On the recursive properties of one kind hybrid power mean involving two-term exponential sums and Gauss sums

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Abstract: The main purpose of this paper is to study the computational problem of one kind hybrid power mean involving two-term exponential sums and quartic Gauss sums using the analytic method and the properties of the classical Gauss sums, and to prove some interesting fourth-order linear recurrence formulae for this problem. As an application of our result, we can also obtain an exact computational formula for one kind congruence equation mod $p$, an odd prime.

Keywords: The quartic Gauss sums, Two-term exponential sums, Hybrid power mean, The fourth-order linear recurrence formula

MSC: 11L05, 11L07

1 Introduction

Let $p \geq 3$ be an odd prime. For any integer $m$ with $(m, p) = 1$, the quartic Gauss sums $B(m) = B(m, p)$ is defined as

$$B(m) = \sum_{a=0}^{p-1} e\left(\frac{ma^k}{p}\right),$$

where as usual, $e(y) = e^{2\pi iy}$.

Recently, some scholars have studied the hybrid power mean problems of various trigonometric sums, and obtained many interesting results. For example, Chen Li and Hu Jiayuan [1] studied the computational problem of the hybrid power mean

$$S_k(p) = \sum_{m=1}^{p-1} \left(\sum_{a=0}^{p-1} e\left(\frac{ma^k}{p}\right)\right)^2 \cdot \left|\sum_{c=1}^{p-1} e\left(\frac{mc + \bar{c}}{p}\right)\right|^2,$$

where $\bar{c}$ denotes the multiplicative inverse of $c$ mod $p$. That is, $c \cdot \bar{c} \equiv 1$ mod $p$.

For $p \equiv 1$ mod 3, they used the elementary method to obtain an interesting third-order linear recurrence formula for $S_k(p)$.

Li Xiaoxue and Hu Jiayuan [2] studied the computational problem of the hybrid power mean

$$\sum_{b=1}^{p-1} \left|\sum_{a=0}^{p-1} e\left(\frac{ba^k}{p}\right)\right|^2 \cdot \left|\sum_{c=1}^{p-1} e\left(\frac{bc + \bar{c}}{p}\right)\right|^2,$$

and proved an exact computational formula for (1).

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Zhang Han and Zhang Wenpeng [3] proved the identity
\[
\sum_{m=1}^{p-1} \sum_{a=0}^{p-1} e \left( \frac{ma^3 + n a}{p} \right) = \begin{cases} 
2p^3 - p^2 & \text{if } 3 \nmid p - 1, \\
2p^3 - 7p^2 & \text{if } 3 \mid p - 1.
\end{cases}
\]

Other related results can also be found in references [4-13].

In this paper, we will consider the calculating problem of the following hybrid power mean:
\[
V_k(p) = \sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4}{p} \right) \right)^k \left( \sum_{b=0}^{p-1} e \left( \frac{mb^4 + b}{p} \right) \right)^3,
\]
where \(k \geq 0\) is an integer.

If \(p = 4h + 3\), then from the properties of the Legendre’s symbol mod \(p\) we have (see [14], formula (30) in Chapter 9)
\[
\sum_{a=0}^{p-1} e \left( \frac{ma^4}{p} \right) = 1 + \sum_{a=1}^{p-1} \left( 1 + \chi_2(a) \right) e \left( \frac{ma^2}{p} \right) = \sum_{a=0}^{p-1} e \left( \frac{ma^2}{p} \right) = i\chi_2(m) \sqrt{p},
\]
where \(\chi_2 = \left( \frac{2}{p} \right)\) denotes the Legendre’s symbol mod \(p\).

So in this case, the problem we considered in (2) is trivial. If \(p = 4h + 1\), then the situation is more complicated. We will use the analytic method and the properties of classical Gauss sums to study this problem, and prove some new interesting fourth-order linear recurrence formulae for (2) with \(p = 4h + 1\). That is, we will give the following four results.

**Theorem 1.1.** Let \(p\) be a prime with \(p = 24h + 1\). Then for any integer \(k \geq 4\), we have the fourth-order linear recurrence formula
\[
V_k(p) = 6pV_{k-2}(p) + 8p\alpha V_{k-3}(p) - p \left( p - 4\alpha^2 \right) V_{k-4}(p),
\]
where the first four values are \(V_0(p) = p^2 - 6p\alpha, V_1(p) = p \left( p^2 - 16p - 4\alpha^2 \right), V_2(p) = p^2 (2p\alpha + 3p - 58\alpha)\) and \(V_3(p) = p^2 (7p^2 + 4p\alpha - 92p - 72\alpha^2)\), \(\alpha = \alpha(p) = \sum_{a=1}^{p-1} \left( \frac{a + \overline{a}}{p} \right)\) is an integer, which satisfies the identity (see Theorem 4-11 in [15])
\[
p = \alpha^2 + \beta^2 \equiv \left( \sum_{a=1}^{p-1} \left( \frac{a + \overline{a}}{p} \right) \right)^2 + \left( \sum_{a=1}^{p-1} \left( \frac{a + r\overline{a}}{p} \right) \right)^2
\]
which \(r\) is any quadratic non-residue mod \(p\).

**Theorem 1.2.** Let \(p\) be a prime with \(p = 24h + 17\). Then for any integer \(k \geq 4\), we have the fourth-order linear recurrence formula
\[
V_k(p) = 6pV_{k-2}(p) + 8p\alpha V_{k-3}(p) - p \left( p - 4\alpha^2 \right) V_{k-4}(p),
\]
where the first four values are \(V_0(p) = -p^2 - 6p\alpha, V_1(p) = p \left( p^2 - 18p - 4\alpha^2 \right), V_2(p) = p^2 (2p\alpha - 3p - 62\alpha)\) and \(V_3(p) = p^2 (7p^2 - 4p\alpha - 106p - 72\alpha^2)\).

**Theorem 1.3.** Let \(p\) be a prime with \(p = 24h + 5\). Then for any integer \(k \geq 4\), we have the fourth-order linear recurrence formula
\[
V_k(p) = -2pV_{k-2}(p) + 8p\alpha V_{k-3}(p) - p \left( 9p - 4\alpha^2 \right) V_{k-4}(p),
\]
where the first four terms are \(V_0(p) = -(p^2 + 6p\alpha), V_1(p) = -p \left( p^2 - 8p + 4\alpha^2 \right), V_2(p) = -p^2 (2p\alpha - p - 22\alpha)\) and \(V_3(p) = p^2 (5p^2 - 6p\alpha - 28p - 36\alpha^2)\).
Theorem 1.4. Let $p$ be a prime with $p = 24h + 13$. Then for any integer $k \geq 4$, we have the fourth-order linear recurrence formula

$$V_k(p) = -2pV_{k-2}(p) + 8p\alpha V_{k-3}(p) - p \left(9p - 4\alpha^2\right) V_{k-4}(p),$$

where the first four terms are $V_0(p) = p^2 - 6p\alpha$, $V_1(p) = -p \left(p^2 - 6p + 4\alpha^2\right)$, $V_2(p) = -p^2 \left(2p\alpha + p - 18\alpha\right)$ and $V_3(p) = p^2 \left(5p^2 + 6p\alpha - 18p - 36\alpha^2\right)$.

From our theorems we may immediately deduce the following:

Corollary 1.5. Let $p$ be a prime with $p \equiv 1 \mod 4$, then we have the identity

$$
\sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e\left(\frac{ma^k}{p}\right) \right) \cdot \left( \sum_{b=0}^{p-1} \left( \frac{mb^4 + b}{p} \right) \right)^3 = \begin{cases} 
p^2 \left(7p^2 + 4p\alpha - 92p - 72\alpha^2\right) & \text{if } p = 24h + 1, \\
p^2 \left(7p^2 - 4p\alpha - 106p - 72\alpha^2\right) & \text{if } p = 24h + 17, \\
p^2 \left(5p^2 - 6p\alpha - 28p - 36\alpha^2\right) & \text{if } p = 24h + 5, \\
p^2 \left(5p^2 + 6p\alpha - 18p - 36\alpha^2\right) & \text{if } p = 24h + 13.
\end{cases}
$$

Note that the estimate $|\alpha| \leq \sqrt{p}$, from Corollary 1.5 we also have the following:

Corollary 1.6. Let $p$ be a prime with $p \equiv 1 \mod 8$, then we have the asymptotic formula

$$
\sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e\left(\frac{ma^k}{p}\right) \right) \cdot \left( \sum_{b=0}^{p-1} \left( \frac{mb^4 + b}{p} \right) \right)^3 = 7p^4 + O\left(p^2\right).
$$

Corollary 1.7. Let $p$ be a prime with $p \equiv 5 \mod 8$, then we have the asymptotic formula

$$
\sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e\left(\frac{ma^k}{p}\right) \right) \cdot \left( \sum_{b=0}^{p-1} \left( \frac{mb^4 + b}{p} \right) \right)^3 = 5p^4 + O\left(p^2\right).
$$

For any prime $p$ with $p \equiv 1 \mod 4$ and any positive integer $k$, let $M_k(p)$ denote the number of the solutions of the congruence equation

$$x_1^4 + x_2^4 + \cdots + x_k^4 + y_1^4 + y_2^4 + y_3^4 \equiv 0 \mod p, \quad y_1 + y_2 + y_3 \equiv 0 \mod p,$$

where $0 \leq x_i, y_j \leq p - 1, i = 1, 2, \ldots, k, j = 1, 2, 3$.

Then from our theorems we can give an exact computational formula for $M_k(p)$. For example, let $H_k(p)$ denote the number of the congruence equation

$$x_1^4 + x_2^4 + \cdots + x_k^4 \equiv 0 \mod p, \quad 0 \leq x_i \leq p - 1, \quad i = 1, 2, \ldots,$$

Then we have the identity

$$V_k(p) = \frac{p^2}{p-1} \cdot M_k(p) - \frac{p}{p-1} \cdot H_k(p).$$

Since $H_k(p)$ has a fourth-order linear recurrence formula (see [8]), so from the above formula and our theorems we can deduce the exact value of $M_k(p)$.

2 Several lemmas

To complete the proofs of our theorems, we need to prove four simple lemmas. Hereafter, we will use many properties of the classical Gauss sums and the fourth-order character mod $p$, all of which can be found in
Proof. First applying trigonometric identity

\[ \tau^2(\lambda) + \tau^2(\bar{\lambda}) = \sqrt{p} \cdot \sum_{a=1}^{p-1} \left( \frac{a + \bar{a}}{p} \right) = 2\sqrt{p} \cdot \alpha, \]

where \( \tau(\lambda) = \sum_{a=1}^{p-1} \lambda(a) e\left( \frac{a}{p} \right) \) denotes the classical Gauss sums, and \( \left( \frac{a}{p} \right) \) is the Legendre’s symbol mod \( p \).

Lemma 2.2. Let \( p \) be a prime with \( p \equiv 1 \mod 4 \), then for any fourth-order character \( \lambda \) mod \( p \), we have the identity

\[
\sum_{m=1}^{p-1} \lambda(m) \left( \sum_{a=0}^{p-1} e\left( \frac{ma^4 + a}{p} \right) \right)^3 = \begin{cases} 
-5p \tau(\lambda) - 2\sqrt{p} \alpha \tau(\bar{\lambda}) & \text{if } p \equiv 1 \mod 8, \\
-p \tau(\lambda) - 2\sqrt{p} \alpha \tau(\bar{\lambda}) & \text{if } p \equiv 5 \mod 8,
\end{cases}
\]

where \( \alpha \) is the same as in Lemma 2.1.

Proof. First applying trigonometric identity

\[
\sum_{m=1}^{q} e\left( \frac{nm}{q} \right) = \begin{cases} 
q & \text{if } q \mid n, \\
0 & \text{if } q \nmid n
\end{cases}
\]

and note that \( \lambda^4 = \chi_0 \), the principal character mod \( p \), we have

\[
\sum_{m=1}^{p-1} \lambda(m) \left( \sum_{a=0}^{p-1} e\left( \frac{ma^4 + a}{p} \right) \right)^3 = \sum_{m=1}^{p-1} \lambda(m) \left( \sum_{a=0}^{p-1} e\left( \frac{ma^4 + a}{p} \right) \right)^2 \\
+ \sum_{m=1}^{p-1} \lambda(m) \left( \sum_{a=0}^{p-1} e\left( \frac{ma^4 + a}{p} \right) \right) \left( \sum_{a=0}^{p-1} e\left( \frac{ma^4 + a}{p} \right) \right)
\]

\[
= \tau(\lambda) \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \lambda(a^4 + b^4 + 1) \sum_{c=0}^{p-1} e\left( \frac{c(a + b + 1)}{p} \right) \\
+ \tau(\lambda) \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \lambda(a^4 + b^4) e\left( \frac{a + b}{p} \right) \\
- \tau(\lambda) \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \lambda(a^4 + b^4 + 1) \sum_{c=0}^{p-1} e\left( \frac{c(a + b + 1)}{p} \right).
\]

From (3) we have

\[
\tau(\lambda) \sum_{a=0}^{p-1} \lambda(a^4 + 1) \sum_{b=0}^{p-1} e\left( \frac{b(a + 1)}{p} \right) = \lambda(2) \tau(\lambda)(p - 1) - \tau(\lambda) \sum_{a=0}^{p-1} \lambda(a^4 + 1)
\]

\[
= \lambda(2) \tau(\lambda)p - \tau(\lambda) \sum_{a=0}^{p-1} \lambda(a^4 + 1).
\]

Note that the identity \( \lambda_2 = \bar{\lambda} \) and

\[
B(m) = \sum_{a=0}^{p-1} e\left( \frac{ma^4}{p} \right) = \chi_2(m) \sqrt{p} + \lambda(m) \tau(\lambda) + \lambda(m) \tau(\bar{\lambda}).
\]
From (6) we have

\[
\tau(\lambda) \sum_{a=0}^{p-1} \chi(a^b + 1) = \sum_{b=1}^{p-1} \lambda(b) \sum_{a=0}^{p-1} e\left(\frac{b(a^b + 1)}{p}\right)
= \sum_{b=1}^{p-1} \lambda(b) \left(\chi_2(b) \sqrt{p} + \chi(\lambda) + \lambda(b) \tau(\lambda)\right)e\left(\frac{b}{p}\right)
= \sqrt{p} \tau(\lambda) - \tau(\lambda) + \sqrt{p} \tau(\lambda) = 2 \sqrt{p} \tau(\lambda) - \tau(\lambda).
\] (7)

If \( p \equiv 5 \mod 8 \), then note that \( \lambda(-1) = -1 \) and \( \tau(\lambda) \tau(\lambda) = -p \), applying (6) and Lemma 2.1 we also have

\[
\sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \chi(a^b + 1 + b^a) = \frac{1}{\tau(\lambda)} \sum_{c=1}^{p-1} \lambda(c) \sum_{a=0}^{p-1} \left(\frac{ca^b + cb^a + c}{p}\right)\]
\[
= \frac{1}{\tau(\lambda)} \sum_{c=1}^{p-1} \lambda(c) e\left(\frac{c}{p}\right) \left(\chi_2(c) \sqrt{p} \tau(\lambda) - \lambda(c) \tau(\lambda)\right)\]
\[
= \frac{1}{\tau(\lambda)} \sum_{c=1}^{p-1} \lambda(c) \left(2 \chi_2(c) \sqrt{p} \alpha - p + 2 \lambda(c) \sqrt{p} \tau(\lambda) + 2 \lambda(c) \sqrt{p} \tau(\lambda)\right)\]
\[
= p + \frac{2 \sqrt{p} (\lambda - 1) \tau(\lambda)}{\tau(\lambda)}.
\] (8)

Note that \( \lambda^2 \equiv \chi_2 \equiv \chi^2 \mod p \) and the congruence \( a + b + 1 \equiv 0 \mod p \) implies the congruence \( a^b + b^a + 1 \equiv 2(a^2 + a + 1)^2 \mod p \). So we have

\[
\sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \chi_2(a^2 + a + 1) = \sum_{a=0}^{p-1} \chi_2(4a^2 + 4a + 4)
= \chi(2) \sum_{a=0}^{p-1} \chi_2(a^2 + 3) = -\chi(2).
\] (9)

Combining (4), (5), (7), (8) and (9) we have the identity

\[
\sum_{m=1}^{p-1} \lambda(m) \left(\frac{ma^b + a}{p}\right)^3 = -p \tau(\lambda) - 2 \sqrt{p} \alpha \tau(\lambda).
\] (10)

If \( p \equiv 1 \mod 8 \), then \( \lambda(-1) = 1 \) and \( \tau(\lambda) \tau(\lambda) = p \), from the method of proving (8) we have

\[
\sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \chi(a^b + 1 + b^a) = \frac{1}{\tau(\lambda)} \sum_{c=1}^{p-1} \lambda(c) \sum_{a=0}^{p-1} \left(\frac{ca^b + cb^a + c}{p}\right)\]
\[
= \frac{1}{\tau(\lambda)} \sum_{c=1}^{p-1} \lambda(c) e\left(\frac{c}{p}\right) \left(\chi_2(b) \sqrt{p} \tau(\lambda) + \lambda(b) \tau(\lambda)\right)\]
\[
= \frac{1}{\tau(\lambda)} \sum_{c=1}^{p-1} \lambda(c) \left(3p + 2 \chi_2(c) \sqrt{p} \alpha + 2 \lambda(c) \sqrt{p} \tau(\lambda) + 2 \lambda(c) \sqrt{p} \tau(\lambda)\right)\]
\[
= 5p + \frac{2 \sqrt{p} (\lambda - 1) \tau(\lambda)}{\tau(\lambda)}.
\] (11)

Combining (4), (5), (7), (8) and (11) we have the identity

\[
\sum_{m=1}^{p-1} \lambda(m) \left(\frac{ma^b + a}{p}\right)^3 = -5p \tau(\lambda) - 2 \sqrt{p} \alpha \tau(\lambda).
\] (12)

Now Lemma 2.2 follows from (10) and (12).
**Lemma 2.3.** Let \( p \) be a prime with \( p \equiv 1 \mod 4 \), then we have the identity

\[
\sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3 = p \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} e \left( \frac{a + b + c}{p} \right)
\]

\[
= \begin{cases} 
   p^3 - 6p & \text{if } p = 24h + 1 \text{ or } p = 24h + 13, \\
   -p^3 + 6p & \text{if } p = 24h + 5 \text{ or } p = 24h + 17.
\end{cases}
\]

**Proof.** From (3) we have

\[
\sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3 = p \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} e \left( \frac{a + b + c}{p} \right)
\]

\[
= p \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} e \left( \frac{a + b}{p} \right) + p \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} e \left( \frac{b(a + 1)}{p} \right)
\]

\[
+p^2 \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} 1 - p \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} 1 - p \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} 1.
\]

(13)

Now we calculate each term in (13). If \( p \equiv 5 \mod 8 \), then note that \( \lambda(-1) = -1 \) we have

\[
p \sum_{a=0}^{p-1} 1 = 0.
\]

(14)

Applying (6) and Lemma 2.1 we have

\[
p^2 \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} 1 = p \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{m=0}^{p-1} e \left( \frac{ma^4 + b}{p} \right)
\]

\[
= p^2 + \sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4}{p} \right) \right) \left( \sum_{a=0}^{p-1} e \left( \frac{m}{p} \right) \right)
\]

\[
= p^2 + \sum_{m=1}^{p-1} \left( 2 \chi_2(c) \sqrt{p} \alpha - p + 2\lambda(c) \sqrt{p} \tau(\lambda) + 2\lambda(c) \sqrt{p} \tau(\lambda) \right) e \left( \frac{m}{p} \right)
\]

\[
= p^2 + 2p \alpha + p + 2\sqrt{p} \tau(\lambda) + 2\sqrt{p} \tau(\lambda) = p^2 + 6p \alpha.
\]

(15)

It is clear that the congruences \( a^4 + b^4 + 1 \equiv 0 \mod p \) and \( a + b + 1 \equiv 0 \mod p \) implies that \( ab \equiv 1 \mod p \) and \( a^3 \equiv b^3 \equiv 1 \mod p \) with \( a \neq b \). So we have

\[
p^2 \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} 1 = p^2 \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} 1 = \begin{cases} 
   0 \text{ if } p \equiv 1 \mod 3, \\
   2p^2 \text{ if } p \equiv 2 \mod 3.
\end{cases}
\]

(16)

Applying (13), (14), (15) and (16) we have the identity

\[
\sum_{m=1}^{p-1} \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right)^3 = \begin{cases} 
   p^3 - 6p \alpha & \text{if } p = 24h + 13, \\
   -p^3 - 6p \alpha & \text{if } p = 24h + 5.
\end{cases}
\]

(17)
If \( p \equiv 1 \mod 8 \), then we also have

\[
p^2 \sum_{a=0}^{p-1} \frac{1}{a^4 + 1 \equiv 0 \mod p} = 4p.
\]

(18)

\[
p^2 \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \frac{1}{a^4 + b^4 \equiv 1 \mod p} = p^2 + \sum_{m=1}^{p-1} \left( 3p + 2\chi_2(c)\sqrt{p}\alpha + 2\lambda(c)\sqrt{p}\tau(\lambda) + 2\lambda(c)\sqrt{p}\tau(\lambda) \right) e\left(\frac{m}{p}\right)
\]

\[
= p^2 + \sum_{m=1}^{p-1} \left( 3p + 2\chi_2(c)\sqrt{p}\alpha + 2\lambda(c)\sqrt{p}\tau(\lambda) \right) e\left(\frac{m}{p}\right)
\]

(19)

\[
p^2 \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \frac{1}{a^4 + b^4 \equiv 1 \mod p} = \begin{cases} 
2p^2 & \text{if } p = 24h + 1, \\
0 & \text{if } p = 24h + 17.
\end{cases}
\]

(20)

Applying (13), (18), (19) and (20) we have

\[
\sum_{m=1}^{p-1} \chi_2(m) \left( \sum_{a=0}^{p-1} e\left( \frac{ma^4 + a}{p} \right) \right)^3 = \begin{cases} 
p^{\frac{1}{2}}(p-6) & \text{if } p = 24h + 1, \\
p^{\frac{1}{2}}(p-8) & \text{if } p = 24h + 17, \\
-p^{\frac{1}{2}}(p-4) & \text{if } p = 24h + 13, \\
-p^{\frac{1}{2}}(p-6) & \text{if } p = 24h + 5.
\end{cases}
\]

(21)

It is clear that Lemma 2.3 follows from (17) and (21). □

**Lemma 2.4.** Let \( p \) be a prime with \( p \equiv 1 \mod 4 \), then we have the identity

\[
\sum_{m=1}^{p-1} \chi_2(m) \left( \sum_{a=0}^{p-1} e\left( \frac{ma^4 + a}{p} \right) \right)^3 = \sum_{m=1}^{p-1} \chi_2(m) \left( \sum_{a=0}^{p-1} e\left( \frac{ma^4 + a}{p} \right) \right)^2
\]

(22)

\[
+ \sum_{m=1}^{p-1} \chi_2(m) \left( \sum_{a=0}^{p-1} e\left( \frac{ma^4 + a}{p} \right) \right)^2 \sum_{c=1}^{p-1} e\left( \frac{mc^4 + c}{p} \right)
\]

\[
+ \sqrt{p} \sum_{a=0}^{p-1} \chi_2(a^4 + 1) \sum_{c=1}^{p-1} e\left( \frac{c(a + 1)}{p} \right)
\]

\[
= -\sqrt{p} \sum_{a=0}^{p-1} \chi_2(a^4 + 1) - \sqrt{p} \sum_{a=0}^{p-1} \chi_2(a^4 + 1) - \sqrt{p} \sum_{a=0}^{p-1} \chi_2(a^4 + 1)
\]

\[
+ p^{\frac{1}{2}} \sum_{a=0}^{p-1} \chi_2(a^4 + 1).
\]

From the properties of fourth-order \( \mod p \) and Lemma 2.1 we have

\[
\sum_{a=0}^{p-1} \chi_2(a^4 + 1) = 1 + \sum_{a=1}^{p-1} \chi_2(a + 1) \left( 1 + \lambda(a) + \chi_2(a) + \chi_2(a) \right)
\]

\[
\sum_{a=0}^{p-1} \chi_2(a^4 + 1) = 1 + \sum_{a=1}^{p-1} \chi_2(a + 1) \left( 1 + \lambda(a) + \chi_2(a) + \chi_2(a) \right)
\]
Combining (22), (23), (24) and (25) we have

\[
\sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \chi_2 \left( a^4 + b^4 + 1 \right) = \chi_2(2) \sum_{a=0}^{p-1} \chi_2 \left( a^8 + (a + 1)^8 + 1 \right)
\]

\[
\chi_2(2) \sum_{a=0}^{p-1} \chi_2 \left( a^4 + 2a^3 + 3a^2 + 2a + 1 \right) = \chi_2(2) \sum_{a=0}^{p-1} \chi_2 \left( (a^2 + a + 1) \right)^2
\]

\[
\begin{cases}
\chi_2(2) p & \text{if } p = 12h + 1, \\
\chi_2(2) (p - 2) & \text{if } p = 12h + 5.
\end{cases}
\]

Note that \( \tau(\lambda) \tau(\overline{\lambda}) = -p \), if \( p = 8h + 5 \). \( \tau(\lambda) \tau(\overline{\lambda}) = p \), if \( p = 8h + 1 \). From the method of proving (15) and (19) we have

\[
\sqrt{p} \sum_{m=1}^{p-1} \sum_{a=0}^{p-1} \chi_2 \left( a^4 + b^4 + 1 \right) = \sum_{m=1}^{p-1} \sum_{a=0}^{p-1} \sum_{m=1}^{p-1} \chi_2(m) e \left( \frac{ma^4 + mb^4 + m}{p} \right)
\]

\[
\sum_{m=1}^{p-1} \chi_2(m) \left( \chi_2(\sqrt{p} + \lambda(m) \tau(\lambda) + \overline{\lambda}(m) \tau(\overline{\lambda}) \right)^2 e \left( \frac{m}{p} \right)
\]

\[
\begin{cases}
7p^{\frac{1}{2}} - 2\sqrt{p} & \text{if } p = 8h + 1, \\
-5p^{\frac{1}{2}} - 2\sqrt{p} & \text{if } p = 8h + 5.
\end{cases}
\]

Combining (22), (23), (24) and (25) we have

\[
\sum_{m=1}^{p-1} \chi_2(m) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3 = \begin{cases} p^{\frac{3}{2}}(p - 6) & \text{if } p = 24h + 1, \\ p^{\frac{3}{2}}(p - 8) & \text{if } p = 24h + 17, \\ -p^{\frac{3}{2}}(p - 4) & \text{if } p = 24h + 13, \\ -p^{\frac{3}{2}}(p - 6) & \text{if } p = 24h + 5. \end{cases}
\]

This proves Lemma 2.4.

\[\square\]

3 Proofs of the theorems

Now we prove our main results. First we prove Theorem 1.1. If \( p = 24h + 1 \), then from Lemmas 2.1, 2.2 and 2.4 we have

\[
V_1(p) = \sum_{m=1}^{p-1} B(m) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3
\]

\[
= \sum_{m=1}^{p-1} \chi_2(m) \sqrt{p} + \overline{\lambda}(m) \tau(\lambda) + \lambda(m) \tau(\overline{\lambda}) \right) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3
\]

\[
= p^{\frac{3}{2}}(p - 6) - 5p^2 - 2\sqrt{p}\alpha \tau^2(\lambda) - 5p^2 - 2\sqrt{p}\alpha \tau^2(\overline{\lambda})
\]

\[
= p \left( p^2 - 16p - 4\alpha^2 \right).
\]

Applying Lemmas 2.1–2.4 we also have

\[
V_2(p) = \sum_{m=1}^{p-1} B^2(m) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3
\]

\[
= \sum_{m=1}^{p-1} \chi_2(m) \sqrt{p} + \overline{\lambda}(m) \tau(\lambda) + \lambda(m) \tau(\overline{\lambda}) \right) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3
\]

\[
= p^{\frac{3}{2}}(p - 6) - 5p^2 - 2\sqrt{p}\alpha \tau^2(\lambda) - 5p^2 - 2\sqrt{p}\alpha \tau^2(\overline{\lambda})
\]

\[
= p \left( p^2 - 16p - 4\alpha^2 \right).
\]
Applying (28) and the method of proving (29) we also have

\[
\begin{align*}
\frac{p}{m+1} & = (p-2) + 2\chi_2(m)\sqrt{p} + 2\lambda(m)\sqrt{\lambda(p-5) + 2\chi_2(m)\sqrt{p}\lambda(\lambda)} + 2\lambda(m)\sqrt{\lambda(p-5) + 2\chi_2(m)\sqrt{p}\lambda(\lambda)} + 2N(m)\sqrt{\lambda(p-5) + 2\chi_2(m)\sqrt{p}\lambda(\lambda)} + 2N(m)\sqrt{\lambda(p-5) + 2\chi_2(m)\sqrt{p}\lambda(\lambda)}) \\
& = p^2 (2p - 3p - 58\alpha).
\end{align*}
\]

So if \( p = 8h + 1 \), then from (6) we have

\[
B^3(m) = (\chi_2(m)\sqrt{p} + \lambda(m)\sqrt{\lambda})^3
= 7\chi_2(m)p^2 + 4p\alpha + 5p(\lambda(m)\tau(\lambda) + \lambda(m)\tau(\lambda)) \]

\[
+ 2(\lambda(m)\tau(\lambda) + \lambda(m)\tau(\lambda))\sqrt{p}\alpha.
\]

If \( p = 24h + 1 \), then from (28), Lemmas 2.1–2.4 we have

\[
V_3(p) = \sum_{m=1}^{p-1} B^3(m) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3
= 7p^3(p - 6) + 4p\alpha(p^2 - 6p\alpha) - 5p\tau(\lambda)(5p\tau(\lambda) + 2\sqrt{p}\alpha\tau(\lambda))
- 5p\tau(\lambda)(5p\tau(\lambda) + 2\sqrt{p}\alpha\tau(\lambda)) - 2\sqrt{p}\alpha\tau(\lambda)(5p\tau(\lambda) + 2\sqrt{p}\alpha\tau(\lambda))
- 2\sqrt{p}\alpha\tau(\lambda)(5p\tau(\lambda) + 2\sqrt{p}\alpha\tau(\lambda))
- 2\sqrt{p}\alpha\tau(\lambda)(5p\tau(\lambda) + 2\sqrt{p}\alpha\tau(\lambda))
= p^2(7p^2 + 4p\alpha - 92p - 72\alpha^2).
\]

If \( p = 24h + 17 \), then from Lemmas 2.1–2.4 we have

\[
V_3(p) = \sum_{m=1}^{p-1} B^3(m) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3
= p^2(p - 8) - 5p^2 - 2\sqrt{p}\alpha\tau(\lambda) - 5p^2 - 2\sqrt{p}\alpha\tau(\lambda)
- p(p^2 - 18p - 4\alpha^2).
\]

Applying (28) and the method of proving (29) we also have

\[
V_3(p) = \sum_{m=1}^{p-1} B^3(m) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3
= 7p^3(p - 8) - 4p\alpha(p^2 + 6p\alpha) - 5p\tau(\lambda)(5p\tau(\lambda) + 2\sqrt{p}\alpha\tau(\lambda))
- 5p\tau(\lambda)(5p\tau(\lambda) + 2\sqrt{p}\alpha\tau(\lambda)) - 2\sqrt{p}\alpha\tau(\lambda)(5p\tau(\lambda) + 2\sqrt{p}\alpha\tau(\lambda))
- 2\sqrt{p}\alpha\tau(\lambda)(5p\tau(\lambda) + 2\sqrt{p}\alpha\tau(\lambda))
= p^2(7p^2 + 4p\alpha - 106p - 72\alpha^2).
\]

Similarly, if \( p = 24h + 5 \), then we have

\[
V_1(p) = \sum_{m=1}^{p-1} B^3(m) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3
= p^2(p - 6) + p^2 - 2\sqrt{p}\alpha\tau(\lambda) + p^2 - 2\sqrt{p}\alpha\tau(\lambda)
= p^2(2p - 8p + 4\alpha^2).
\]
\[ V_2(p) = \sum_{m=1}^{p-1} B^2(m) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3 \]
\[ = \sum_{m=1}^{p-1} \left( \chi_2(m) \sqrt{\rho} + \chi(m) \tau(\lambda) + \lambda(m) \tau(\lambda) \right)^2 \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3 \]
\[ = \sum_{m=1}^{p-1} \left( 2\chi_2(c) \sqrt{\rho} - p + 2\lambda(c) \sqrt{\rho} \tau(\lambda) + 2\lambda(c) \sqrt{\rho} \tau(\lambda) \right) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3 \]
\[ = -p^2 \left( 2p\alpha - p - 22\alpha \right). \quad (34) \]

If \( p = 24h + 5 \), then from (6) we have
\[ B^3(m) = \left( \chi_2(m) \sqrt{\rho} + \chi(m) \tau(\lambda) + \lambda(m) \tau(\lambda) \right)^3 \]
\[ = -5\chi_2(m)p^2 + 6p\alpha + p \left( \chi(m) \tau(\lambda) + \lambda(m) \tau(\lambda) \right) \]
\[ + 2 \left( \chi(m) \tau(\lambda) + \lambda(m) \tau(\lambda) \right) \sqrt{\rho}. \quad (35) \]

So from (35) and the method of proving (29) we have
\[ V_3(p) = \sum_{m=1}^{p-1} B^3(m) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3 \]
\[ = 5p^3 (p - 6) - 6p\alpha \left( p^2 + 6p\alpha \right) - p\tau(\lambda) \left( p\tau(\lambda) + 2\sqrt{\rho} \alpha \tau(\lambda) \right) \]
\[ - p\tau(\lambda) \left( p\tau(\lambda) + 2\sqrt{\rho} \alpha \tau(\lambda) \right) - 2\sqrt{\rho} \alpha \tau(\lambda) \left( p\tau(\lambda) + 2\sqrt{\rho} \alpha \tau(\lambda) \right) \]
\[ = p^2 \left( 5p^2 - 6p\alpha - 28p - 36\alpha^2 \right). \quad (36) \]

If \( p = 24h + 13 \), then (35), Lemmas 2.1–2.4 we have
\[ V_1(p) = \sum_{m=1}^{p-1} B(m) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3 \]
\[ = -p^2 (p - 4) + p^2 - 2\sqrt{\rho} \alpha \tau(\lambda) + p^2 - 2\sqrt{\rho} \alpha \tau(\lambda) \]
\[ = -p \left( p^2 - 6p + 4\alpha^2 \right). \quad (37) \]

\[ V_2(p) = \sum_{m=1}^{p-1} B^2(m) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3 \]
\[ = \sum_{m=1}^{p-1} \left( \chi_2(m) \sqrt{\rho} + \chi(m) \tau(\lambda) + \lambda(m) \tau(\lambda) \right)^2 \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3 \]
\[ = \sum_{m=1}^{p-1} \left( 2\chi_2(c) \sqrt{\rho} - p + 2\lambda(c) \sqrt{\rho} \tau(\lambda) + 2\lambda(c) \sqrt{\rho} \tau(\lambda) \right) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3 \]
\[ = -p^2 \left( 2p\alpha + p - 18\alpha \right). \quad (38) \]

\[ V_3(p) = \sum_{m=1}^{p-1} B^3(m) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3 \]
\[ = 5p^3 (p - 4) + 6p\alpha \left( p^2 - 6p\alpha \right) - p\tau(\lambda) \left( p\tau(\lambda) + 2\sqrt{\rho} \alpha \tau(\lambda) \right) \]
\[ - p\tau(\lambda) \left( p\tau(\lambda) + 2\sqrt{\rho} \alpha \tau(\lambda) \right) - 2\sqrt{\rho} \alpha \tau(\lambda) \left( p\tau(\lambda) + 2\sqrt{\rho} \alpha \tau(\lambda) \right) \]
\[ = p^2 \left( 5p^2 + 6p\alpha - 18p - 36\alpha^2 \right). \quad (39) \]
Finally, note that if \( p = 8h + 1 \), then from (6) and direct calculation (or see Lemma 3 in [7]) we have the identity
\[
B^4(m) = 6pB^2(m) + 8p\alpha B(m) - p\left(p - 4\alpha^2\right). \quad (40)
\]
For any prime \( p = 24h + 1 \) and integer \( k \geq 4 \), from (26), (27), (29) and (40) we may immediately deduce the fourth-order linear recurrence formula
\[
V_k(p) = \frac{p-1}{m=1} B^4(m) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3
= \sum_{m=1}^{p-1} B^{k-4}(m) \left( 6pB^2(m) + 8p\alpha B(m) - p\left(p - 4\alpha^2\right) \right) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3
= 6pV_{k-2}(p) + 8p\alpha V_{k-3}(p) - p\left(p - 4\alpha^2\right) V_{k-4}(p),
\]
where the first four values \( V_0(p) = p^2 - 6p\alpha, V_1(p) = p\left(p^2 - 16p - 4\alpha^2\right), V_2(p) = p^3 - 62p - 4\alpha^2 \) and \( V_3(p) = p^4 \left( 7p^2 + 4p\alpha - 92p - 72\alpha^2 \right) \).

This proves Theorem 1.1.

If \( p = 24h + 17 \), then from (30), (31), (32) and (40) we have
\[
V_k(p) = 6pV_{k-2}(p) + 8p\alpha V_{k-3}(p) - p\left(p - 4\alpha^2\right) V_{k-4}(p),
\]
where the first four values \( V_0(p) = -p^2 - 6p\alpha, V_1(p) = p\left(p^2 - 18p - 4\alpha^2\right), V_2(p) = p^2 \left( 2p\alpha - 3p - 62\alpha \right) \) and \( V_3(p) = p^3 \left( 7p^2 - 4p\alpha - 106p - 72\alpha^2 \right) \).

This proves Theorem 1.2.

If \( p = 8h + 5 \), then from (6) and direct calculation (or see Lemma 3 in [7]) we also have
\[
B^4(m) = -2pB^2(m) + 8p\alpha B(m) - p\left(9p - 4\alpha^2\right), \quad (41)
\]
For any prime \( p = 24h + 5 \) and integer \( k \geq 4 \), from (33), (34), (35) and (41) we can deduce the fourth-order linear recurrence formula
\[
V_k(p) = -2pV_{k-2}(p) + 8p\alpha V_{k-3}(p) - p\left(9p - 4\alpha^2\right) V_{k-4}(p),
\]
where the first four terms are \( V_0(p) = -p^2 - 6p\alpha, V_1(p) = -p\left(p^2 - 8p + 4\alpha^2\right), V_2(p) = -p^2 \left( 2p\alpha - p - 22\alpha \right) \) and \( V_3(p) = p^2 \left( 5p^2 - 6p\alpha - 28p - 36\alpha^2 \right) \).

This proves Theorem 1.3.

If \( p = 24h + 13 \), then from (37), (38), (39) and (41) we also have
\[
V_k(p) = -2pV_{k-2}(p) + 8p\alpha V_{k-3}(p) - p\left(9p - 4\alpha^2\right) V_{k-4}(p),
\]
where the first four terms are \( V_0(p) = p^2 - 6p\alpha, V_1(p) = -p\left(p^2 - 6p + 4\alpha^2\right), V_2(p) = -p^2 \left( 2p\alpha + p - 18\alpha \right) \) and \( V_3(p) = p^2 \left( 5p^2 + 6p\alpha - 18p - 36\alpha^2 \right) \).

This completes the proofs of our all results.

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**References**


