A note on the three-way generalization of the Jordan canonical form

Abstract: The limit point $X$ of an approximating rank-$R$ sequence of a tensor $Z$ can be obtained by fitting a decomposition $(S, T, U) \cdot G$ to $Z$. The decomposition of the limit point $X = (S, T, U) \cdot G$ with $G = \text{blockdiag}(G_1, \ldots, G_m)$ can be seen as a three order generalization of the real Jordan canonical form. The main aim of this paper is to study under what conditions we can turn $G_j$ into canonical form if some of the upper triangular entries of the last three slices of $G_j$ are zeros. In addition, we show how to turn $G_j$ into canonical form under these conditions.

Keywords: Low-rank tensor approximations, Jordan canonical form, Tensor decomposition

MSC: 15A18, 15A69

1 Introduction

A tensor can be regarded as a higher-order generalization of a matrix, which takes the form

$$A = (a_{i_1 \cdots i_m}) \quad a_{i_1 \cdots i_m} \in R \quad 1 \leq i_1, \ldots, i_m \leq n$$

Such a multi-array $A$ is said to be an $m$th-order $n$-dimensional square real tensor with $n^m$ entries $a_{i_1 \cdots i_m}$. In this paper, we only consider the case $m = 3$ and real-valued three-way arrays.

Definition 1.1 ([1]). Let $A$ be a $m$th-order $n$-dimensional tensor. The mode-$k$ matrix (or $k$th matrix unfolding) $A_{(k)} \in R^{n \times n^{m-1}}$ is a matrix containing the element $a_{i_1 \cdots i_m}$.

When $A$ is a 3rd-order $n$-dimensional tensor, its mode-$k$ matrices are:

$$A_{(1)} = [A(:, 1, 1) \ldots A(:, n, 1)A(:, 1, 2) \ldots A(:, n, 2) \ldots A(:, n, n)];$$

$$A_{(2)} = [A(1, :, 1) \ldots A(1, :, n)A(2, :, 1) \ldots A(2, :, n) \ldots A(n, :, n)];$$

$$A_{(3)} = [A(1, 1, :) \ldots A(n, 1, :)A(1, 2, :) \ldots A(n, 2, :) \ldots A(n, n, :)].$$

Definition 1.2 ([2]). The multilinear rank of an $I \times J \times K$ array is defined as the triplet (mode-1 rank, mode-2 rank, mode-3 rank). The mode-$k$ rank of a tensor $A$ is defined as the rank of mode-$k$ matrix.

Obviously, a three order tensor has 3 mode-$k$ ranks and the different mode-$k$ ranks of tensor are not necessarily the same [3]. In addition, the rank and the mode-$k$ rank of a same tensor are not necessarily equal even though all the mode-$k$ ranks are equal.

*Corresponding Author: Lu-Bin Cui: School of Mathematics and Information Sciences, Henan Normal University, XinXiang, Henan 453007, China, E-mail: cuilubinrx@163.com
Ming-Hui Li: School of Mathematics and Information Sciences, Henan Normal University, XinXiang, Henan 453007, China, E-mail: 909732996@qq.com

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Let
\[ S_R(I, J, K) = \{ Y \in \mathbb{R}^{I \times J \times K} | \text{rank}(Y) \leq R \} \]

Fitting the CP decomposition [4] to \( Z \) boils down to solving the following minimization problem:
\[ \text{Minimize} \| Z - Y \| \text{ subject to } Y \in S_R(I, J, K) \]

Hence, we are looking for a best rank-R approximation [5] to \( Z \). In [6], A. Stegeman has shown that such a best rank-R approximation may not exist due to the set \( S_R(I, J, K) \) not being closed for \( R \geq 2 \). In this case, we are trying to compute the approximation results in diverging rank-1 terms [7]. This phenomenon can be seen as a three-way generalization of approximate diagonalization of a nondiagonalizable matrix. In [6, 8], A. Stegeman has shown that, analogous to the matrix case, the limit point of the approximating rank-R sequence satisfies a three-way generalization of the real Jordan canonical form. \([6, 9]\) show that the limit point \( X \) is a boundary point of \( S_R(I, J, K) \) and can be obtained by fitting a decomposition \((S, T, U) \cdot G = Z\) to \( Z \), with \( G = \text{blockdiag}(G_1, \ldots, G_m) \) and core block \( G_j \) of size \( d_j \times d_j \times d_j \) and in sparse canonical form. The decomposition of \( X \) has been introduced in \([10–12]\), where the block terms are \((S_j, T_j, U_j) \cdot G_j\). Nondiverging rank-1 terms have an associated core block with \( d_j = 1 \), and core blocks with \( d_j \geq 2 \) are the limit of a group of \( d_j \) diverging rank-1 terms.

For groups of two, or three, or four diverging rank-1 terms, \([6, 8]\) have shown limit point \( X = \sum_{j=1}^{m} X_j \) and its decomposition \( X = (S, T, U) \cdot G = \sum_{j=1}^{m} (S_j, T_j, U_j) \cdot G_j \) have the following results.

**Lemma 1.3** ([13]). For a group of \( d_j = 2 \) diverging rank-1 terms, the limit \( X_j \) can be written as \( X_j = (S_j, T_j, U_j) \cdot G_j \) with \( S_j, T_j, U_j \) of rank 2, and \( 2 \times 2 \times 2 \) array \( G_j \) given by
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix},
\]
we have \( \text{rank}(G_j) = 3 \). Here, we denote the \( 2 \times 2 \times 2 \) array \( G_j \) with \( 2 \times 2 \) slices \( G_1 \) and \( G_2 \) as \([G_1](G_2)\). (3) is referred to as the canonical form of a boundary array of \( S_2(2, 2, 2) \).

**Lemma 1.4** ([6]). For a group of \( d_j = 3 \) diverging rank-1 terms, and \( \min(I, J, K) \geq 3 \), almost all limits \( X_j \) with multilinear rank \((3, 3, 3)\) can be written as \( X_j = (S_j, T_j, U_j) \cdot G_j \) with \( S_j, T_j, U_j \) of rank 3, and \( 3 \times 3 \times 3 \) array \( G_j \) given by
\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
\]
where \( * \) denotes a nonzero entry. We have \( \text{rank}(X_j) = \text{rank}(G_j) = 5 \).

**Lemma 1.5** ([8]). For a group of \( d_j = 4 \) diverging rank-1 terms, and \( \min(I, J, K) \geq 4 \), almost all limits \( X_j \) with multilinear rank \((4, 4, 4)\) can be written as \( X_j = (S_j, T_j, U_j) \cdot G_j \) with \( S_j, T_j, U_j \) of rank 4, and \( 4 \times 4 \times 4 \) array \( G_j \) given by
\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
\]
where \( * \) denotes a nonzero entry. We have \( \text{rank}(X_j) = \text{rank}(G_j) \geq 7 \).

**Remark 1.6.** The proof of Lemma 1.5 in [8] has shown that \( G_j \) has multilinear rank \((4,4,4)\).

However, the proof of Lemma 1.5 in [8] does not take account of the cases that some of the upper triangular entries of the last three slices of \( G_j \) are zeros. To make up for this defect, we assume that some of the upper triangular entries of the last three slices of \( G_j \) are zeros and study whether we can turn \( G_j \) into the canonical form (5). Firstly, we consider the following two examples.
Example 1.7. Let \( \mathcal{G}_j \) be a \( 4 \times 4 \times 4 \) array that satisfies the conditions of Lemma 1.5, and the (1,4) entries of the last three slices are equal to zeros, i.e.,

\[
\mathcal{G}_j = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

By subtracting 3 times the first slice of \( \mathcal{G}_j \) from slice 2,3,4, then we obtain an all-zero diagonal in slice 2,3,4. Similarly, by subtracting 2 times the second slice from the third slice and 3 times the second slice from the fourth slice, \( \mathcal{G}_j \) is of the form

\[
\mathcal{G}_j = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

Next, by subtracting 2 times the third slice from the fourth slice, we can turn the (1,3) and (2,4) entries of slice four into zeros. By subtracting 5 times the third slice from the second slice, we can turn the (2, 4) entry of slice two into zero. Then we obtain the following form

\[
\mathcal{G}_j = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

In each slice of the above \( \mathcal{G}_j \), we add 23/2 times row 2 to row 1 and subtract 23/2 times column 1 from column 2. Then we obtain the following form

\[
\mathcal{G}_j = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

It is apparent from this example that if some of the upper triangular entries of the last three slices of \( \mathcal{G}_j \) are zeros, we can’t turn \( \mathcal{G}_j \) into canonical form (5).

Example 1.8. Let \( \mathcal{G}_j \) be a \( 4 \times 4 \times 4 \) array that satisfies the conditions of Lemma 1.5, and the (1,4) entries of the second and the third slices are equal to zeros, i.e.,

\[
\mathcal{G}_j = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

Using the same method as Example 1.7, we can obtain

\[
\mathcal{G}_j = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

Now, the fourth slice only has its (1,4) entry nonzero, we normalize it to one. Then by subtracting the 23/2 times the fourth slice from the third slice, the (1,4) entry of the third slice can be turned into zero. Then we obtain

\[
\mathcal{G}_j = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]
According to this example, we see that if some of the upper triangular entries of the last three slices of \( G_j \) are zeros, we can turn \( G_j \) into canonical form (5).

A natural question is under what conditions we can turn \( G_j \) into canonical form (5) if some of the upper triangular entries of the last three slices of \( G_j \) are zeros. The answer to this question is the main contribution of this paper.

Remark 1.9. It is worth noting that if some of the upper triangular entries of the last three slices of \( G_j \) are zeros, the entry \( e \) of canonical form (5) may be zero. Therefore, in this paper, we mainly consider the following canonical form

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & e & 0 & 0 & h & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & f & 0 & 0 & 0 & i & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & g & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix},
\]

(6)

where \( e, f, g, h, i \) may be zero, and there must be at least one nonzero entry in every slice.

Now, we prove why we require that there must be at least one nonzero entry in every slice. According to the definition of the mode-\( k \) matrix of a tensor, (5) is actually the mode-1 matrix of \( G_j \). Because the first slice of \( G_j \) is an identity matrix, it follows that the mode-1 rank of \( G_j \) is always 4, no matter what values \( e, f, g, h, i \) take.

The mode-2 matrix of \( G_j \) is

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix},
\]

Since each row of this matrix has a nonzero entry 1, it follows that mode-2 rank of \( G_j \) is always 4, no matter what values \( e, f, g, h, i \) take.

The mode-3 matrix of \( G_j \) is

\[
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix},
\]

For this matrix, if there exists a row that its elements are all zeros, then the mode-3 rank of \( G_j \) does not equal to 4. This contradicts the fact that the mode-3 rank of \( G_j \) is 4. Consequently, \( e, f, g \) can’t be zero at the same time, and \( h, i \) can’t be zero at the same time. This implies that there must be at least one nonzero entry in every slice.

Definition 1.10. For a matrix \( A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\
a_{21} & a_{22} & a_{23} & a_{24} \\
a_{31} & a_{32} & a_{33} & a_{34} \\
a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \), we call the main diagonal elements of matrix \( A \) as the 1st diagonal elements, and \( a_{12}, a_{23}, a_{34} \) as the 2nd diagonal elements, and \( a_{13}, a_{24} \) as the 3rd diagonal elements, and \( a_{14} \) as the 4th diagonal element. The \( k \)th diagonal element of \( A \) is called zero if the elements of the \( k \)th diagonal are all zeros. The \( k \)th diagonal elements of \( A \) is called nonzero if the \( k \)th diagonal of \( A \) has a nonzero element.

The remaining of this paper is organized as follows. In section 2, we study some properties related to \( G_j \). In section 3, we discuss under what conditions can we turn the 2nd diagonal elements of any two slices of the last three slices of \( G_j \) into zeros. In section 4, based on the results of the third section, we analyze under what conditions can we turn the 3rd and 4th diagonal elements of the last three slices of \( G_j \) into zeros. Finally, some concluding remarks are given in section 5.
2 Some properties related to $G_j$

Before we discuss what conditions we need to turn $G_j$ into canonical form (6), we first give some properties related to $G_j$.

**Property 2.1.** If $G_j$ satisfies the conditions of Lemma 1.5, then it can be turned into

$$\begin{bmatrix}
1 & 0 & 0 & 0 & a_2 & e_2 & h_2 & j_2 \\
0 & 1 & 0 & 0 & b_2 & f_2 & i_2 & 0 \\
0 & 0 & 1 & 0 & c_2 & g_2 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & d_2 \\
\end{bmatrix}, \tag{7}
$$

and we can obtain the following three conclusions: (1) $a_p = b_p = c_p = d_p \ (p = 2, 3, 4)$ hold for almost all $G_j$; (2) The equations

$$
\begin{align*}
e_3 f_2 &= f_3 e_2, \\
e_4 f_2 &= f_4 e_2 \\
e_3 f_3 &= f_3 e_3, \\
e_4 f_3 &= f_4 e_3,
\end{align*}
$$

hold for almost all $G_j$; (3) The equations

$$
\begin{align*}
e_3 i_2 - e_3 i_3 + h_3 g_2 - g_3 h_2 &= 0 \\
e_4 i_2 - e_4 i_3 + h_4 g_2 - g_4 h_2 &= 0 \\
e_3 i_3 - e_3 i_4 + h_3 g_3 - g_3 h_3 &= 0
\end{align*}
$$

holds for almost all $G_j$.

**Proof.** The proof of Lemma 3.3 in [8] has shown that $G_j$ can be turned into (7) if it satisfies the conditions of Lemma 3.3.

The first conclusion has been proved in the second conclusion. Similarly to the proof of the vectors, $(e_p, f_p, g_p)$ are proportional for $p = 2, 3, 4$ in Lemma 3.3 of the [8]. Write $A^{(n)}$ in terms of $p = 3$ and compute $Y_2^{(n)} = A^{(n)} C_2^{(n)} (A^{(n)})^{-1}$, which yields matrix (A.10). The entries in this matrix equal those of $Y_2^{(n)}$ in (A.2). It follows that $e_3 f_2 = f_3 e_2, f_3 g_2 = g_3 f_2$. When we write $A^{(n)}$ in terms of $p = 4$ and compute $Y_2^{(n)} = A^{(n)} C_2^{(n)} (A^{(n)})^{-1}$, this yields a matrix similar with (A.10). The entries in this matrix equal those of $Y_3^{(n)}$ in (A.2). It follows that $e_4 f_2 = f_4 e_2, f_4 g_2 = g_4 f_2$. Analogously, when writing $A^{(n)}$ in terms of $p = 3$ and compute $Y_4^{(n)} = A^{(n)} C_4^{(n)} (A^{(n)})^{-1}$, we obtain that $e_4 f_3 = f_4 e_3, f_4 g_3 = g_4 f_3$.

Now we prove the third conclusion. Similarly to the proof of the vectors $(h_3 - \alpha h_2, i_3 - \alpha i_2)$ and $(h_4 - \beta h_2, i_4 - \beta i_2)$ are proportional for $p = 3, 4$. Write $A^{(n)}$ in terms of $p = 3$ and compute $Y_2^{(n)} = A^{(n)} C_2^{(n)} (A^{(n)})^{-1}$, which yields matrix (A.10). The entries in this matrix equal those of $Y_2^{(n)}$ in (A.2). It follows that $e_3 i_2 - e_3 i_3 + h_3 g_2 - g_3 h_2 = 0$. We write $A^{(n)}$ in terms of $p = 4$ and compute $Y_2^{(n)} = A^{(n)} C_2^{(n)} (A^{(n)})^{-1}$, which yields a matrix similar with (A.10). The entries in this matrix equal those of $Y_3^{(n)}$ in (A.2). It follows that $e_4 i_2 - e_4 i_3 + h_4 g_2 - g_4 h_2 = 0$. Analogously, when writing $A^{(n)}$ in terms of $p = 3$ and compute $Y_4^{(n)} = A^{(n)} C_4^{(n)} (A^{(n)})^{-1}$, we obtain that $e_4 i_3 - e_3 i_4 + h_3 g_3 - g_4 h_3 = 0$.

**Remark 2.2.** According to the first conclusion of Property 2.1, if $G_j$ satisfies the conditions of Lemma 1.5, by subtracting $a_p$ times the first slice of $G_j$ from slice $p = 2, 3, 4$, then it can be further turned into

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & e_2 & h_2 & j_2 \\
0 & 1 & 0 & 0 & f_2 & i_2 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & g_2 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{bmatrix},
$$

**Remark 2.3.** According to the second conclusion of the Property 2.1, if $f_2 f_4 i_4 = 0$, the vectors $(e_p, f_p, g_p) \ (p = 2, 3, 4)$ are proportional. If $f_2 = f_4 = 0$, the vectors $(e_p, f_p, g_p) \ (p = 2, 3, 4)$ are disproportionate.
By the first conclusion of Property 2.1, we have turned the 2nd diagonal elements of the last three slices of $G_j$ into zeros. In the next section, we mainly consider turning the 2nd diagonal elements of any two slices of the last three slices of $G_j$ into zeros. Noting that if $f_2 f_3 f_4 \neq 0$, the vectors $(e_p, f_p, g_p)$ $(p = 2, 3, 4)$ are proportionate. This implies that we can directly turn the 2nd diagonal elements of any two slices of the last three slices of $G_j$ into zeros if $f_2 f_3 f_4 \neq 0$. However, in this paper we assume some of the upper triangular entries of the last three slices of $G_j$ are zeros. This means the vectors $(e_p, f_p, g_p)$ $(p = 2, 3, 4)$ may not be proportionate. Consequently, we first discuss all the combinations of $e_p, f_p, g_p$ $(p = 2, 3, 4)$ when some of them are equal to zeros. On the other hand, noting that $e_p, f_p, g_p$ $(p = 2, 3, 4)$ satisfies equations (8), it is easy to get the following conclusion.

**Property 2.4.** There are 91 combinations of $e_p, f_p, g_p$ $(p = 2, 3, 4)$ when some of them are equal to zeros and satisfy the equations (8).

**Proof.** We traverse $e_p, f_p, g_p$ $(p = 2, 3, 4)$ based on the number of nonzero entries, and remove some of the combinations that do not meet the conditions (8).

If one of $e_p, f_p, g_p$ $(p = 2, 3, 4)$ is not equal to zero, and the remaining eight of them are equal to zeros, this yields 9 combinations. For convenience, we only write nonzero entry in each combination, i.e., $e_2, e_3, e_4, f_2, f_3, f_4, g_2, g_3, g_4$. For example, $e_2$ represents $e_2 \neq 0$, $e_3 = e_4 = f_2 = f_3 = f_4 = g_2 = g_3 = g_4 = 0$.

If two of $e_p, f_p, g_p$ $(p = 2, 3, 4)$ are not equal to zeros, and the remaining seven of them are equal to zeros, this yields $C_7^2$ combinations. However, some combinations do not satisfy conditions (8). For example, if two of $e_p, f_p, g_p$ $(p = 2, 3, 4)$ are not equal to zeros, and the remaining seven of them are equal to zeros, this yields $C_7^2$ combinations. However, some combinations do not satisfy conditions (8). For example, combination $e_2 \neq 0, f_2 \neq 0, e_3 = e_4 = f_2 = f_3 = f_4 = g_2 = g_3 = g_4 = 0$ contradicts the condition $f_2 e_2 = e_2 f_2$ of (8). Thus, we remove this combination. Similarly, we remove other combinations that do not meet conditions (8). Finally, by traversing we obtain the following 24 combinations that satisfy conditions (8): $e_2 e_3, e_2 e_4, e_3 e_4, f_2 f_3, f_2 f_4, f_3 f_4, g_2 g_3, g_2 g_4, g_3 g_4, e_2 f_2, e_2 g_2, f_2 g_2, f_3 f_4, e_3 g_3, f_3 g_3, f_4 g_4, e_4 f_4, e_4 g_4, f_4 g_4, e_2 g_3, e_2 g_4, e_3 g_3, e_3 g_4, e_4 g_4$.

Analogously, if three of $e_p, f_p, g_p$ $(p = 2, 3, 4)$ are not equal to zeros, and the remaining six of them are equal to zeros, there are 24 combinations that satisfy conditions (8), i.e., $e_2 e_3 e_4, f_2 f_3 f_4, g_2 g_3 g_4, e_2 f_3 g_3, e_2 f_4 g_4, e_3 f_2 g_3, e_3 f_4 g_4, e_4 f_2 g_3, e_4 f_3 g_4, e_4 f_4 g_4, e_2 f_3 g_4, e_2 f_4 g_4, e_3 f_2 g_4, e_3 f_4 g_4, e_4 f_2 g_4, e_4 f_3 g_4, e_4 f_4 g_4, e_2 f_3 g_4, e_2 f_4 g_4, e_3 f_2 g_4, e_3 f_4 g_4, e_4 f_2 g_4, e_4 f_3 g_4, e_4 f_4 g_4, e_2 f_3 g_4, e_2 f_4 g_4, e_3 f_2 g_4, e_3 f_4 g_4, e_4 f_2 g_4, e_4 f_3 g_4, e_4 f_4 g_4, e_2 f_3 g_4, e_2 f_4 g_4, e_3 f_2 g_4, e_3 f_4 g_4, e_4 f_2 g_4, e_4 f_3 g_4, e_4 f_4 g_4$.

If four of $e_p, f_p, g_p$ $(p = 2, 3, 4)$ are not equal to zeros, and the remaining five of them are equal to zeros, there are 21 combinations that satisfy conditions (8), i.e., $e_2 e_3 e_4, f_2 f_3 f_4, g_2 g_3 g_4, e_2 e_3 e_4, f_2 f_3 f_4, g_2 g_3 g_4, e_2 e_3 e_4, f_2 f_3 f_4, g_2 g_3 g_4$. For example, $e_2 e_3 e_4$ represents $e_2 \neq 0, e_3 \neq 0, e_4 \neq 0$.

If five of $e_p, f_p, g_p$ $(p = 2, 3, 4)$ are not equal to zeros, and the remaining four of them are equal to zeros, there are 6 combinations that satisfy conditions (8), i.e., $e_2 e_3 g_2 g_3 g_4, e_2 e_4 g_2 g_3 g_4, e_3 e_4 g_2 g_3 g_4, e_2 e_4 g_2 g_3 g_4$.

If six of $e_p, f_p, g_p$ $(p = 2, 3, 4)$ are not equal to zeros, and the remaining three of them are equal to zeros, there are 6 combinations that satisfy conditions (8), i.e., $e_2 e_3 e_4 f_2 f_3 f_4, e_2 e_3 e_4 f_2 f_3 f_4, e_2 e_3 e_4 f_2 f_3 f_4, e_2 e_3 e_4 f_2 f_3 f_4, e_2 e_3 e_4 f_2 f_3 f_4, e_2 e_3 e_4 f_2 f_3 f_4$.

If seven (or eight) of $e_p, f_p, g_p$ $(p = 2, 3, 4)$ are not equal to zeros, the remaining two (or one) of them are equal to zeros (or zero), there is no combination that satisfies conditions (8).

The last one combination is that $e_p, f_p, g_p$ $(p = 2, 3, 4)$ are all nonzero, i.e., $e_2 e_3 e_4 f_2 f_3 f_4 g_2 g_3 g_4$.

In the next section, we will study in which combinations of the 91 combinations in Property 2.4 we turn the 2nd diagonal elements of any two slices of the last three slices into zeros.

**Remark 2.5.** Noting that if $e_p, f_p, g_p$ $(p = 2, 3, 4)$ are all zeros, this combination also satisfies the conditions (8). However, another question arises: Under this combination, can we turn $G_j$ into canonical form? Our answer is negative. In fact, if $e_p, f_p, g_p$ $(p = 2, 3, 4)$ are all zeros, then the 2nd diagonal elements of the last three slices of $G_j$ are all zeros. This means in the process of transforming $G_j$ into canonical form, the 2nd diagonal elements of the last three slices of $G_j$ are always zeros. However, in canonical form (6), there exist a slice whose 2nd diagonal elements are nonzero. Therefore, if $e_p, f_p, g_p$ $(p = 2, 3, 4)$ are all zeros, we can’t turn $G_j$ into canonical form.
3 Make the 2nd diagonal of any two slices of the last three slices of $\mathcal{G}_j$ zero

In this section we analyze under what conditions we can turn the 2nd diagonal elements of any two slices of the last three slices of $\mathcal{G}_j$ into zeros.

It is worth noting that if 2nd diagonal elements of any two slices of the last three slices of $\mathcal{G}_j$ can be turned into zeros, then we can continue to analyze the 3rd and 4th diagonal elements. If the 2nd diagonal elements of any two slices of the last three slices of $\mathcal{G}_j$ cannot be turned into zeros, it is meaningless to continue to analyze the 3rd and 4th diagonal elements. Therefore, in this section, we only consider the 2nd diagonal elements of the last three slices of $\mathcal{G}_j$.

Another thing we should pay attention to is why we discuss the conditions that turn the 2nd diagonal elements of any two slices of the last three slices of $\mathcal{G}_j$ into zeros, instead of the conditions that turn the 2nd diagonal elements of the 3rd and the 4th slices into zeros. In fact, if the 2nd diagonal elements of the second slice and the fourth slice (or the second slice and the third slice) are equal to zeros, we can exchange the second slice and the third slice (or the second slice and the fourth slice). The specific operation will be discussed in detail in the next section and so is omitted here.

Now, based on the Property 2.4, we discuss under what conditions we can turn 2nd diagonal elements of any two slices of the last slices of $\mathcal{G}_j$ into zeros. For convenience of the following discussion, we divide the 91 combinations of Property 2.4 into three categories. We regard each category as a set. Each element of the set represents a combination, which can be represented by the nonzero entries of $e_p, f_p, g_p$ ($p = 2, 3, 4$). For example, the element $e_2$ of set $T_1$ represents $e_2 \neq 0, e_3 = e_4 = f_2 = f_3 = f_4 = g_2 = g_3 = g_4 = 0$. The element $e_2 g_3$ of set $T_2$ represents $e_2 \neq 0, g_3 \neq 0, e_3 = e_4 = f_2 = f_3 = f_4 = g_2 = g_4 = 0$.

\[ T_1 = \{ e_2, e_3, e_4, g_2, g_3, g_4, e_2e_4, e_3e_4, g_2g_3, g_2g_4, g_3g_4, e_2f_2, e_2g_2, f_3j_3, g_3j_3, e_4f_4, \]
\[ e_4g_4, f_4g_4, e_2e_4e_3e_4, g_2g_3g_4, e_2j_2g_3, e_3j_3g_4, e_4j_4g_4, e_2e_3f_2f_3, e_2e_4f_4f_3, e_2e_4f_4f_3g_4, \]
\[ f_2f_4g_2f_4g_3g_4, e_2e_3e_4f_3f_4g_3g_4, e_2f_2g_3j_3, e_2f_4g_4j_4, e_2g_2e_4f_4g_4, \}

\[ T_2 = \{ e_2g_3, e_2g_4, e_3g_2, e_3g_4, e_4g_2, e_4g_3, e_2g_3g_4, e_2g_4g_3, e_3g_2g_3, e_3g_2g_4, e_4g_2g_3, e_4g_2g_4, \}
\[ e_4g_3g_4, e_2e_3g_2, e_2e_3g_3, e_2e_4g_2, e_2e_4g_3, e_2e_4g_4, e_3e_4g_2, e_3e_4g_3, e_3e_4g_4, \]
\[ e_2e_3g_2g_3, e_2e_3g_2g_4, e_2e_3g_3g_4, e_2e_4g_2g_3, e_2e_4g_3g_4, e_2e_4g_4g_3, e_2e_4g_4g_4, \}
\[ e_2e_3g_2g_3g_4, e_2e_3g_2g_4g_3, e_2e_3g_3g_4g_3, e_2e_4g_2g_3g_4, e_2e_4g_3g_4g_3, e_2e_4g_4g_3g_4, \}
\[ e_2e_3g_2g_3g_4g_3, e_2e_3g_2g_4g_3g_4, \}

\[ T_3 = \{ f_2, f_3, f_4, f_2f_3, f_2f_4, f_3f_4, f_2f_3f_4, e_2e_4g_2g_3, e_2e_4g_2g_4, e_3e_4g_3g_4, e_2e_4g_2g_4g_3, \}

Now we show that under each combination of $T_1$ we can turn 2nd diagonal elements of any two slices of the last three slices of $\mathcal{G}_j$ into zeros. For each combination in $T_1$, nonzero entries in $e_p, f_p, g_p$ ($p = 2, 3, 4$) are either in the same slice or in the same position of different slices.

In fact, if nonzero entries are in the same slice, there are just two slices of the last three slices of $\mathcal{G}_j$ whose 2nd diagonal elements are zeros. For example, the element $e_2f_3g_3$ of $T_1$ represents $e_3 \neq 0, f_3 \neq 0, g_3 \neq 0, e_2 = e_4 = f_2 = f_4 = g_2 = g_4 = 0$. From Remark 2.2, $G_j$ is of the form

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & h_2 & j_2 & 0 & e_3 & h_3 & j_3 & 0 & 0 & h_4 & j_4 \\
0 & 1 & 0 & 0 & 0 & 0 & j_2 & 0 & f_3 & i_3 & 0 & 0 & 0 & i_4 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & g_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

From the above $G_j$, we can see that 2nd diagonal elements of the second and fourth slices are zeros.

If nonzero entries lie in the same position of different slices, it will yield two possibilities. The first case is that there exist two slices whose 2nd diagonal elements are nonzero. For these two slices, by subtracting one
slice from the other slice we can turn the 2nd diagonal elements of one of them into zeros. After this, we can obtain two slices of the last three slices of \( G_j \) whose 2nd diagonal elements are equal to zeros. For example, the element \( e_3 e_4 f_3 f_4 \) in \( T_1 \) represents \( e_3 \neq 0, e_4 \neq 0, f_3 \neq 0, f_4 \neq 0, e_2 = f_2 = g_2 = g_3 = g_4 = 0 \). From Remark 2.2, \( G_j \) is of the form

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & h_2 & j_2 & 0 & e_3 & h_3 & j_3 & 0 & e_4 & h_4 & j_4 \\
0 & 1 & 0 & 0 & 0 & 0 & i_2 & 0 & f_3 & i_3 & 0 & f_4 & i_4 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}.
\]

For this \( G_j \), through subtracting \( e_4 / e_3 \) times the third slice from the fourth slice (or \( e_3 / e_4 \) times the fourth slice from the third slice), we can turn \( e_3 \) and \( f_4 \) (or \( e_2 \) and \( f_3 \)) into zeros because \( e_4 f_3 = f_4 e_3 \) holds for almost all \( G_j \) in (8). Then we get the second and fourth slices (or the second and third slices) whose 2nd diagonal elements are zeros. The second case is that there exist three slices whose 2nd diagonal elements are nonzero. For these three slices, through subtracting one slice from another two slices, then we can turn the 2nd diagonal elements of two of them into zeros. For example, the element \( e_2 e_3 e_4 f_3 f_4 \) in \( T_1 \) represents \( e_2 = 0, e_3 = 0, e_4 = 0, f_2 = 0, f_3 = 0, f_4 = 0, g_2 = g_3 = g_4 = 0 \). From Remark 2.2, \( G_j \) is of the form

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & e_2 & h_2 & j_2 & 0 & e_3 & h_3 & j_3 & 0 & e_4 & h_4 & j_4 \\
0 & 1 & 0 & 0 & 0 & 0 & f_2 & i_2 & 0 & f_3 & i_3 & 0 & f_4 & i_4 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}.
\]

For this \( G_j \), through subtracting \( e_2 / e_3 \) times the second slice from the third slice and \( e_4 / e_2 \) times the second slice from the fourth slice (or \( e_2 / e_3 \) times the third slice from the second slice and \( e_4 / e_2 \) times the third slice from the fourth slice), we can turn \( e_3, f_4, f_3 \) (or \( e_2, f_3, e_4, f_4 \) or \( e_2, f_3, e_4, f_4 \)) into zeros because \( e_3 f_2 = f_3 e_2, e_4 f_4 = f_3 e_2, e_4 f_4 = f_3 e_2 \) holds for almost all \( G_j \) in (8). Then we get the third and fourth (or the second and fourth or the second and third) slices whose 2nd diagonal elements are zeros.

Now we show that under each combination of \( T_2 \), we can’t turn 2nd diagonal elements of any two slices of the last three slices of \( G_j \) into zeros. For each combination in \( T_2 \), we find that nonzero entries in \( e_p, f_p, g_p \) (\( p = 2, 3, 4 \)) are in different position of different slices, which results in at least two slices whose 2nd diagonal elements can’t be turned into zeros. For example, the element \( e_2 g_3 \) in \( T_2 \) represents \( e_2 = 0, g_3 = 0, e_3 = e_4 = f_2 = f_3 = f_4 = g_2 = g_4 = 0 \). From Remark 2.2, \( G_j \) is of the form

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & e_2 & h_2 & j_2 & 0 & 0 & h_3 & j_3 & 0 & 0 & h_4 & j_4 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & i_2 & 0 & 0 & i_3 & 0 & 0 & i_4 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}.
\]

For this \( G_j \), if we can turn \( e_2 \) into zero, then we get the second and fourth slices whose 2nd diagonal elements are all zeros; if we can turn \( g_1 \) into zero, then we get the third and fourth slices whose 2nd diagonal elements are all zeros. On the other hand, \( e_2 \) can be turned into zero only by the \((1,2)\) entry of the third slice or the \((1,2)\) entry of the fourth slice; \( g_1 \) can be turned into zero only by the \((3,4)\) entry of the second slice or the \((3,4)\) entry of the fourth slice. However, \((1,2)\) entries of the third and fourth slices are all zeros; \((3,4)\) entries of the second and fourth slices are all zeros; so \( e_2 \) and \( g_1 \) can’t be turned into zeros. This implies that if nonzero entries are in different position of different slices, we can’t turn any two slices of the last three slices whose 2nd diagonal elements into zeros. Therefore, it makes no sense to continue to analyze the 3rd and 4th diagonal elements. Consequently, in the next section, we no longer consider all the combinations in \( T_2 \).

For each combination of \( T_3 \), we have not found the relationship between \( e_p, f_p, g_p \) and \( h_p, i_p, j_p \) for \( p = 2, 3, 4 \). Consequently, in the next section analysis we no longer consider all the combinations in \( T_3 \).
4 Make the 3rd and 4th diagonal of the last three slices of $\mathcal{G}_j$ zero

In this section we analyze under what conditions we can turn the 3rd and the 4th diagonal elements of the last three slices of $\mathcal{G}_j$ into zeros.

Our main idea is as follows. For each combination in $T_1$, combining with expression (9), we can obtain the relationship between $e_p, f_p, g_p$ and $h_p, t_p$ ($p = 2, 3, 4$). Through the relationship between them, we give conditions that can turn the 3rd and 4th diagonal elements of the last three slices of $\mathcal{G}_j$ into zeros. Under these conditions, we can turn $\mathcal{G}_j$ into canonical form (6).

**Theorem 4.1.** If one of $e_p, f_p, g_p$ ($p = 2, 3, 4$) is not equal to zero, the remaining eight of them are equal to zeros; there are 6 combinations in $T_1$, i.e., $e_p, g_p$ ($p = 2, 3, 4$). If $e_x \neq 0$, the other of $e_p, f_p, g_p$ ($p = 2, 3, 4$) are equal to zeros, under conditions $i_x = i_y = i_z = 0, h_x h_y = h_x z$, or if $g_x \neq 0$, the other of $e_p, f_p, g_p$ ($p = 2, 3, 4$) are equal to zeros, under conditions $h_x h_y h_z = 0, j_x i_x \neq i_x j_x$, where $x, y, z \in \{2, 3, 4\}$ and $x \neq y \neq z$, we can turn $\mathcal{G}_j$ into canonical form (6).

**Proof.** Here, we only consider the combination that $e_2 \neq 0$, the remaining eight of $e_p, f_p, g_p$ ($p = 2, 3, 4$) are equal to zeros. The combinations that $e_3 \neq 0$ (or $e_4 \neq 0$ or $g_2 \neq 0$ or $g_3 \neq 0$ or $g_4 \neq 0$), the remaining eight of $e_p, f_p, g_p$ ($p = 2, 3, 4$) are equal to zeros can be proved in a similar way as the combination that $e_2 \neq 0$, the remaining eight of $e_p, f_p, g_p$ ($p = 2, 3, 4$) are equal to zeros.

If $e_2 \neq 0, e_3 = e_4 = f_2 = f_3 = f_4 = g_2 = g_3 = g_4 = 0$, according to (9), we have $i_3 = i_4 = 0$, then $\mathcal{G}_j$ is of the form

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & e_2 & h_2 & j_2 & 0 & 0 & h_3 & j_3 & 0 & 0 & h_4 & j_4 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Because the mode-3 rank of $\mathcal{G}_j$ is 4, then $h_3 j_4 \neq j_3 h_4$. Next, we show why the mode-3 rank of $\mathcal{G}_j$ is not equal to 4 if $h_3 j_4 = j_3 h_4$. Suppose $h_3 = 0$, then $j_3 h_4 = 0$. This yields three possibilities. The first case is $j_3 = 0, h_4 \neq 0$, then all the entries of the third slice are equal to zeros. The second case is $j_3 \neq 0, h_4 = 0$, if $i_4 = 0$, then all the entries of the fourth slice are equal to zeros; if $i_4 \neq 0$, by subtracting $j_4/j_3$ times the third slice from the fourth slice (or $j_3/j_4$ times the fourth slice from the third slice), we can turn $j_4$ (or $j_3$) into zero, then all the entries of the fourth (or third) slice are equal to zeros. The third case is $j_3 = 0, h_4 = 0$, then all the entries of the third slice are equal to zeros. The situations where we suppose that $j_4 = 0$ or $j_3 = 0$ or $h_4 = 0$ can be dealt with analogously. Suppose $h_3, j_4, j_3, h_4$ are nonzero, by subtracting $h_4/h_3$ times the third slice from the fourth slice (or $h_3/h_4$ times the fourth slice from the third slice), we can turn $h_4$ and $j_4$ (or $h_3$ and $j_3$) into zeros. Then all the entries of the third slice (or the fourth slice) are equal to zeros. From the above discussion we can see that, if $h_3 j_4 = j_3 h_4$, there always exists a slice whose entries are all equal to zeros. This implies that the mode-3 rank of $\mathcal{G}_j$ is 3. Thus we draw the conclusion that $h_3 j_4 \neq j_3 h_4$.

Now we prove that $i_2$ must be equal to zero, because $i_2$ can be turned into zero only through $(2,3)$ or $(3,4)$ entry of the second slice or through $(2,4)$ entry of the third or $(2,4)$ entry of the fourth slice. However, $(2,3)$ and $(3,4)$ entries of the second slice, $(2,4)$ entry of the third and fourth slices are all equal to zeros. This means in the process of transforming $\mathcal{G}_j$ into canonical form, $i_2$ can’t be turned into zero. Therefore, we must have $i_2$ equal to zero.

Now we prove how we turn $\mathcal{G}_j$ into canonical form if $e_2 \neq 0, e_3 = e_4 = f_2 = f_3 = f_4 = g_2 = g_3 = g_4 = 0$, under conditions that $i_2 = i_3 = i_4 = 0$ and $h_3 j_4 \neq j_3 h_4$. In fact, under these conditions, $\mathcal{G}_j$ have the following form

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & e_2 & h_2 & j_2 & 0 & 0 & h_3 & j_3 & 0 & 0 & h_4 & j_4 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$
Firstly, we discuss how to standardize the last two slices. Because $h_3j_h \neq j_3h_4$, suppose $h_3 = 0$, then $j_3h_4 \neq 0$. If $j_4 \neq 0$, then the third slice only has its $(1, 4)$ entry $j_3$ nonzero, the fourth slice only has its $(2, 3)$ entry $h_4$ nonzero. We normalize them to one. By exchanging the third slice and the fourth slice, then we obtain the following form

$$
\begin{bmatrix}
  1 & 0 & 0 & 0 & e_2 & h_2 & j_2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
  0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

(10)

If $j_4 \neq 0$, by subtracting $j_4/j_3$ times the third slice from the fourth slice, we can turn $j_4$ into zero. After this, the third slice only has $j_3$ is nonzero, the fourth slice only has $h_4$ is nonzero. Similarly, we normalize them to one. By exchanging the third slice and the fourth slice, then we obtain (10). The situations where we suppose that $j_4 = 0$ or $j_3 = 0$ or $h_4 = 0$ can be dealt with analogously.

Next, we discuss how to standardize the second slice. If $h_2 = j_2 = 0$, then we have transformed $G_j$ into canonical form. If $h_2, j_2$ are nonzero, we can turn them into zeros. The specific method is: $h_2$ can be turned into zero by subtracting $h_2$ times the third slice from the second slice, $j_2$ can be turned into zero by subtracting $j_2$ times the fourth slice from the second slice.

Consequently, if $e_x \neq 0$, the remaining eight of $e_p, f_p, g_p \ (p = 2, 3, 4)$ are equal to zeros, under conditions $i_x = i_y = i_z = 0$ and $h_2j_2 \neq j_2h_2$, where $x = 2, y = 3, z = 4$ or $x = 2, y = 4, z = 3$, we can turn $G_j$ into canonical form.

\[\square\]

**Remark 4.2.** In fact, theorem 4.1 contains the following six cases.

Let $x = 2, y = 3, z = 4$ or $x = 2, y = 4, z = 3$.

- **Case 1:** If $e_2 \neq 0, e_3 = e_4 = f_2 = f_3 = f_4 = g_2 = g_3 = g_4 = 0$, under conditions $i_2 = i_3 = i_4 = 0$ and $h_3j_4 \neq j_3h_4$, we can turn $G_j$ into canonical form.

- **Case 2:** If $g_2 \neq 0, e_2 = e_3 = e_4 = f_2 = j_3 = j_4 = g_3 = g_4 = 0$, under conditions $h_2 = h_3 = h_4 = 0$ and $j_3i_2 \neq i_2j_3$, we can turn $G_j$ into canonical form.

Let $x = 3, y = 2, z = 4$ or $x = 3, y = 4, z = 2$.

- **Case 3:** If $e_3 \neq 0, e_2 = e_4 = f_2 = f_3 = f_4 = g_2 = g_3 = g_4 = 0$, under conditions $i_2 = i_3 = i_4 = 0$ and $h_2j_4 \neq j_2h_4$, we can turn $G_j$ into canonical form.

- **Case 4:** If $g_3 \neq 0, e_2 = e_3 = e_4 = f_2 = f_3 = f_4 = g_2 = g_3 = g_4 = 0$, under conditions $h_2 = h_3 = h_4 = 0$ and $j_2i_4 \neq i_2j_4$, we can turn $G_j$ into canonical form.

Let $x = 4, y = 2, z = 3$ or $x = 4, y = 3, z = 2$.

- **Case 5:** If $e_4 \neq 0, e_2 = e_3 = f_2 = f_3 = f_4 = g_2 = g_3 = g_4 = 0$, under conditions $i_2 = i_3 = i_4 = 0$ and $h_2j_3 \neq j_2h_3$, we can turn $G_j$ into canonical form.

- **Case 6:** If $g_4 \neq 0, e_2 = e_3 = e_4 = f_2 = f_3 = f_4 = g_2 = g_3 = g_4 = 0$, under conditions $h_2 = h_3 = h_4 = 0$ and $j_2i_3 \neq i_2j_3$, we can turn $G_j$ into canonical form.

For the sake of simplicity, we write the above six cases as follows:

- $e_x \neq 0, \text{the other are zero, } i_x = i_y = i_z = 0, h_3j_4 \neq j_3h_4$;
- $g_x \neq 0, \text{the other are zero, } h_x = h_y = h_z = 0, i_2j_2 \neq j_2i_2$;

where $x, y, z \in \{2, 3, 4\}$ and $x \neq y \neq z$.

The following theorem is also written in a similar way, and we don’t repeat it. For example, $e_xe_y \neq 0, i_x = i_y = i_z = 0, h_3j_4 \neq j_3h_4$, where $x = 2, y = 3, z = 4$, in Theorem 4.3 represents if $e_2 \neq 0, e_3 \neq 0, e_4 = f_2 \neq f_3 \neq f_4 = g_2 = g_3 = g_4 = 0$, under conditions $i_2 = i_3 = i_4 = 0$ and $h_3j_4 \neq j_3h_4$, can we turn $G_j$ into canonical form.

**Theorem 4.3.** If two of $e_p, f_p, g_p \ (p = 2, 3, 4)$ are not equal to zeros, the remaining seven of them are equal to zeros, there are 15 combinations in $T_1$, i.e. $e_2e_3, e_2e_4, e_3e_4, g_2g_3, g_2g_4, g_3g_4, e_1f_2, e_1f_3, e_1f_4, f_2g_2, f_3g_3, f_4g_4,$
For each combinations, we can turn $G_j$ into canonical form under the following conditions:

$$
e_x e_y = 0, i_x = i_y = 0, h_j z_j \ne \tilde{h}_j z_j;$$
$$g_x g_y = 0, h_x = h_y = 0, i_j z_j \ne \tilde{i}_j z_j;$$
$$e_x f_x = 0, i_x = i_z = 0, h_y z_j \ne \tilde{j}_y h_z;$$
$$f_x g_x = 0, h_y = h_z = 0, i_j z_j = \tilde{j}_y i_z;$$
$$e_x g_x = 0, i_x = h_y z_j = 0, i_z = i_z = 0;$$
$$(e_x g_x \ne 0, i_y h_y i_z = 0, h_y i_y = i_x h_y, h_j z_j \ne \tilde{j}_y h_z);$$

where $x, y, z \in \{2, 3, 4\}$ and $x = y = z$.

**Proof.** Firstly, we discuss the combination that $e_x e_3 \ne 0$, the remaining seven of $e_p, f_p, g_p (p = 2, 3, 4)$ are equal to zeros. The combinations $e_x e_4, e_3 e_4, g_2 g_3, g_2 g_4, g_3 g_4$ can be similar as in the discussion with $e_x e_3$.

If $e_x e_3 = 0$, $e_4 = f_2 = f_3 = f_4 = g_2 = g_3 = g_4 = 0$, according to (9), we can obtain $i_4 = 0$ and $e_2 i_3 = e_3 i_2$. Because $e_x e_3 \ne 0$, then $i_2$ and $i_3$ are zero or nonzero at the same time. This yields two possibilities.

The first case is $i_2 = i_3 = 0$. Then $G_j$ is of the form

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & e_2 & h_j & h_2 & 0 & e_3 & h_3 & j_3 & 0 & 0 & h_a & j_a \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

By subtracting $e_3/e_2$ times the second slice from the third slice, we can turn $e_3$ into zero. Then we obtain the following form:

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & e_2 & h_j & h_2 & 0 & 0 & \tilde{h}_3 & j_3 & 0 & 0 & h_a & j_a \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
$$

where $\tilde{h}_3 = h_3 - e_3 h_2/e_2, \tilde{j}_3 = j_3 - e_3 j_2/e_2$. According to Theorem 4.1, under condition $\tilde{h}_3 j_4 = \tilde{j}_3 h_a$, we can turn $G_j$ into canonical form. Analogously, by subtracting $e_2/e_3$ times the third slice from the second slice, we can turn $e_2$ into zero, under condition $\tilde{h}_2 j_4 = \tilde{j}_2 h_a$, where $\tilde{h}_2 = h_2 - e_2 h_1/e_3, \tilde{j}_2 = j_2 - e_2 j_3/e_3$, we can also turn $G_j$ into canonical form.

The second case is $i_2 i_3 \ne 0$. By subtracting $e_3/e_2$ times the second slice from the third slice, we can turn $e_3$ into zeros according to $e_2 i_3 = e_3 i_2$. Then we obtain the following form:

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & e_2 & h_j & h_2 & 0 & 0 & \tilde{h}_3 & j_3 & 0 & 0 & h_a & j_a \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
$$

where $\tilde{h}_3 = h_3 - e_3 h_2/e_2, \tilde{j}_3 = j_3 - e_3 j_2/e_2$. According to Theorem 4.1, only under conditions $i_2 = 0$ and $\tilde{h}_3 j_4 = \tilde{j}_3 h_a$, we can turn $G_j$ into canonical form. This contradicts the fact that $i_2 i_3 \ne 0$.

Thus we conclude that if $e_x e_y = 0$, the remaining seven of $e_p, f_p, g_p (p = 2, 3, 4)$ are equal to zeros, under conditions $i_x = i_y = i_z = 0, h_j z_j = \tilde{j}_y h_z$, where $x = 2, y = 3, z = 4$ or $x = 3, y = 2, z = 4$, we can turn $G_j$ into canonical form (6).

Next, we discuss the combination that $e_x f_2 = 0$, the remaining seven of $e_p, f_p, g_p (p = 2, 3, 4)$ are equal to zeros. The combinations $f_j g_2, e_3 f_3, g_3 f_3, e_4 f_4, f_j g_4$ can be similar as in the discussion with $e_x f_2$.

If $e_x f_2 \ne 0$, $e_3 = e_4 = f_3 = f_4 = g_3 = g_4 = 0$, according to (9), we can obtain $i_3 = i_4 = 0$. Because the mode-3 rank of $G_j$ is 4, then $h_j j_4 \ne \tilde{j}_4 h_a$. The way to standardize the last two slices is the same as in Theorem 4.1, so we don't repeat it here. Now, we show how to standardize the second slice. The way to turn $h_2$ and $j_2$ into zeros is the same as in Theorem 4.1. It remains to consider how to turn $i_2$ into zero. In fact, if $i_2 = 0$, then we have transformed $G_j$ into canonical form. If $i_2 \ne 0$, for every slice of $G_j$, by subtracting $i_2/f_2$
times column 3 from column 4 and adding $i_2/f_2$ times row 4 to row 3, we can turn $i_2$ into zero. Then the (1,4) entry of slice three is turned into $-i_2/f_2$. Next, by subtracting $-i_2/f_2$ times the fourth slice form the third slice, we can turn (1,4) entry of slice three into zero.

Consequently, if $e_{i_2}f_x \neq 0$, the remaining seven of $e_p, f_p, g_p (p = 2, 3, 4)$ are equal to zeros, under conditions $i_y = i_x = 0$ and $h_{jy} = h_{jx}$, where $x = 2, y = 3, z = 4$ or $x = 2, y = 4, z = 3$, we can turn $G_j$ into canonical form.

Next, we discuss the combination that $e_2g_2 \neq 0$, the remaining seven of $e_p, f_p, g_p (p = 2, 3, 4)$ are equal to zeros. The combinations $e_3g_3, e_4g_4$ can be similar as in the discussion with $e_2g_2$.

If $e_2g_2 \neq 0$, $e_3 = e_4 = f_2 = f_3 = f_4 = g_3 = g_4 = 0$, according to (9), we have $e_2i_3 = h_3g_2, e_3i_4 = h_4g_2$. Because $e_2g_2 \neq 0$, so we must have $i_3$ and $h_3$ being zero or nonzero at the same time, $i_4$ and $h_4$ being zero or nonzero at the same time. This yields four possibilities.

The first case is $i_3 = h_3 = 0, i_4h_4 \neq 0$. Then $G_j$ is of the form

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & e_2 & h_2 & j_2 & 0 & 0 & j_3 & 0 & 0 & h_4 & j_4 \\
0 & 1 & 0 & 0 & 0 & 0 & i_2 & 0 & 0 & 0 & 0 & 0 & i_4 \\
0 & 0 & 1 & 0 & 0 & 0 & g_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Because the mode-3 rank of $G_j$ is 4, then $j_3 \neq 0$. Next, we normalize $j_3$ to one, and it can be used to turn $j_2$ and $j_4$ into zeros if $j_2$ and $j_4$ are nonzero. After this, by exchanging the third slice and the fourth slice, then we can obtain the following form

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & e_2 & h_2 & 0 & 0 & h_4 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & i_2 & 0 & 0 & i_4 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & g_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

If $h_2$ and $i_2$ are equal to zeros, then we have turned $G_j$ into canonical form. If one of $h_3, i_2$ is equal to zero, we can’t turn $G_j$ into canonical form. In fact, if $h_2 = 0$, $i_2 \neq 0$, for every slice, by subtracting $i_2/g_2$ times row 3 from row 2 and adding $i_2/g_2$ times column 2 to column 3, we can turn $i_2$ into zero. But meanwhile, $h_2$ is turned into nonzero. Similarly, if $h_2 \neq 0, i_2 = 0$, while turning $h_2$ into zero, $i_2$ is turned into nonzero. Similarly to the discussion of the above, if $h_2$ and $i_2$ are nonzero, after turn $h_2$ into zero, $i_2$ is turned into $i_2 = i_2 + h_2g_2/e_2$. While turning $i_2$ into zero, $h_2$ is turned into nonzero. If we add a restriction condition $h_2i_4 = i_2h_4$, by subtracting $h_4/h_2$ times the third slice from the second slice, we can turn $h_2$ and $i_2$ into zeros. Hence, we have turned $G_j$ into canonical form. From the above discussion we can see that, if $i_3 = h_3 = 0, i_4h_4 \neq 0$, only under condition $h_2i_4 = i_2h_4$, we can turn $G_j$ into canonical form.

The second case is $i_3h_3 \neq 0, i_4 = h_4 = 0$. This situation can be similar as in the discussion with $i_3 = h_3 = 0, i_4h_4 \neq 0$.

The third case is $i_3 = h_3 = i_4 = h_4 = 0$. Then $G_j$ is of the form

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & e_2 & h_2 & j_2 & 0 & 0 & 0 & j_3 & 0 & 0 & j_4 \\
0 & 1 & 0 & 0 & 0 & 0 & i_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & g_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

If one of $j_3$, $j_4$ is equal to zero, then the mode-3 rank of $G_j$ is equal to 3. If $j_3$ and $j_4$ are all equal to zeros, then the mode-3 rank of $G_j$ is equal to 2. If $j_3$ and $j_4$ are nonzero, by subtracting $j_3/j_4$ times the third slice from the fourth slice, we can turn $j_4$ into zero. Then the mode-3 rank of $G_j$ is equal to 3. Consequently, if $i_3 = h_3 = i_4 = h_4 = 0$, we can’t turn $G_j$ into canonical form.

The fourth case is $i_3h_3i_4h_4 \neq 0$. It follows from $e_2i_3 = h_3g_2, e_2i_4 = h_4g_2$ that $i_3h_4 = i_4h_3$. This means the vectors $(h_3, i_3)$ and $(h_4, i_4)$ are proportional. If the vectors $(h_3, j_3)$ and $(h_4, j_4)$ are also proportional, then the mode-3 rank of $G_j$ is not equal to 4. Therefore, we conclude that $h_3j_4 = h_4j_3$. Similarly to the discussion of the case that $i_3 = h_3 = 0, i_4h_4 \neq 0$. If we add a restriction condition $h_2i_4 = i_2h_4$ or $h_2i_3 = i_2h_3$, we can turn $h_2$ and $i_2$ into zero. Hence, if $i_3h_3i_4h_4 \neq 0$, under conditions $h_3j_4 = h_4j_3, h_2i_4 = i_2h_4$ or $h_2j_3 = h_4j_3, h_2i_3 = i_2h_3$, we can turn $G_j$ into canonical form.
Based on the above argument, we draw the conclusion that if $e_xg_x ≠ 0$, the remaining seven of $e_p, f_p, g_p$ (p=2,3,4) are equal to zeros, under conditions $i_y = h_y = 0, i_zh_z = 0, j_y = 0, h_xz = i_xh_x$ or $i_y, h_xz = 0, h_xh_y = i_xh_y$, $h_xj_x ≠ j_yh_z$, where $x = 2, y = 3, z = 4$ or $x = 2, y = 4, z = 3$, we can turn $G_j$ into canonical form. \[ \Box \]

**Theorem 4.4.** If three of $e_p, f_p, g_p$ (p = 2, 3, 4) are not equal to zeros, the remaining six of them are equal to zeros, there have 5 combinations in $T_1$, i.e. $e_2e_3e_4, g_2g_3g_4, e_2f_2g_2, e_3f_3g_3, e_4f_4g_4$. For each combinations, $G_j$ can be turned into canonical form under the following conditions:

$$ e_xe_ye_z ≠ 0, i_x = i_y = i_z = 0, \tilde{h}_yj_y ≠ \tilde{h}_zj_z; $$

$$ g_xg_yg_z ≠ 0, h_x = h_y = h_z = 0, \tilde{i}_yj_y ≠ \tilde{i}_zj_z; $$

$$ e_xf_xg_x ≠ 0, i_y = h_y = 0, i_xh_x = 0, j_y ≠ 0; $$

$$ (e_xf_xg_x ≠ 0, i_yh_xi_xh_z = 0, j_yh_z ≠ h_yj_z); $$

where $x, y, z ∈ \{2, 3, 4\}$ and $x ≠ y ≠ z$.

**Proof.** Firstly, we discuss the combination that $e_2e_3e_4 ≠ 0$, the remaining six of $e_p, f_p, g_p$ (p = 2, 3, 4) are equal to zeros. The combination $g_2g_3g_4$ can be similar as in the discussion with $e_2e_3e_4$.

If $e_2e_3e_4 ≠ 0, f_2 = f_3 = f_4 = g_2 = g_3 = g_4 = 0$, according to (9), we have $e_2i_2 = i_3e_2$, $e_4i_2 = e_2i_4$, $e_4i_3 = e_4i_4$. Because $e_2e_3e_4 ≠ 0$, so we must have $i_2, i_3, i_4$ being zero or nonzero at the same time. This yields two possibilities. The first case is $i_2 = i_3 = i_4 = 0$. The second case is $i_2i_3i_4 ≠ 0$. Similarly to the discussion of the situation $e_2e_3 ≠ 0$ in Theorem 4.3, only for the case that $i_2 = i_3 = i_4 = 0$ we can turn $G_j$ into canonical form. Now, we show if $i_2 = i_3 = i_4 = 0$, under what conditions we can turn $G_j$ into canonical form. In fact, if $i_2 = i_3 = i_4 = 0, then G_j$ is of the form

$$ \begin{bmatrix} 1 & 0 & 0 & 0 & e_2 & h_x & j_x & 0 & e_3 & h_y & j_y & 0 & e_4 & h_z & j_z \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. $$

Through subtracting $e_3/e_2$ times the second slice from the third slice and $e_4/e_2$ times the second slice from the fourth slice, we can turn $e_3$ and $e_4$ into zeros. Then $G_j$ is of the form

$$ \begin{bmatrix} 1 & 0 & 0 & 0 & e_2 & h_x & j_x & 0 & e_3 & h_y & j_y & 0 & e_4 & h_z & j_z \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, $$

where $\tilde{h}_x = h_x - e_3h_x/e_2, \tilde{h}_y = j_y = e_3j_y/e_2, \tilde{h}_z = i_xh_x = e_4h_4/e_2$. According to the mode-3 rank of $G_j$ is 4, then $\tilde{h}_yj_y ≠ \tilde{h}_zj_z$. Thus we draw the conclusion that if $e_2e_3e_4 ≠ 0$, the remaining six of $e_p, f_p, g_p$ (p = 2, 3, 4) are equal to zeros, under the conditions $i_x = i_y = i_z = 0, \tilde{h}_yj_y ≠ \tilde{h}_zj_z$, where $x, y, z ∈ \{2, 3, 4\}$ and $x ≠ y ≠ z$, we can turn $G_j$ into canonical form. \[ \Box \]

Next, we discuss the combination that $e_2f_2g_2 ≠ 0$, the remaining six of $e_p, f_p, g_p$ (p = 2, 3, 4) are equal to zeros. The combinations $e_2f_3g_3, e_3f_4g_4$ can be similar as in the discussion with $e_2f_2g_2$.

If $e_2f_2g_2 ≠ 0, e_3 = e_4 = f_3 = f_4 = g_3 = g_4 = 0$, according to (9), we have $e_2i_2 = h_2g_2, e_3i_3 = h_3g_3$. Because $e_2g_2 ≠ 0$, then $i_2$ and $h_2$ are zero or nonzero at the same time, $i_3$ and $h_3$ are zero or nonzero at the same time. This yields four possibilities. The proof of these four cases is almost identical as the case that $e_2g_2 ≠ 0$ in Theorem 4.3, the major change is that $h_2$ can be turned into zero by $e_2, i_2$ can be turned into zero by $f_2$. Consequently, if $e_2f_2g_2 ≠ 0$, the remaining six of $e_p, f_p, g_p$ (p = 2, 3, 4) are zeros, under conditions $i_y = h_y = 0, i_xh_x ≠ j_yh_z = 0, h_xj_x ≠ j_yh_z$, where $x = 2, y = 3, z = 4$ or $x = 2, y = 4, z = 3$, we can turn $G_j$ into canonical form. \[ \Box \]

**Theorem 4.5.** If four of $e_p, f_p, g_p$ (p = 2, 3, 4) are not equal to zero, the remaining five of them are equal to zeros, there have 6 combinations in $T_1$, i.e. $e_2e_3f_3, e_2e_4f_4, e_3e_4f_3, f_3f_4g_3, f_3f_4g_4, f_3f_4g_3g_4$. For each
Proof. Here, we only discuss the combination that $e_2 e_3 f_3 f_4 = 0$, the remaining five of $e_p, f_p, g_p \ (p = 2, 3, 4)$ are equal to zeros. The combinations $e_2 e_a f_s f_4, e_1 e_2 f_s f_4, f_2 f_3 g_3 g_5, f_3 f_4 g_3 g_5$ can be similar as in the discussion with $e_2 e_3 f_4 f_4$.

If $e_2 e_3 f_3 f_3 \neq 0$, then the remaining five of $e_p, f_p, g_p \ (p = 2, 3, 4)$ are equal to zeros, and situations can be discussed as the situation that $e_1 f_1 h_1 = 0$ or $e_2 i_2 = e_2 i_2$. The proof of this case is almost identical as the combination that $e_2 e_3 = 0$ in Theorem 4.3, the major change is that $i_2$ (or $i_3$) can be turned into zero by $f_2$ (or $f_3$) if $i_2$ (or $i_3$) is nonzero.

Thus we conclude that if $e_2 e_3 f_3 f_3 \neq 0$, the remaining five of $e_p, f_p, g_p \ (p = 2, 3, 4)$ are equal to zeros, under conditions $i_2 = 0, i_4, i_5$ being zero or not at the same time, $h_2 = h_2$, where $x = 2, y = 3, z = 4$ or $x = 3, y = 2, z = 4$, we can turn $G_j$ into canonical form.

Theorem 4.6. If six of $e_p, f_p, g_p \ (p = 2, 3, 4)$ are not equal to zero, the remaining three of them are equal to zeros, there are six combinations in $T_1$, i.e., $e_2 e_3 e_4 f_3 f_4, e_2 f_3 f_4 g_3 g_5, e_3 f_3 f_3 g_3 g_5, e_3 f_2 g_3 e_2 s_3 g_4, e_3 f_2 f_2 e_2 f_4 g_4, e_3 f_3 g_3 e_3 f_4 g_4$. For each combination, $G_j$ can be turned into canonical form under the following conditions:

$$e_i e_j e_k f_s f_t f_u = 0, i_2, i_6, i_7, f_2 being zero or nonzero at same time, \tilde{h}_2 = \tilde{h}_2;$$

$$f_s f_u g_2 g_3 g_5 = 0, h_2, h_5, h_7 being zero or nonzero at same time, \tilde{j}_2 = \tilde{j}_2;$$

$$e_i f_s e_k g_v g_r = 0, \tilde{h}_2 = \tilde{h}_2, h_2 = h_2;$$

$$e_i f_s e_j g_v g_r = 0, \tilde{h}_2 = \tilde{h}_2, h_2 = h_2;$$

$$e_i f_s e_j g_v g_r = 0, \tilde{j}_2 = \tilde{j}_2, h_2 = h_2;$$

$$e_i f_s e_j g_v g_r = 0, \tilde{j}_2 = \tilde{j}_2, h_2 = h_2;$$

where $x, y, z \in \{2, 3, 4\}$ and $x = y = z$.

Proof. Firstly, we discuss the combination that $e_2 e_3 e_4 f_3 f_4 = 0$, the remaining three of $e_p, f_p, g_p \ (p = 2, 3, 4)$ are equal to zeros. The combination $f_2 f_3 f_4 g_3 g_5$ is similar as in the discussion with $e_2 e_3 f_4 f_4$. If $e_2 e_3 e_4 f_3 f_4 \neq 0$, then $e_2 i_3 = e_2 i_3, e_2 i_2 = e_2 i_2, e_2 i_3 = e_2 i_3$. The proof of this combination is almost identical as the combination that $e_2 e_3 e_4 = 0$ in Theorem 4.4, the major change is that $i_2$ (or $i_3$) can be turned into zero by $f_2$ (or $f_3$) if $i_2$ (or $i_3$) is nonzero. Thus we conclude that if $e_2 e_3 e_4 f_3 f_4 \neq 0$, the remaining three of $e_p, f_p, g_p \ (p = 2, 3, 4)$ are equal to zeros, under conditions $i_2 = 0, i_2, i_3$ being zero or not at the same time, $h_2 = h_2$, where $x, y, z \in \{2, 3, 4\}$ and $x = y = z$, we can turn $G_j$ into canonical form.

Next, we discuss the combination that $e_2 f_2 g_2 e_3 f_3 g_3 = 0$, the remaining three of $e_p, f_p, g_p \ (p = 2, 3, 4)$ are equal to zeros. The combinations $e_2 f_2 g_2 e_3 f_3 g_3, e_2 f_2 e_2 e_4 g_4, e_2 f_2 e_2 e_2 g_4, e_2 f_2 e_2 e_4 g_4$ can be similar as in the discussion with $e_2 f_2 g_2 e_3 f_3 g_3$. If $e_2 f_2 g_2 e_3 f_3 g_3 \neq 0, e_4 f_4 = g_4 = 0$, according to (9), we have $e_2 i_2 = e_2 i_2, e_2 i_2 = e_2 i_2, e_2 i_2 = e_2 i_2$. Because $e_2 e_3 g_3 = 0$ and $e_2 / e_3 / e_3 g_3 = 0$ and $e_2 g_3 g_3 = 0$, then we obtain $e_2 \tilde{i}_3 = g_2 \tilde{h}_3$, where $e_2 \tilde{i}_3 = e_2 \tilde{i}_3, h_3 = h_3 - \alpha g_2$. On the other hand, because $e_2 e_3 g_3 = 0$, we obtain $e_2 \tilde{i}_3 = g_2 \tilde{h}_3$, where $e_2 \tilde{i}_3 = e_2 \tilde{i}_3, h_3 = h_3 - \alpha g_2$. This yields four possibilities. These four situations can be discussed as the situation that $e_2 f_2 g_2 = 0$, the remaining six of $e_p, f_p, g_p \ (p = 2, 3, 4)$ are equal to zeros.

Consequently, if $e_2 f_2 g_2 e_3 f_3 g_3 \neq 0$, the remaining three of $e_p, f_p, g_p \ (p = 2, 3, 4)$ are equal to zeros, under conditions $h_2 = h_2$, where $x = 2, y = 3, z = 4$, or $x = 2, y = 4, z = 3$, we can turn $G_j$ into canonical form.
Theorem 4.7. If $e_p, f_p, g_p$ ($p = 2, 3, 4$) are all nonzero, $\mathcal{G}_i$ can be turned into canonical form under the following conditions:

$$e_x e_y e_z f_x f_y f_z g_x g_y g_z \neq 0, \tilde{h}_y = \tilde{i}_y = 0, \tilde{h}_z = \tilde{i}_z \neq 0, \tilde{j}_y \neq 0;$$

$$(e_x e_y e_z f_x f_y f_z g_x g_y g_z \neq 0, \tilde{h}_y \tilde{i}_z \tilde{j}_z \neq 0, \tilde{j}_y \tilde{h}_z \neq \tilde{h}_y \tilde{j}_z);$$

where $x, y, z \in \{2, 3, 4\}$ and $x \neq y \neq z$.

Proof. According to the second conclusion of Property 2.1, if $e_p, f_p, g_p$ ($p = 2, 3, 4$) are all nonzero, the vectors $(e_p, f_p, g_p)$ ($p = 2, 3, 4$) being proportional. Then we can turn $e_3, e_4, f_3, f_4, g_3, g_4$ into zeros due to the vectors $(e_p, f_p, g_p)$ ($p = 2, 3, 4$) being proportional. Then we obtain the following for the last three slices of $\mathcal{G}_i$:

$$\begin{bmatrix}
0 & e_2 & h_2 & j_2 & 0 & 0 & h_3 & j_3 & 0 & 0 & h_4 & j_4 \\
0 & f_2 & i_2 & 0 & 0 & 0 & i_3 & 0 & 0 & i_4 \\
0 & 0 & g_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

where $\tilde{h}_3 = h_3 - \alpha h_2, \tilde{f}_3 = f_3 - \alpha f_2, h_4 = h_4 - \beta h_2, \tilde{j}_3 = j_3 - \beta j_2$ and $\alpha = e_3/e_2, \beta = e_4/e_2$.

Now we show that equality $e_2 \tilde{i}_3 = g_2 \tilde{h}_3$ and $e_2 \tilde{j}_3 = g_2 \tilde{h}_3$ holds. Firstly, we write $e_3 = \alpha e_2, g_3 = \alpha g_2, e_4 = \beta e_2, g_4 = \beta g_2$. Next, by substituting them into the first two equations of (9), we obtain $e_2 \tilde{i}_3 = g_2 \tilde{h}_3$ and $e_2 \tilde{j}_3 = g_2 \tilde{h}_3$. Because $e_2 g_2 \neq 0$, then $\tilde{i}_3$ and $\tilde{h}_3$ are zero or nonzero at the same time, $\tilde{i}_4$ and $\tilde{h}_4$ are zero or nonzero at the same time. This yields four possibilities. These four situation can be discussed just as the situation that $e_2 e_3 f_3 f_3 f_3 \neq 0$, the remaining three of $e_p, f_p, g_p$ ($p = 2, 3, 4$) are equal to zeros.

Consequently, if $e_x e_y e_z f_x f_y f_z g_x g_y g_z$ are nonzero, under conditions $\tilde{h}_y = \tilde{i}_y = 0, \tilde{h}_z = \tilde{i}_z \neq 0, \tilde{j}_y \neq 0$ or $\tilde{h}_y \tilde{i}_z \tilde{j}_z \neq 0, \tilde{j}_y \tilde{h}_z \neq \tilde{h}_y \tilde{j}_z$, we can turn $\mathcal{G}_i$ into canonical form.

\section{Concluding remarks}

In this paper, we have studied under what conditions we can turn $\mathcal{G}_i$ into canonical form (6) if some of the upper triangular entries of the last three slices of $\mathcal{G}_i$ are zeros. In addition, we have shown how to turn $\mathcal{G}_i$ into canonical form under these conditions. In the future, it will be interesting to explore the connection between our three order generalization of the Jordan canonical form and eigenvectors for three order arrays.

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\section*{References}

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