1, $B_1(X)$ does not have the property $S_{fn}(S, S)$.

(c). $Z$ is not countably selective sequentially separable.
5 Question of A. Bella and C. Costantini

In [5], Question 2.7, it is asked to find a compact $T_2$ sequentially separable space which is not selective sequentially separable.

The following theorem answers this question.

**Theorem 5.1.** ($b < q$) There is a consistent example of a compact $T_2$ sequentially separable space which is not selective sequentially separable.

**Proof.** Let $D$ be a discrete space of size $b$. Since $b < q$, a space $2^b$ is sequentially separable (see Proposition 3 in [13]).

We claim that $2^b$ is not selective sequentially separable.

On the contrary, suppose that $2^b$ is selective sequentially separable. Since $\text{non}(S_{fin}(B_I, B_f)) = b$ (see Theorem 1 and Theorem 27 in [15]), there is a set of reals $X$ such that $|X| = b$ and $X$ does not have the property $S_{fin}(B_I, B_f)$. Hence there exists sequence $(A_n : n \in \mathbb{N})$ of elements of $B_I$ that for any sequence $(B_n : n \in \mathbb{N})$ of finite sets such that for each $n, B_n \subseteq A_n$, we have that $\bigcup_{n \in \mathbb{N}} B_n \notin B_f$.

Consider an identity mapping $\text{id} : D \rightarrow X$ from the space $D$ onto the space $X$. Denote $C_n^i = \text{id}^{-1}(A_n^i)$ for each $A_n^i \in A_n$ and $i \in \mathbb{N}$. Let $C_n = \{ C_n^i \}_{i \in \mathbb{N}}$ (i.e. $C_n = \text{id}^{-1}(A_n)$) and let $S = \{ s_n \}_{n \in \mathbb{N}}$ be a countable sequentially dense subset of $B(D, \{0, 1\}) = 2^b$.

For each $n \in \mathbb{N}$ we consider a countable sequentially dense subset $S_n = \{ f_n^i \}_{i \in \mathbb{N}}$ of $B(D, \{0, 1\})$ where

$S_n = \{ f_n^i \}_{i \in \mathbb{N}} : f_n^i \in B(D, 2) : f_n^i \upharpoonright C_n^i = h_i \text{ and } f_n^i \upharpoonright (X \setminus C_n^i) = 1 \text{ for } i \in \mathbb{N}$.

Since $C_n = \{ C_n^i \}_{i \in \mathbb{N}}$ is a Borel $\gamma$-cover of $D$ and $S$ is a countable sequentially dense subset of $B(D, \{0, 1\})$, we have that $S_n$ is a countable sequentially dense subset of $B(D, \{0, 1\})$ for each $n \in \mathbb{N}$.

Indeed, let $h \in B(D, \{0, 1\})$, there is a sequence $\{ s_n \}_{n \in \mathbb{N}} \subset S$ such that $\{ s_n \}_{n \in \mathbb{N}}$ converges to $h$. We claim that $\{ f_n^i \}_{n \in \mathbb{N}}$ converges to $h$. Let $K = \{ x_1, \ldots, x_k \}$ be a finite subset of $D$, $\epsilon = \{ \epsilon_1, \ldots, \epsilon_k \}$ where $\epsilon_j \in \{0, 1\}$ for $j = 1, \ldots, k$, and $W = \{ h, K, \epsilon \} : \{ g \in B(D, \{0, 1\}) : |g(x_j) - h(x_j)| < \epsilon_j \text{ for } j = 1, \ldots, k \}$ be a base neighborhood of $h$, then there is a number $m_0$ such that $K \subset C_n^i$ for $i > m_0$ and $h_n \in W$ for $s > m_0$. Since $f_n^i \upharpoonright K = h_n \upharpoonright K$ for each $s > m_0, f_n^i \in W$ for each $s > m_0$. It follows that a sequence $\{ f_n^i \}_{n \in \mathbb{N}}$ converges to $h$.

Since $B(D, \{0, 1\})$ is selective sequentially separable, there is a sequence $\{ F_n \} = \{ f_n^1, \ldots, f_n^{k(n)} \} : n \in \mathbb{N}$ such that for each $n, F_n \subset S_n$, and $\bigcup_{n \in \mathbb{N}} F_n$ is a countable sequentially dense subset of $B(D, \{0, 1\})$.

For $0 \in B(D, \{0, 1\})$ there is a sequence $\{ f_n^j \}_{j \in \mathbb{N}} \subset \bigcup_{n \in \mathbb{N}} F_n$ such that $\{ f_n^j \}_{j \in \mathbb{N}}$ converges to $0$. Consider a sequence $\{ C_n^j \}_{j \in \mathbb{N}}$. Then

1. $C_n^j \subset C_n^i$;
2. $\{ C_n^j \}_{j \in \mathbb{N}}$ is a $\gamma$-cover of $D$.

Indeed, let $K$ be a finite subset of $D$ and $U = \{0, K, \{0\} \}$ be a base neighborhood of $0$, then there is a number $j_0$ such that $f_n^j \in U$ for every $j > j_0$. It follows that $K \subset C_n^j$ for every $j > j_0$. Hence, $\{ A_n^j = \text{id}(C_n^j) : j \in \mathbb{N} \} \in B_f$ in the space $X$, a contradiction.

Let $\mu = \min\{ k : 2^k \text{ is not selective sequentially separable} \}$. It is well-known that $p \leq \mu \leq q$ (see [3]).

**Theorem 5.2.** $\mu = \min\{ b, q \}$.

**Proof.** Let $k < \min\{ b, q \}$. Then, by Proposition 3 in [13], $2^k$ is a sequentially separable space.

Let $X$ be a set of reals such that $|X| = k$ and $X$ be a $Q$-set.

Analogously to the proof of implication (2) $\Rightarrow$ (1) in Theorem 3.1, we can claim that $B(X, \{0, 1\}) = 2^X = 2^k$ is selectively sequentially separable.

It follows that $\mu \geq \min\{ b, q \}$.

Since $\mu \leq q$, we suppose that $\mu > b$ and $b < q$. Then, by Theorem 5.1, $2^b$ is not selectively sequentially separable. It follows that $\mu = \min\{ b, q \}$.

In [3], Question 4.12: is it the case $\mu \in \{ p, q \}$?
A partial positive answer to this question is the existence of the following models of set theory (Theorem 8 in [1]):
1. $\mu = p = b < q$;
2. $p < \mu = b = q$;
and
3. $\mu = p = q < b$.

The author does not know whether, in general, the answer can be negative. In this regard, the following question is of interest.

**Question.** Is there a model of set theory in which $p < b < q$?

**References**