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On the recursive properties of one kind hybrid power mean involving two-term exponential sums and Gauss sums

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Abstract: The main purpose of this paper is to study the computational problem of one kind hybrid power mean involving two-term exponential sums and quartic Gauss sums using the analytic method and the properties of the classical Gauss sums, and to prove some interesting fourth-order linear recurrence formulae for this problem. As an application of our result, we can also obtain an exact computational formula for one kind congruence equation mod $p$, an odd prime.

Keywords: The quartic Gauss sums, Two-term exponential sums, Hybrid power mean, The fourth-order linear recurrence formula

MSC: 11L05, 11L07

1 Introduction

Let $p \geq 3$ be an odd prime. For any integer $m$ with $(m, p) = 1$, the quartic Gauss sums $B(m) = B(m, p)$ is defined as

$$B(m) = \sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right),$$

where as usual, $e(y) = e^{2\pi i y}$.

Recently, some scholars have studied the hybrid power mean problems of various trigonometric sums, and obtained many interesting results. For example, Chen Li and Hu Jiayuan [1] studied the computational problem of the hybrid power mean

$$S_k(p) = \sum_{m=1}^{p-1}\left(\sum_{a=0}^{p-1} e\left(\frac{ma}{p}\right)\right)^k \sum_{c=1}^{p-1} e\left(\frac{mc + \bar{c}}{p}\right),$$

where $\bar{c}$ denotes the multiplicative inverse of $c$ mod $p$. That is, $c \cdot \bar{c} \equiv 1 \text{ mod } p$.

For $p \equiv 1 \text{ mod } 3$, they used the elementary method to obtain an interesting third-order linear recurrence formula for $S_k(p)$.

Li Xiaoxue and Hu Jiayuan [2] studied the computational problem of the hybrid power mean

$$\sum_{b=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ba^4}{p}\right)\right|^2 \cdot \left|\sum_{c=1}^{p-1} e\left(\frac{bc + \bar{c}}{p}\right)\right|^2,$$

and proved an exact computational formula for (1).

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Zhang Han and Zhang Wenpeng [3] proved the identity

\[ \sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e\left( \frac{ma^3 + na}{p} \right) \right)^4 = \begin{cases} 2p^3 - p^2 & \text{if } 3 \nmid p - 1, \\ 2p^3 - 7p^2 & \text{if } 3 \mid p - 1. \end{cases} \]

Other related results can also be found in references [4-13].

In this paper, we will consider the calculating problem of the following hybrid power mean:

\[ V_k(p) = \sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} e\left( \frac{ma^4}{p} \right) \right)^k \left( \sum_{b=0}^{p-1} e\left( \frac{mb^4 + b}{p} \right) \right)^3, \tag{2} \]

where \( k \geq 0 \) is an integer.

If \( p = 4h + 3 \), then from the properties of the Legendre’s symbol mod \( p \) we have (see [14], formula (30) in Chapter 9)

\[ \sum_{a=0}^{p-1} e\left( \frac{ma^4}{p} \right) = 1 + \sum_{a=1}^{p-1} (1 + \chi_2(a)) e\left( \frac{ma^2}{p} \right) = \sum_{a=0}^{p-1} e\left( \frac{ma^2}{p} \right) = i\chi_2(m) \sqrt{p}, \]

where \( \chi_2 \left( \frac{a}{p} \right) \) denotes the Legendre’s symbol mod \( p \).

So in this case, the problem we considered in (2) is trivial. If \( p = 4h + 1 \), then the situation is more complicated. We will use the analytic method and the properties of classical Gauss sums to study this problem, and prove some new interesting fourth-order linear recurrence formulae for (2) with \( p = 4h + 1 \). That is, we will give the following four results.

**Theorem 1.1.** Let \( p \) be a prime with \( p = 24h + 1 \). Then for any integer \( k \geq 4 \), we have the fourth-order linear recurrence formula

\[ V_k(p) = 6pV_{k-2}(p) + 8p\alpha V_{k-3}(p) - p \left( p - 4\alpha^2 \right) V_{k-4}(p), \]

where the first four values are \( V_0(p) = p^2 - 6p\alpha \), \( V_1(p) = p \left( p^2 - 16p - 4\alpha^2 \right) \), \( V_2(p) = p^2 (2p\alpha + 3p - 58\alpha) \) and \( V_3(p) = p^2 \left( 7p^2 + 4p\alpha - 92p - 72\alpha^2 \right) \), \( \alpha = \alpha(p) = \sum_{a=1}^{p-1} \left( \frac{a + \alpha}{p} \right) \) is an integer, which satisfies the identity (see Theorem 4-11 in [15])

\[ p = \alpha^2 + \beta^2 + \left( \sum_{a=1}^{p-1} \left( \frac{a + \alpha}{p} \right) \right)^2 + \left( \sum_{a=1}^{p-1} \left( \frac{a + \alpha}{p} \right) \right)^2, \]

which \( r \) is any quadratic non-residue mod \( p \).

**Theorem 1.2.** Let \( p \) be a prime with \( p = 24h + 17 \). Then for any integer \( k \geq 4 \), we have the fourth-order linear recurrence formula

\[ V_k(p) = 6pV_{k-2}(p) + 8p\alpha V_{k-3}(p) - p \left( p - 4\alpha^2 \right) V_{k-4}(p), \]

where the first four values are \( V_0(p) = -p^2 - 6p\alpha \), \( V_1(p) = p \left( p^2 - 18p - 4\alpha^2 \right) \), \( V_2(p) = p^2 (2p\alpha - 3p - 62\alpha) \) and \( V_3(p) = p^2 \left( 7p^2 - 4p\alpha - 106p - 72\alpha^2 \right) \).

**Theorem 1.3.** Let \( p \) be a prime with \( p = 24h + 5 \). Then for any integer \( k \geq 4 \), we have the fourth-order linear recurrence formula

\[ V_k(p) = -2pV_{k-2}(p) + 8p\alpha V_{k-3}(p) - p \left( 9p - 4\alpha^2 \right) V_{k-4}(p), \]

where the first four terms are \( V_0(p) = -(p^2 + 6p\alpha) \), \( V_1(p) = -p \left( p^2 - 8p + 4\alpha^2 \right) \), \( V_2(p) = -p^3 (2p\alpha - p - 22\alpha) \) and \( V_3(p) = p^3 \left( 5p^2 + 6p\alpha - 28p - 36\alpha^2 \right) \).
Theorem 1.4. Let \( p \) be a prime with \( p = 24h + 13 \). Then for any integer \( k \geq 4 \), we have the fourth-order linear recurrence formula

\[
V_k(p) = -2pV_{k-2}(p) + 8p\alpha V_{k-3}(p) - p\left(9p - 4\alpha^2\right)V_{k-4}(p),
\]

where the first four terms are \( V_0(p) = p^2 - 6p\alpha, V_1(p) = -p\left(p^2 - 6p + 4\alpha^2\right), V_2(p) = -p^2\left(2p\alpha + p - 18\alpha\right) \) and \( V_3(p) = p^2\left(5p^2 + 6p\alpha - 18p - 36\alpha^2\right) \).

From our theorems we may immediately deduce the following:

Corollary 1.5. Let \( p \) be a prime with \( p \equiv 1 \) mod 4, then we have the identity

\[
\sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} \left( \frac{ma^6}{p} \right) \right)^3 \cdot \left( \sum_{b=0}^{p-1} \left( \frac{mb^6 + b}{p} \right) \right)^3 = \begin{cases} p^2 \left( 7p^2 + 4p\alpha - 92p - 72\alpha^2 \right) & \text{if } p = 24h + 1, \\ p^2 \left( 7p^2 - 4p\alpha - 106p - 72\alpha^2 \right) & \text{if } p = 24h + 17, \\ p^2 \left( 5p^2 - 6p\alpha - 28p - 36\alpha^2 \right) & \text{if } p = 24h + 5, \\ p^2 \left( 5p^2 + 6p\alpha - 18p - 36\alpha^2 \right) & \text{if } p = 24h + 13. 
\end{cases}
\]

Note that the estimate \(|\alpha| \leq \sqrt{p}\), from Corollary 1.5 we also have the following:

Corollary 1.6. Let \( p \) be a prime with \( p \equiv 1 \) mod 8, then we have the asymptotic formula

\[
\sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} \left( \frac{ma^6}{p} \right) \right)^3 \cdot \left( \sum_{b=0}^{p-1} \left( \frac{mb^6 + b}{p} \right) \right)^3 = 7p^4 + O\left(p^2\right).
\]

Corollary 1.7. Let \( p \) be a prime with \( p \equiv 5 \) mod 8, then we have the asymptotic formula

\[
\sum_{m=1}^{p-1} \left( \sum_{a=0}^{p-1} \left( \frac{ma^6}{p} \right) \right)^3 \cdot \left( \sum_{b=0}^{p-1} \left( \frac{mb^6 + b}{p} \right) \right)^3 = 5p^4 + O\left(p^2\right).
\]

For any prime \( p \) with \( p \equiv 1 \) mod 4 and any positive integer \( k \), let \( M_k(p) \) denote the number of the solutions of the congruence equation

\[
x_1^4 + x_2^4 + \cdots + x_k^4 + y_1^4 + y_2^4 + y_3^4 \equiv 0 \mod p, \quad y_1 + y_2 + y_3 \equiv 0 \mod p,
\]

where \( 0 \leq x_i, y_j \leq p - 1, i = 1, 2, \ldots, k, j = 1, 2, 3 \).

Then from our theorems we can give an exact computational formula for \( M_k(p) \). For example, let \( H_k(p) \) denote the number of the congruence equation

\[
x_1^4 + x_2^4 + \cdots + x_s^4 \equiv 0 \mod p, \quad 0 \leq x_i \leq p - 1, \quad i = 1, 2, \ldots s.
\]

Then we have the identity

\[
V_k(p) = \frac{p^2}{p - 1} \cdot M_k(p) - \frac{p}{p - 1} \cdot H_k(p).
\]

Since \( H_k(p) \) has a fourth-order linear recurrence formula (see [8]), so from the above formula and our theorems we can deduce the exact value of \( M_k(p) \).

2 Several lemmas

To complete the proofs of our theorems, we need to prove four simple lemmas. Hereafter, we will use many properties of the classical Gauss sums and the fourth-order character mod \( p \), all of which can be found in
lemmas concerning Elementary Number Theory or Analytic Number Theory, such as references [7], [14] or [15]. Some important results related to Gauss sums can also be found in [16] and [17]. These contents will not be repeated here. First we have the following:

**Lemma 2.1.** Let $p$ be a prime with $p \equiv 1 \mod 4$, let $\lambda$ be any fourth-order character mod $p$, then we have

$$\tau^2(\lambda) + \tau^2(\bar{\lambda}) = \sqrt{p} \cdot \sum_{a=1}^{p-1} \left( \frac{a + \bar{a}}{p} \right) = 2 \sqrt{p} \cdot \alpha,$$

where $\tau(\lambda) = \sum_{a=1}^{p-1} \lambda(a) e\left( \frac{a}{p} \right)$ denotes the classical Gauss sums, and $\left( \frac{a}{p} \right)$ is the Legendre’s symbol mod $p$.

**Proof.** In fact this is Lemma 2 of [18], so its proof is omitted. $\square$

**Lemma 2.2.** Let $p$ be a prime with $p \equiv 1 \mod 4$, then for any fourth-order character $\lambda$ mod $p$, we have the identity

$$\sum_{m=1}^{p-1} \lambda(m) \left( \sum_{a=0}^{p-1} e\left( \frac{ma^4 + a}{p} \right) \right)^3 = \begin{cases} -5p \tau(\lambda) - 2 \sqrt{p} \alpha \tau(\bar{\lambda}) & \text{if } p \equiv 1 \mod 8, \\ -p \tau(\lambda) + 2 \sqrt{p} \alpha \tau(\bar{\lambda}) & \text{if } p \equiv 5 \mod 8, \end{cases}$$

where $\alpha$ is the same as in Lemma 2.1.

**Proof.** First applying trigonometric identity

$$\sum_{m=1}^{q} e\left( \frac{nm}{q} \right) = \begin{cases} q & \text{if } q \mid n, \\ 0 & \text{if } q \nmid n \end{cases} \quad (3)$$

and note that $\lambda^4 = \lambda_0$, the principal character mod $p$, we have

$$\sum_{m=1}^{p-1} \lambda(m) \left( \sum_{a=0}^{p-1} e\left( \frac{ma^4 + a}{p} \right) \right)^2 = \sum_{m=1}^{p-1} \lambda(m) \left( \sum_{a=0}^{p-1} e\left( \frac{ma^4 + a}{p} \right) \right)$$

$$+ \sum_{m=1}^{p-1} \lambda(m) \left( \sum_{a=0}^{p-1} e\left( \frac{ma^4 + a}{p} \right) \right)^2 \left( \sum_{a=0}^{p-1} e\left( \frac{ma^4 + a}{p} \right) \right)$$

$$= \tau(\lambda) \sum_{a=0}^{p-1} \bar{\lambda}(a^4 + b^4 + 1) \sum_{c=1}^{p-1} e\left( \frac{c(a + b + 1)}{p} \right)$$

$$+ \tau(\lambda) \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \bar{\lambda}(a^4 + b^4) e\left( \frac{a + b}{p} \right)$$

$$= \tau(\lambda) p \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \bar{\lambda}(a^4 + b^4 + 1) - \tau(\lambda) \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \bar{\lambda}(a^4 + b^4 + 1)$$

$$- \tau(\lambda) \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \bar{\lambda}(a^4 + 1) \sum_{b=1}^{p-1} e\left( \frac{b(a + 1)}{p} \right). \quad (4)$$

From (3) we have

$$\tau(\lambda) \sum_{a=0}^{p-1} \bar{\lambda}(a^4 + 1) \sum_{b=1}^{p-1} e\left( \frac{b(a + 1)}{p} \right) = \bar{\lambda}(2) \tau(\lambda)(p - 1) - \tau(\lambda) \sum_{a=0}^{p-1} \bar{\lambda}(a^4 + 1)$$

$$= \bar{\lambda}(2) \tau(\lambda)p - \tau(\lambda) \sum_{a=0}^{p-1} \bar{\lambda}(a^4 + 1). \quad (5)$$

Note that the identity $\lambda\chi_2 = \bar{\lambda}$ and

$$B(m) = \sum_{a=0}^{p-1} e\left( \frac{ma^4}{p} \right) = \chi_2(m) \sqrt{p} + \bar{\lambda}(m) \tau(\lambda) + \lambda(m) \tau(\bar{\lambda}). \quad (6)$$
From (6) we have
\[
\tau(\lambda) \sum_{a=0}^{p-1} \lambda(a^4 + 1) = \sum_{b=1}^{p-1} \lambda(b) \sum_{a=0}^{p-1} e\left(\frac{b}{p}\right) \left(\frac{a^4 + 1}{p}\right)
= \sum_{b=1}^{p-1} \lambda(b) \left(\chi_2(b)\sqrt{p} + \lambda(b)\tau(\lambda) + \lambda(b)\tau(\lambda)\right)e\left(\frac{b}{p}\right)
= \sqrt{p}\tau(\lambda) - \tau(\lambda) + \sqrt{p}\tau(\lambda) = 2\sqrt{p}\tau(\lambda) - \tau(\lambda)
\]
(7)

If \( p \equiv 5 \mod 8 \), then note that \( \lambda(-1) = -1 \) and \( \tau(\lambda)\tau(\lambda) = -p \), applying (6) and Lemma 2.1 we also have
\[
\sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \lambda(a^4 + b^4 + 1) = \frac{1}{\tau(\lambda)} \sum_{c=1}^{p-1} \lambda(c) \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} e\left(\frac{c}{p}\right) \left(\frac{a^4 + b^4 + c}{p}\right)
= \frac{1}{\tau(\lambda)} \sum_{c=1}^{p-1} \lambda(c) e\left(\frac{c}{p}\right) \left(\frac{\chi_2(c)\sqrt{p} + \lambda(c)\tau(\lambda) + \lambda(c)\tau(\lambda)}{p}\right)^2
= \frac{1}{\tau(\lambda)} \sum_{c=1}^{p-1} \lambda(c) \left(2\chi_2(c)\sqrt{p}\alpha - p + 2\lambda(c)\sqrt{p}\tau(\lambda) + 2\lambda(c)\sqrt{p}\tau(\lambda)\right)e\left(\frac{c}{p}\right)
= p + \frac{2\sqrt{p}(\alpha - 1)\tau(\lambda)}{\tau(\lambda)}
\]
(8)

Note that \( \lambda^2 \equiv \chi_2 = \chi^2 \) and the congruence \( a + b + 1 \equiv 0 \mod p \) implies the congruence \( a^4 + b^4 + 1 \equiv 2(a^2 + a + 1)^2 \mod p \). So we have
\[
\sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \lambda(a^4 + b^4 + 1) = \sum_{a=0}^{p-1} \lambda(2) \sum_{a=0}^{p-1} \lambda_2(4a^2 + 4a + 4)
= \lambda(2) \sum_{a=0}^{p-1} \lambda_2((2a + 1)^2 + 3) = \lambda(2) \sum_{a=0}^{p-1} \lambda_2(a^2 + 3) = -\lambda(2)
\]
(9)

Combining (4), (5), (7), (8) and (9) we have the identity
\[
\sum_{m=1}^{p-1} \lambda(m) \left(\sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right)\right)^3 = -p\tau(\lambda) - 2\sqrt{p}\alpha\tau(\lambda).
\]
(10)

If \( p \equiv 1 \mod 8 \), then \( \lambda(-1) = 1 \) and \( \tau(\lambda)\tau(\lambda) = p \), from the method of proving (8) we have
\[
\sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \lambda(a^4 + b^4 + 1) = \frac{1}{\tau(\lambda)} \sum_{c=1}^{p-1} \lambda(c) \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} e\left(\frac{c}{p}\right) \left(\frac{a^4 + b^4 + c}{p}\right)
= \frac{1}{\tau(\lambda)} \sum_{c=1}^{p-1} \lambda(c) e\left(\frac{c}{p}\right) \left(\chi_2(c)\sqrt{p} + \lambda(c)\tau(\lambda) + \lambda(c)\tau(\lambda)\right)^2
= \frac{1}{\tau(\lambda)} \sum_{c=1}^{p-1} \lambda(c) \left(3p + 2\chi_2(c)\sqrt{p}\alpha + 2\lambda(c)\sqrt{p}\tau(\lambda) + 2\lambda(c)\sqrt{p}\tau(\lambda)\right)e\left(\frac{c}{p}\right)
= 5p + \frac{2\sqrt{p}(\alpha - 1)\tau(\lambda)}{\tau(\lambda)}
\]
(11)

Combining (4), (5), (7), (8) and (11) we have the identity
\[
\sum_{m=1}^{p-1} \lambda(m) \left(\sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right)\right)^3 = -5p\tau(\lambda) - 2\sqrt{p}\alpha\tau(\lambda).
\]
(12)

Now Lemma 2.2 follows from (10) and (12).
Lemma 2.3. Let $p$ be a prime with $p \equiv 1 \mod 4$, then we have the identity

$$\left(\sum_{m=1}^{p-1} \left(\sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right)\right) \right)^3 = \begin{cases} p^2 - 6p\alpha & \text{if } p = 24h + 1 \text{ or } p = 24h + 13, \\ -p^2 - 6p\alpha & \text{if } p = 24h + 5 \text{ or } p = 24h + 17. \end{cases}$$

Proof. From (3) we have

$$\sum_{m=1}^{p-1} \left(\sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right)\right)^3 = p^2 \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} e\left(\frac{a + b + c}{p}\right)$$

$$= p \sum_{a=0}^{p-1} e\left(\frac{a}{p}\right) + p \sum_{a=0}^{p-1} e\left(\frac{b(a+1)}{p}\right) + p \sum_{a=0}^{p-1} e\left(\frac{c(a+b+1)}{p}\right)$$

$$= p + p^2 \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} 1 - p \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} 1 - p \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} 1. \quad (13)$$

Now we calculate each term in (13). If $p \equiv 5 \mod 8$, then note that $\lambda(-1) = -1$ we have

$$p \sum_{a=0}^{p-1} 1 = 0. \quad (14)$$

Applying (6) and Lemma 2.1 we have

$$p \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} 1 = \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{m=0}^{p-1} e\left(\frac{m(a^4 + b^4 + 1)}{p}\right)$$

$$= p^2 + \sum_{m=1}^{p-1} \left(\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right)^2 e\left(\frac{m}{p}\right)$$

$$= p^2 + \sum_{m=1}^{p-1} \left(2\chi_2(c)\sqrt{p}\alpha - p + 2\lambda(c)\sqrt{p}\tau(\lambda) + 2\overline{\lambda}(c)\sqrt{p}\tau(\overline{\lambda})\right) e\left(\frac{m}{p}\right)$$

$$= p^2 + 2p\alpha + p + 2\sqrt{p}\tau(\lambda) + 2\sqrt{p}\tau(\overline{\lambda}) = p^2 + p + 6p\alpha. \quad (15)$$

It is clear that the congruences $a^4 + b^4 + 1 \equiv 0 \mod p$ and $a + b + 1 \equiv 0 \mod p$ implies that $ab \equiv 1 \mod p$ and $a^3 \equiv b^3 \equiv 1 \mod p$ with $a \neq b$. So we have

$$p^2 \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} 1 = p^2 \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} 1 = \begin{cases} 2p^2 & \text{if } p \equiv 1 \mod 3, \\ 0 & \text{if } p \equiv 2 \mod 3. \end{cases} \quad (16)$$

Applying (13), (14), (15) and (16) we have the identity

$$\sum_{m=1}^{p-1} \left(\sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right)\right)^3 = \begin{cases} p^2 - 6p\alpha & \text{if } p = 24h + 13, \\ -p^2 - 6p\alpha & \text{if } p = 24h + 5. \end{cases} \quad (17)$$
If \( p \equiv 1 \mod 8 \), then we also have

\[
p \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \frac{a^4 + b^4 + 1}{p} = 4p.
\]

(18)

Applying (13), (18), (19) and (20) we have

\[
p \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{m=0}^{p-1} \left( m \left( a^4 + b^4 + 1 \right) \right) = p^2 + \sum_{m=1}^{p-1} \left( 3p + 2 \chi_2(c) \sqrt{p} \alpha + 2 \lambda(c) \sqrt{p} \tau(\lambda) + 2 \tau(c) \sqrt{p} \tau(\lambda) \right) e \left( \frac{m}{p} \right)
\]

(19)

\[
p \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{m=0}^{p-1} \left( \frac{ma^4 + a}{p} \right)^3 = \begin{cases} p^2 - 6p\alpha & \text{if } p = 24h + 1, \\ -p^2 - 6p\alpha & \text{if } p = 24h + 17. \end{cases}
\]

(21)

It is clear that Lemma 2.3 follows from (17) and (21).

Lemma 2.4. Let \( p \) be a prime with \( p \equiv 1 \mod 4 \), then we have the identity

\[
\sum_{m=1}^{p-1} \chi_2(m) \left( \sum_{a=0}^{p-1} \left( \frac{ma^4 + a}{p} \right) \right)^3 = \begin{cases} p^3(p - 6) & \text{if } p = 24h + 1, \\ p^3(p - 8) & \text{if } p = 24h + 17, \\ -p^3(p - 4) & \text{if } p = 24h + 13, \\ -p^3(p - 6) & \text{if } p = 24h + 5. \end{cases}
\]

Proof. From the properties of the Legendre’s symbol mod \( p \) we have

\[
\sum_{m=1}^{p-1} \chi_2(m) \left( \sum_{a=0}^{p-1} \left( \frac{ma^4 + a}{p} \right) \right)^3 = \sum_{m=1}^{p-1} \chi_2(m) \left( \sum_{a=0}^{p-1} \left( \frac{ma^4 + a}{p} \right) \right)^2
\]

(22)

\[
+ \sum_{m=1}^{p-1} \chi_2(m) \left( \sum_{a=0}^{p-1} \left( \frac{ma^4 + a}{p} \right) \right)^2 \sum_{c=1}^{p-1} \left( \frac{mc^4 + c}{p} \right)
\]

\[
= \sqrt{p} \sum_{a=0}^{p-1} e \left( \frac{a}{p} \right) + \sqrt{p} \sum_{a=0}^{p-1} \chi_2 \left( a^4 + 1 \right) \sum_{b=1}^{p-1} e \left( \frac{b(a+1)}{p} \right)
\]

\[
+ \sqrt{p} \sum_{a=0}^{p-1} \chi_2 \left( a^4 + b^4 + 1 \right) \sum_{c=1}^{p-1} e \left( \frac{c(a+b+1)}{p} \right)
\]

\[
= -\sqrt{p} + \chi_2(2) p^2 - \sqrt{p} \sum_{a=0}^{p-1} \chi_2 \left( a^4 + 1 \right) - \sqrt{p} \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \chi_2 \left( a^4 + b^4 + 1 \right)
\]

\[
+ p^2 \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \chi_2 \left( a^4 + b^4 + 1 \right).
\]

From the properties of fourth-order mod \( p \) and Lemma 2.1 we have

\[
\sum_{a=0}^{p-1} \chi_2 \left( a^4 + 1 \right) = 1 + \sum_{a=1}^{p-1} \chi_2(a+1) \left( 1 + \lambda(a) + \chi_2(a) + \bar{\chi}(a) \right)
\]
Combining (22), (23), (24) and (25) we have

\[ \frac{1}{\sqrt{p}} \left( \tau^2(\lambda) + \tau^2(\lambda^*) - 1 \right) = 2\alpha - 1. \]  

(23)

\[ \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \chi_2 \left( a^4 + b^4 + 1 \right) = \sum_{a=0}^{p-1} \chi_2 \left( a^4 + (a + 1)^4 + 1 \right) \]

\[ = \chi_2(2) \sum_{a=0}^{p-1} \chi_2 \left( a^4 + 2a^3 + 3a^2 + 2a + 1 \right) = \chi_2(2) \sum_{a=0}^{p-1} \chi_2 \left( (a^2 + a + 1)^2 \right) \]

\[ = \begin{cases} 
\chi_2(2)p & \text{if } p = 12h + 1, \\
\chi_2(2)(p-2) & \text{if } p = 12h + 5.
\end{cases} \]  

(24)

Note that \( \tau(\lambda) \tau(\lambda^*) = -p \), if \( p = 8h + 5 \). \( \tau(\lambda) \tau(\lambda^*) = p \), if \( p = 8h + 1 \). From the method of proving (15) and (19) we have

\[ \sqrt{p} \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \chi_2 \left( a^4 + b^4 + 1 \right) = \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{m=1}^{p-1} \chi_2(m) \left( \chi_2\sqrt{p} + \lambda(m)\tau(\lambda) + \lambda(m)\tau(\lambda^*) \right)^2 e \left( \frac{m}{p} \right) \]

\[ = \begin{cases} 
7p^{3/2} - 2\sqrt{p}\alpha & \text{if } p = 8h + 1, \\
-5p^{3/2} - 2\sqrt{p}\alpha & \text{if } p = 8h + 5.
\end{cases} \]  

(25)

Combining (22), (23), (24) and (25) we have

\[ \sum_{m=1}^{p-1} \chi_2(m) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3 = \begin{cases} 
p^{3/2}(p-6) & \text{if } p = 24h + 1, \\
p^{3/2}(p-8) & \text{if } p = 24h + 17, \\
-p^{3/2}(p-4) & \text{if } p = 24h + 13, \\
-p^{3/2}(p-6) & \text{if } p = 24h + 5.
\end{cases} \]

This proves Lemma 2.4. \( \square \)

3 Proofs of the theorems

Now we prove our main results. First we prove Theorem 1.1. If \( p = 24h + 1 \), then from Lemmas 2.1, 2.2 and 2.4 we have

\[ V_1(p) = \sum_{m=1}^{p-1} B(m) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3 \]

\[ = \sum_{m=1}^{p-1} (\chi_2(m)\sqrt{p} + \lambda(m)\tau(\lambda) + \lambda(m)\tau(\lambda^*)) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3 \]

\[ = p^{3/2}(p-6) - 5p^{3/2} - 2\sqrt{p}\alpha^2 \tau(\lambda) - 5p^{3/2} - 2\sqrt{p}\alpha^2 \tau(\lambda^*) \]

\[ = p \left( p^2 - 16p - 4\alpha^2 \right). \]  

(26)

Applying Lemmas 2.1–2.4 we also have

\[ V_2(p) = \sum_{m=1}^{p-1} B^2(m) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3 \]

\[ = \sum_{m=1}^{p-1} (\chi_2(m)\sqrt{p} + \lambda(m)\tau(\lambda) + \lambda(m)\tau(\lambda^*))^2 \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3 \]
Applying (28) and the method of proving (29) we also have

\[
= \sum_{m=1}^{p-1} \left( 3p + 2\chi_2(m)\sqrt{p\alpha} + 2\lambda(m)\sqrt{p\tau(\lambda)} + 2\lambda(m)\sqrt{p\tau(\lambda)} \right) \left( \sum_{a=0}^{p-1} \left( \frac{ma^4 + a}{p} \right) \right)^3
\]

\[
= p^7 \left( 2p\alpha + 3p - 58\alpha \right). \tag{27}
\]

If \( p = 8h + 1 \), then from (6) we have

\[
B^1(m) = \left( \chi_2(m)\sqrt{p} + \lambda(m)\tau(\lambda) + \lambda(m)\tau(\lambda) \right)^3
\]

\[
= 7\chi_2(m)p^2 + 4p\alpha + 5p\left( \lambda(m)\tau(\lambda) + \lambda(m)\tau(\lambda) \right) + 2\left( \lambda(m)\tau(\lambda) + \lambda(m)\tau(\lambda) \right) \sqrt{p\alpha}.
\] \tag{28}

So if \( p = 24h + 1 \), then from (28), Lemmas 2.1–2.4 we have

\[
V_3(p) = \sum_{m=1}^{p-1} B^3(m) \left( \sum_{a=0}^{p-1} \left( \frac{ma^4 + a}{p} \right) \right)^3
\]

\[
= 7p^3(p - 6) + 4p\alpha \left( p^2 - 6p\alpha \right) - 5p\tau(\lambda)(5p\tau(\lambda) + 2\sqrt{p\alpha\tau(\lambda)})
\]

\[
- 5p\tau(\lambda)(5p\tau(\lambda) + 2\sqrt{p\alpha\tau(\lambda)}) - 2\sqrt{p\alpha\tau(\lambda)}(5p\tau(\lambda) + 2\sqrt{p\alpha\tau(\lambda)})
\]

\[
= p^2 \left( 7p^2 + 4p\alpha - 92p - 72\alpha \right). \tag{29}
\]

If \( p = 24h + 17 \), then from Lemmas 2.1–2.4 we have

\[
V_1(p) = \sum_{m=1}^{p-1} B(m) \left( \sum_{a=0}^{p-1} \left( \frac{ma^4 + a}{p} \right) \right)^3
\]

\[
= p^2(p - 8) - 5p^2 - 2\sqrt{p\alpha\tau(\lambda)}p^2 - 5p^2 - 2\sqrt{p\alpha\tau(\lambda)}p^2
\]

\[
= p \left( p^2 - 18p - 4\alpha \right). \tag{30}
\]

\[
V_2(p) = \sum_{m=1}^{p-1} B^2(m) \left( \sum_{a=0}^{p-1} \left( \frac{ma^4 + a}{p} \right) \right)^3
\]

\[
= \sum_{m=1}^{p-1} \left( 3p + 2\chi_2(c)\sqrt{p\alpha} + 2\lambda(c)\sqrt{p\tau(\lambda)} + 2\lambda(c)\sqrt{p\tau(\lambda)} \right) \left( \sum_{a=0}^{p-1} \left( \frac{ma^4 + a}{p} \right) \right)^3
\]

\[
= p^2 \left( 2p\alpha - 3p - 62\alpha \right). \tag{31}
\]

Applying (28) and the method of proving (29) we also have

\[
V_3(p) = \sum_{m=1}^{p-1} B^3(m) \left( \sum_{a=0}^{p-1} \left( \frac{ma^4 + a}{p} \right) \right)^3
\]

\[
= 7p^3(p - 8) - 4p\alpha \left( p^2 + 6p\alpha \right) - 5p\tau(\lambda)(5p\tau(\lambda) + 2\sqrt{p\alpha\tau(\lambda)})
\]

\[
- 5p\tau(\lambda)(5p\tau(\lambda) + 2\sqrt{p\alpha\tau(\lambda)}) - 2\sqrt{p\alpha\tau(\lambda)}(5p\tau(\lambda) + 2\sqrt{p\alpha\tau(\lambda)})
\]

\[
= p^2 \left( 7p^2 - 4p\alpha - 106p - 72\alpha \right). \tag{32}
\]

Similarly, if \( p = 24h + 5 \), then we have

\[
V_1(p) = \sum_{m=1}^{p-1} B(m) \left( \sum_{a=0}^{p-1} \left( \frac{ma^4 + a}{p} \right) \right)^3
\]

\[
= -p^3(p - 6) + p^2 - 2\sqrt{p\alpha\tau(\lambda)}p^2 + p^2 - 2\sqrt{p\alpha\tau(\lambda)}p^2
\]

\[
= -p \left( p^2 - 8p + 4\alpha \right). \tag{33}
\]
So from (35) and the method of proving (29) we have

\[
V_2(p) = \sum_{m=1}^{p-1} B^2(m) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3
\]

\[
= \sum_{m=1}^{p-1} \left( \chi_2(m) \sqrt{p} + \bar{\lambda}(m) \tau(\lambda) + \lambda(m) \tau(\bar{\lambda}) \right)^2 \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3
\]

\[
= \sum_{m=1}^{p-1} \left( 2\chi_2(c) \sqrt{p} \alpha - p + 2\lambda(c) \sqrt{p} \tau(\lambda) + 2\bar{\lambda}(c) \sqrt{p} \tau(\bar{\lambda}) \right) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3
\]

\[
= -p^2 \left( 2p\alpha - p - 22\alpha \right).
\]

(34)

If \( p = 24h + 5 \), then from (6) we have

\[
B^3(m) = \left( \chi_2(m) \sqrt{p} + \bar{\lambda}(m) \tau(\lambda) + \lambda(m) \tau(\bar{\lambda}) \right)^3
\]

\[
= -5\chi_2(m)p^3 + 6p\alpha + p \left( \bar{\lambda}(m) \tau(\lambda) + \lambda(m) \tau(\bar{\lambda}) \right) + 2 \left( \bar{\lambda}(m) \tau(\bar{\lambda}) + \lambda(m) \tau(\lambda) \right) \sqrt{p} \alpha.
\]

(35)

So from (35) and the method of proving (29) we have

\[
V_3(p) = \sum_{m=1}^{p-1} B^3(m) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3
\]

\[
= 5p^3(p - 6) - 6p\alpha \left( p^2 + 6p\alpha \right) - p\tau(\bar{\lambda}) \left( p\tau(\lambda) + 2\sqrt{p} \alpha \tau(\bar{\lambda}) \right) - p\tau(\lambda) \left( p\tau(\lambda) + 2\sqrt{p} \alpha \tau(\lambda) \right) - 2\sqrt{p} \alpha \tau(\bar{\lambda}) \left( p\tau(\lambda) + 2\sqrt{p} \alpha \tau(\lambda) \right)
\]

\[
= p^2 \left( 5p^2 - 6p\alpha - 28p - 36\alpha^2 \right).
\]

(36)

If \( p = 24h + 13 \), then (35), Lemmas 2.1–2.4 we have

\[
V_1(p) = \sum_{m=1}^{p-1} B(m) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3
\]

\[
= -p^2(p - 4) + p^2 - 2\sqrt{p} \alpha \tau^2(\lambda) + p^2 - 2\sqrt{p} \alpha \tau^2(\bar{\lambda})
\]

\[
= -p \left( p^2 - 6p + 4\alpha^2 \right).
\]

(37)

\[
V_2(p) = \sum_{m=1}^{p-1} B^2(m) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3
\]

\[
= \sum_{m=1}^{p-1} \left( \chi_2(m) \sqrt{p} + \bar{\lambda}(m) \tau(\lambda) + \lambda(m) \tau(\bar{\lambda}) \right)^2 \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3
\]

\[
= \sum_{m=1}^{p-1} \left( 2\chi_2(c) \sqrt{p} \alpha - p + 2\lambda(c) \sqrt{p} \tau(\lambda) + 2\bar{\lambda}(c) \sqrt{p} \tau(\bar{\lambda}) \right) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3
\]

\[
= -p^2 \left( 2p\alpha + p - 18\alpha \right).
\]

(38)

\[
V_3(p) = \sum_{m=1}^{p-1} B^3(m) \left( \sum_{a=0}^{p-1} e \left( \frac{ma^4 + a}{p} \right) \right)^3
\]

\[
= 5p^3(p - 4) + 6p\alpha \left( p^2 - 6p\alpha \right) - p\tau(\bar{\lambda}) \left( p\tau(\lambda) + 2\sqrt{p} \alpha \tau(\bar{\lambda}) \right) - p\tau(\lambda) \left( p\tau(\lambda) + 2\sqrt{p} \alpha \tau(\lambda) \right) - 2\sqrt{p} \alpha \tau(\bar{\lambda}) \left( p\tau(\lambda) + 2\sqrt{p} \alpha \tau(\lambda) \right)
\]

\[
= p^2 \left( 5p^2 + 6p\alpha - 18p - 36\alpha^2 \right).
\]

(39)
Finally, note that if \( p = 8h + 1 \), then from (6) and direct calculation (or see Lemma 3 in [7]) we have the identity
\[
B^4(m) = 6pB^2(m) + 8p\alpha B(m) - p\left(p - 4\alpha^2\right).
\]
For any prime \( p = 24h + 1 \) and integer \( k \geq 4 \), from (26), (27), (29) and (40) we may immediately deduce the fourth-order linear recurrence formula
\[
V_k(p) = \sum_{m=1}^{p-1} B^4(m) \left( \sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right)\right)^3
\]
\[
= \sum_{m=1}^{p-1} B^{4-k}(m) \left( 6pB^2(m) + 8p\alpha B(m) - p\left(p - 4\alpha^2\right) \right) \left( \sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right)\right)^3
\]
\[
= 6pV_{k-2}(p) + 8p\alpha V_{k-3}(p) - p\left(p - 4\alpha^2\right) V_{k-4}(p),
\]
where the first four values \( V_0(p) = p^2 - 6p\alpha, V_1(p) = p\left(p^2 - 16p - 4\alpha^2\right), V_2(p) = p^2\left(2p\alpha + 3p - 58\alpha\right) \) and \( V_3(p) = p^2\left(7p^2 + 4p\alpha - 92p - 72\alpha^2\right) \).
This proves Theorem 1.1.
If \( p = 24h + 17 \), then from (30), (31), (32) and (40) we have
\[
V_k(p) = 6pV_{k-2}(p) + 8p\alpha V_{k-3}(p) - p\left(p - 4\alpha^2\right) V_{k-4}(p),
\]
where the first four values \( V_0(p) = -p^2 - 6p\alpha, V_1(p) = p\left(p^2 - 18p - 4\alpha^2\right), V_2(p) = p^2\left(2p\alpha - 3p - 62\alpha\right) \) and \( V_3(p) = p^2\left(7p^2 - 4p\alpha - 106p - 72\alpha^2\right) \).
This proves Theorem 1.2.
If \( p = 8h + 5 \), then from (6) and direct calculation (or see Lemma 3 in [7]) we also have
\[
B^4(m) = -2pB^2(m) + 8p\alpha B(m) - p\left(9p - 4\alpha^2\right).
\]
For any prime \( p = 24h + 5 \) and integer \( k \geq 4 \), from (33), (34), (35) and (41) we can deduce the fourth-order linear recurrence formula
\[
V_k(p) = -2pV_{k-2}(p) + 8p\alpha V_{k-3}(p) - p\left(9p - 4\alpha^2\right) V_{k-4}(p),
\]
where the first four terms are \( V_0(p) = -\left(p^2 + 6p\alpha\right), V_1(p) = -p\left(p^2 - 8p + 4\alpha^2\right), V_2(p) = -p^2\left(2p\alpha - p - 22\alpha\right) \) and \( V_3(p) = p^2\left(5p^2 - 6p\alpha - 28p - 36\alpha^2\right) \).
This proves Theorem 1.3.
If \( p = 24h + 13 \), then from (37), (38), (39) and (41) we also have
\[
V_k(p) = -2pV_{k-2}(p) + 8p\alpha V_{k-3}(p) - p\left(9p - 4\alpha^2\right) V_{k-4}(p),
\]
where the first four terms are \( V_0(p) = p^2 - 6p\alpha, V_1(p) = -p\left(p^2 - 6p + 4\alpha^2\right), V_2(p) = -p^2\left(2p\alpha + p - 18\alpha\right) \) and \( V_3(p) = p^2\left(5p^2 + 6p\alpha - 18p - 36\alpha^2\right) \).
This completes the proofs of our all results.

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References