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Two asymptotic expansions for gamma function developed by Windschitl’s formula

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**Abstract:** In this paper we develop Windschitl’s approximation formula for the gamma function by giving two asymptotic expansions using a little known power series. In particular, for \( n \in \mathbb{N} \) with \( n \geq 4 \), we have

\[
\Gamma(x + 1) = \sqrt{2\pi x} \left( \frac{x}{e} \right)^x \left( x \sinh \frac{1}{x} \right)^{x/2} \exp \left( \sum_{k=3}^{n-1} \frac{a_n}{x^{2k-1}} + R_n(x) \right)
\]

with

\[
|R_n(x)| \leq \frac{|B_{2n}|}{2n(2n-1)} \frac{1}{x^{2n-1}}
\]

for all \( x > 0 \), where \( a_n \) has a closed-form expression, \( B_{2n} \) is the Bernoulli number. Moreover, we present some approximation formulas for the gamma function related to Windschitl’s approximation, which have higher accuracy.

**Keywords:** Gamma function, Windschitl’s formula, Asymptotic expansion

**MSC:** 33B15, 41A60, 41A10, 41A20

1 Introduction

It is known that the Stirling’s formula

\[
n! \sim \sqrt{2\pi n} \left( \frac{n}{e} \right)^n
\]

for \( n \in \mathbb{N} \) has various applications in probability theory, statistical physics, number theory, combinatorics and other branches of science. As a generalization of the factorial function, the gamma function \( \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \) for \( x > 0 \) is no exception. Thus, many scholars pay attention to find various better approximations for the factorial or gamma function, for example, Ramanujan [1, P. 339], Burnside [2], Gosper [3], Alzer [4, 5], Windschitl (see [6, 7]), Smith [6], Batir [8, 9], Mortici [10–15] Nemes [16, 17], Qi et al. [18, 19], Chen [20], Yang et al. [21–23], Lu et al. [24, 25].

As an asymptotic expansion of Stirling’s formula (1), one has the Stirling’s series for the gamma function [26, p. 257, Eq. (6.1.40)]

\[
\Gamma(x + 1) \sim \sqrt{2\pi x} \left( \frac{x}{e} \right)^x \exp \left( \sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)x^{2n-1}} \right)
\]

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as \( x \to \infty \), where \( B_{2n} \) for \( n \in \mathbb{N} \cup \{0\} \) is the Bernoulli number. It was proved in [4, Theorem 8] by Alzer (see also [27, Theorem 2]) that for given integer \( n \in \mathbb{N} \), the function

\[
F_n(x) = \ln \Gamma(x+1) - \left(x + \frac{1}{2}\right) \ln x + x - \frac{1}{2} \ln (2\pi) - \sum_{k=1}^{n} \frac{B_{2k}}{2k(2k-1)} x^{2k-1}
\]

is strictly completely monotonic on \((0, \infty)\) if \( n \) is even, and so is \(-F_n(x)\) if \( n \) is odd. It follows that the double inequality

\[
\exp\left(\sum_{k=1}^{2n} \frac{B_{2k}}{2k(2k-1)} x^{2k-1}\right) < \frac{\Gamma(x+1)}{\sqrt{2\pi x (x/e)^x}} < \exp\left(\sum_{k=1}^{2n-1} \frac{B_{2k}}{2k(2k-1)} x^{2k-1}\right)
\]

(3) holds for all \( x > 0 \).

Another asymptotic expansion is the Laplace series (see [26, p. 257, Eq. (6.13)])

\[
\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x}\right)^{x/2}, \quad \text{as } x \to \infty.
\]

(5)

As shown in [20, Eq. (3.8)], the rate of Windschitl’s approximation \( W_0(x) \) converging to \( \Gamma(x+1) \) is like \( x^{-5} \) as \( x \to \infty \), and is like \( x^{-7} \) if replacing \( W_0(x) \) with

\[
W_1(x) = \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x}\right)^{x/2} + \frac{1}{810x^6},
\]

(6)

by an easy check. These show that \( W_0(x) \) and \( W_1(x) \) are excellent approximations for the gamma function. Recently, Lu, Song and Ma [32] extended Windschitl’s formula to the following asymptotic expansion

\[
\Gamma(n+1) \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left[n \sinh \left(\frac{1}{n} + \frac{a_7}{n^7} + \frac{a_9}{n^9} + \frac{a_{11}}{n^{11}} + \ldots\right)\right]^{n/2}
\]

(7)

as \( n \to \infty \) with \( a_7 = 1/810, a_9 = -67/42525, a_{11} = 19/8505, \ldots \). An explicit formula for determining the coefficients of \( n^{-k} (n \in \mathbb{N}) \) was given in [34, Theorem 1] by Chen. Other two asymptotic expansions

\[
\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x}\right)^{x/2 + \sum_{i=1}^{\infty} \frac{d_i}{x^{2i}}}.
\]

(8)

\[
\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x}\right)^{x/2} + \frac{1}{x} \left(1 + \sum_{n=1}^{\infty} \frac{b_n}{x^n}\right)
\]

(9)

as \( x \to \infty \) were presented in the papers [34, Theorem 2], [36], respectively.

Inspired by the asymptotic expansions (7), (8), (9) and Windschitl’s approximation formula (6), the first aim of this paper is to further present the following two asymptotic expansions related to Windschitl’s one (5): as \( x \to \infty \),

\[
\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x}\right)^{x/2} \exp\left(\sum_{n=1}^{\infty} \frac{a_n}{x^{2n-1}}\right),
\]

(10)

\[
\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x}\right)^{x/2} \left(1 + \sum_{n=1}^{\infty} \frac{b_n}{x^n}\right).
\]

(11)

It is worth pointing out that those coefficients in (10) have a closed-form expression, which is due to a little known power series expansion of \( \ln \left(\frac{x}{e}\sinh t\right) \) (Lemma 2.1). We also give an estimate of the remainder in the asymptotic expansion (10). Incidentally, we provide a more explicit coefficients formula in Chen’s asymptotic expansion (8). These results (Theorems 1–4) are presented in Section 2.

The second aim of this paper is to give some closed approximation formulas for the gamma function generated by truncating five asymptotic series just mentioned, and compare the accuracy of them by numeric computations and some inequalities. These results (Table 1 and Theorem 5) are listed in Section 3.
2 Asymptotic expansions

To obtain the explicit coefficients formulas in the asymptotic expansions (10), (11) and (8), and to estimate the remainder in the asymptotic expansions (10), we first give a lemma.

Lemma 2.1. For $|t| < \pi$, we have
\[ \ln \frac{\sinh t}{t} = \sum_{n=1}^{\infty} \frac{2^{2n}B_{2n}}{(2n)!} t^{2n}. \] (12)

Moreover, for $n \in \mathbb{N}$, the double inequality
\[ \sum_{k=1}^{2n} \frac{2^k B_{2k}}{2k (2k)!} t^{2k} < \ln \frac{\sinh t}{t} < \sum_{k=1}^{2n-1} \frac{2^k B_{2k}}{2k (2k)!} t^{2k} \] (13)
holds for all $t > 0$.

Proof. It was listed in [26, p. 85, Eq. (4.5.64), (4.5.65), (4.5.67)] that
\[ \coth t = \sum_{n=0}^{\infty} \frac{2^{2n}B_{2n}}{(2n)!} t^{2n-1} |t| < \pi. \]

Then we obtain that for $|t| < \pi$,
\[ \ln \frac{\sinh t}{t} = \int_{0}^{t} \left( \coth x - \frac{1}{x} \right) dx = \int_{0}^{t} \left( \sum_{n=0}^{\infty} \frac{2^{2n}B_{2n}}{(2n)!} x^{2n-1} - \frac{1}{x} \right) dx = \sum_{n=1}^{\infty} \frac{2^{2n}B_{2n}}{(2n)!} t^{2n}. \]
The double inequality (13) was proved in [23, Corollary 1], and the proof is completed. \hfill \Box

Theorem 2.2. As $x \to \infty$, the asymptotic expansion
\[ \Gamma'(x+1) \sim \sqrt{2\pi x} \left( \frac{x}{e} \right)^x \left( x \sinh \frac{1}{x} \right)^{x/2} \exp \left( \sum_{n=3}^{\infty} \frac{a_n}{x^{2n-1}} \right) \]
\[ = \sqrt{2\pi x} \left( \frac{x}{e} \right)^x \left( x \sinh \frac{1}{x} \right)^{x/2} \exp \left[ \frac{1}{1620x^5} - \frac{11}{18900x^7} + \frac{143}{170100x^9} - \frac{2260261}{1178793000x^{11}} + \cdots \right] \]
holds with
\[ a_n = \frac{2n(2n-2)! - 2^{2n-1}}{2n(2n)!} B_{2n}, \] (14)
where $B_{2n}$ is the Bernoulli number.

Proof. By the asymptotic expansion (2) and Lemma 2.1 we have that as $x \to \infty$,
\[ \ln \Gamma'(x+1) = \ln \sqrt{2\pi} - \left( x + \frac{1}{2} \right) \ln x + x \sim \sum_{n=1}^{\infty} \frac{a'_n}{x^{2n-1}}, \]
\[ \frac{x}{2} \ln \left( x \sinh \frac{1}{x} \right) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{a''_n}{x^{2n-1}}, \]
where
\[ a'_n = \frac{B_{2n}}{2n(2n-1)} \quad \text{and} \quad a''_n = \frac{2^{2n}B_{2n}}{2n(2n)!}. \]
Let
\[ \Gamma'(x+1) \sim \sqrt{2\pi x} \left( \frac{x}{e} \right)^x \left( x \sinh \frac{1}{x} \right)^{x/2} \exp \left( w_0(x) \right) \quad \text{as} \quad x \to \infty. \]
Then we have that as $x \to \infty$,
\[ w_0(x) = \left( \ln \Gamma'(x+1) - \ln \sqrt{2\pi} - \left( x + \frac{1}{2} \right) \ln x + x \right) - \frac{x}{2} \ln \left( x \sinh \frac{1}{x} \right) \]
An easy computation yields $a_1 = a_2 = 0$ and

$$a_3 = \frac{1}{1620}, \quad a_4 = -\frac{11}{18900}, \quad a_5 = \frac{143}{170100}, \quad a_6 = -\frac{2260261}{1178793000}$$

which completes the proof.

The following theorem offers an estimate of the remainder in the asymptotic expansion (10).

**Theorem 2.3.** For $n \in \mathbb{N}$ with $n \geq 4$, let

$$\Gamma (x + 1) = \sqrt{2\pi x} \left( \frac{x}{e} \right)^x \left( x \sinh \frac{1}{x} \right)^{x/2} \exp \left( \sum_{k=3}^{n-1} \frac{a_k}{x^{2k-1}} + R_n (x) \right),$$

where $a_k$ is given by (14). Then we have

$$|R_n (x)| \leq \frac{|B_{2n}|}{2n (2n-1) x^{2n-1}}$$

for all $x > 0$.

**Proof.** We have

$$R_n (x) = \ln \Gamma (x + 1) - \ln \sqrt{2\pi} - \left( x + \frac{1}{2} \right) \ln x + x - x \ln \left( x \sinh \frac{1}{x} \right) - \sum_{k=3}^{n-1} \frac{a_k}{x^{2k-1}}.$$

If $n = 2m + 1$ for $m \geq 2$ then by inequalities (3) and (13) we have

$$R_{2m} (x) = \left( \ln \Gamma (x + 1) - \ln \sqrt{2\pi} - \left( x + \frac{1}{2} \right) \ln x + x - x \ln \left( x \sinh \frac{1}{x} \right) - \sum_{k=3}^{2m+1} \frac{a_k}{x^{2k-1}} \right) - \frac{1}{2} \frac{B_{2m+1}}{x^{2m+1}},$$

$$R_{2m} (x) < \sum_{k=3}^{2m+1} a_k \frac{a'''}{x^{2k-1}} - \frac{1}{2} \frac{B_{2m+1}}{x^{2m+1}} - \sum_{k=3}^{2m} \frac{a_m}{x^{2k-1}} = \frac{a'''_{2m+1}}{x^{2m+1}},$$

where the last equalities in the above two inequalities hold due to $a_m - a''/2 = a_m$. It follows that

$$|R_n (x)| < \max \left( \left| \frac{B_{2n}}{2n (2n-1)} \right|, \left| \frac{a'''}_{2m+1}}{x^{2m+1}} \right| \right) = \max \left( \frac{1}{2} |a''|, |a'''| \right) \frac{1}{x^{2n-1}}.$$

The calculations also hold for the case when $n = 2m$, $m \geq 2$.

Since

$$\frac{|a''|}{a''/2} = \left| \frac{B_{2n}}{2n (2n-1)} \right| \left| \frac{1}{2} \frac{2^n B_{2n}}{(2n-1)!} \right| = 2 \frac{(2n)!}{(2n-1)!} \frac{B_{2n}}{2n} = a'''' \quad \text{and} \quad \frac{a'''_{2m+1}}{a'''} = 1 - \frac{1}{2} \frac{(2n+3)(n-1)}{2n} > 0,$$

it is derived that $a'''' > a'' = 45$ for $n \geq 4$, so we obtain

$$\max \left( \frac{1}{2} |a''|, |a'''| \right) = \left| a'' \right| = \frac{|B_{2n}|}{2n (2n-1)},$$

which completes the proof. □
Remark 2.4. Since $B_{2n+1} = 0$ for $n \in \mathbb{N}$, the asymptotic series $w_0(x)$ can also be written as
\[
 w_0(x) = \sum_{n=1}^{\infty} \frac{2n(2n-2)! - 2^{2n-1} B_{2n}}{2n(2n)!} = \sum_{n=1}^{\infty} \frac{(n+1)(n-1)! - 2^n B_{n+1}}{(n+1)(n+1)!} x^n = \sum_{n=1}^{\infty} a_n^* x^n,
\]
where
\[
a_n^* = \frac{(n+1)(n-1)! - 2^n B_{n+1}}{(n+1)(n+1)!}.
\]
(15)

Now we establish the second Windschitl type asymptotic series for the gamma function.

Theorem 2.5. As $x \to \infty$, the asymptotic expansion (9)
\[
\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x}\right)^{x/2} \left(1 + \sum_{n=1}^{\infty} b_n x^n\right)
\]
holds with $b_0 = 1$, $b_1 = b_2 = b_3 = b_4 = 0$ and for $n \geq 5$,
\[
b_n = \frac{1}{n} \sum_{k=1}^{n} \left(\frac{1}{k+1} - \frac{2^k}{(k+1)^k(k-1)!}\right) B_{k+1} b_{n-k}.
\]
(16)

Proof. It was proved in [29, Lemma 3] that as $x \to \infty$,
\[
\exp \left(\sum_{n=1}^{\infty} a_n x^{-n}\right) \sim \sum_{n=0}^{\infty} b_n x^{-n}
\]
with $b_0 = 1$ and
\[
b_n = \frac{1}{n} \sum_{k=1}^{n} k a_k b_{n-k} \text{ for } n \geq 1.
\]
(17)

Substituting $a_n^*$ given in (15) into (17) gives recurrence formula (16).

An easy verification shows that $b_n = 0$ for $1 \leq n \leq 4$, $b_5 = b_8 = 0$ and
\[
b_5 = \frac{1}{1620}, \quad b_7 = -\frac{11}{18900}, \quad b_9 = \frac{143}{170100}, \quad b_{10} = \frac{1}{5248800},
\]
which completes the proof. □

The following theorem improves Chen’s result [34, Theorem 2].

Theorem 2.6. As $x \to \infty$, the asymptotic expansion
\[
\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x}\right)^{x/2} (1 + \sum_{n=2}^{\infty} c_n x^{-n})
\]
holds with $c_0 = 1$, $c_1 = 0$ and for $n \geq 2$,
\[
c_n = \frac{6B_{2n+2}}{(n+1)(2n+1)} - 6 \sum_{k=1}^{n} \frac{2^{2k+2} B_{2k+2}}{(k+1)(2k+2)} c_{n-k}.
\]

Proof. The asymptotic expansion (8) can be written as
\[
\ln \Gamma(x+1) - \ln \sqrt{2\pi x} - x \ln x + x \sim \left(\frac{x}{2} + \sum_{j=0}^{\infty} \frac{r_j}{x^j}\right) \ln \left(x \sinh \frac{1}{x}\right),
\]
which, by (2) and (12), is equivalent to
\[
\sum_{n=1}^{\infty} \frac{B_{2n}}{2n (2n - 1) x^{2n-1}} \sim \left( \frac{x}{2} + \sum_{j=0}^{\infty} \frac{r_j}{x^j} \right) \left( \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n}}{2n (2n)!} \frac{1}{x^{2n}} \right).
\]

Since the left hand side and the second factor of the right hand side are odd and even, respectively, the asymptotic expansion \(x/2 + \sum_{j=0}^{\infty} r_j x^{-j}\) has to be odd, and so \(r_{2n} = 0\) for \(n \in \mathbb{N} \cup \{0\}\). Then, the asymptotic expansion (8) has the form of (18), which is equivalent to
\[
\sum_{n=1}^{\infty} \frac{B_{2n}}{2n (2n - 1) x^{2n-1}} \sim \frac{x}{2} \left( \sum_{n=0}^{\infty} \frac{c_n}{x^{2n}} \right) \left( \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n}}{2n (2n)!} \frac{1}{x^{2n}} \right).
\]

It can be written as
\[
\sum_{n=0}^{\infty} \frac{B_{2n+2}}{(n+1) (2n+1) x^{2n+1}} \sim \left( \sum_{n=0}^{\infty} \frac{c_n}{x^{2n}} \right) \left( \sum_{n=1}^{\infty} \frac{2^{2n+2} B_{2n+2}}{2(n+1) (2n+2)!} \frac{1}{x^{2n}} \right).
\]

Comprising coefficients of \(x^{-2n}\) gives
\[
\frac{B_{2n+2}}{(n+1) (2n+1)} = \sum_{k=0}^{n} \frac{2^{2k+2} B_{2k+2}}{2 (k+1) (2k+2)!} c_{n-k},
\]
which yields \(c_0 = 1\) and for \(n \geq 1\),
\[
c_n = \frac{6B_{2n+2}}{(n+1) (2n+1)} - 6 \sum_{k=1}^{n} \frac{2^{2k+2} B_{2k+2}}{2 (k+1) (2k+2)!} c_{n-k}.
\]

A straightforward computation leads to
\[
c_1 = 0, \ c_2 = \frac{1}{135}, \ c_3 = \frac{191}{28350}, \ c_4 = \frac{25127}{2551500}, \ c_5 = \frac{19084273}{841995000},
\]
which ends the proof. \(\square\)

**Remark 2.7.** Chen’s recurrence formula of coefficients \(r_j\) given in [34, Theorem 2] may be complicated, since he was unaware of the power series (12).

### 3 Numeric comparisons and inequalities

If the series in (10), (11), (9) (18) are truncated at \(n = 3, 5, 3, 2\), respectively, then we obtain four Windschitl type approximation formulas:
\[
\Gamma(x+1) \sim \sqrt{2\pi x} \left( \frac{x}{e} \right)^x \left( x \sinh \frac{1}{x} \right)^{x/2} \exp \left( \frac{1}{1620x^7} \right) =: W_{01}(x), \quad (19)
\]
\[
\Gamma(x+1) \sim \sqrt{2\pi x} \left( \frac{x}{e} \right)^x \left( x \sinh \frac{1}{x} \right)^{x/2} \left( 1 + \frac{1}{1620x^7} \right) =: W_{01}'(x), \quad (20)
\]
\[
\Gamma(x+1) \sim \sqrt{2\pi x} \left( \frac{x}{e} \right)^x \left( x \sinh \frac{1}{x} + \frac{1}{810x^5} \right)^{x/2} =: W_1(x), \quad (21)
\]
\[
\Gamma(x+1) \sim \sqrt{2\pi x} \left( \frac{x}{e} \right)^x \left( x \sinh \frac{1}{x} + \frac{1}{150x^3} \right)^{1/2} =: W_{c1}(x), \quad (22)
\]
as \(x \to \infty\). Also, we denote Lu et al.’s one [32, Theorem 1.8] by
\[
W_{11}(x) = \sqrt{2\pi x} \left( \frac{x}{e} \right)^x \left( x \sinh \left( \frac{1}{x} + \frac{1}{810x^5} \right) \right)^{x/2}.
\]

In this section, we aim to compare the five closed approximation formulas listed above.
3.1 Numeric comparisons

We easily obtain

\[
\lim_{x \to \infty} \ln' \left( \frac{x+1}{x} \right) = -\frac{163}{340 \, 200},
\]

\[
\lim_{x \to \infty} \ln' \left( \frac{x+1}{x} \right) = -\frac{191}{340 \, 200},
\]

\[
\lim_{x \to \infty} \ln' \left( \frac{x+1}{x} \right) = -\frac{268}{340 \, 200}.
\]

These show that the rates of the five approximation formulas converging to \( \ln' (x + 1) \) are all like \( x^{-7} \) as \( x \to \infty \), and \( W_i(x) \) are the best of all five approximations, which can also be seen from the following Table 1.

| Table 1. Comparisons among \( W_1(x) \), \( W_{c1}(x) \), \( W_{01}(x) \), \( W_{01}(x) \), \( W_{11}(x) \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( x \) | \( W_1(x) - (x+1) \) | \( W_{c1}(x) - (x+1) \) | \( W_{01}(x) - (x+1) \) | \( W_{11}(x) - (x+1) \) |
| 1 | 1.832 × 10^{−6} | 2.562 × 10^{−6} | 2.754 × 10^{−6} | 4.686 × 10^{−6} |
| 2 | 2.668 × 10^{−6} | 3.292 × 10^{−6} | 3.449 × 10^{−6} | 5.030 × 10^{−6} |
| 5 | 5.743 × 10^{−9} | 6.791 × 10^{−9} | 7.054 × 10^{−9} | 9.681 × 10^{−9} |
| 10 | 4.711 × 10^{−13} | 5.532 × 10^{−11} | 5.738 × 10^{−11} | 7.794 × 10^{−11} |
| 20 | 3.727 × 10^{−13} | 4.370 × 10^{−13} | 4.531 × 10^{−13} | 6.138 × 10^{−13} |
| 50 | 6.129 × 10^{−16} | 7.182 × 10^{−16} | 7.464 × 10^{−16} | 1.008 × 10^{−15} |
| 100 | 4.791 × 10^{−18} | 5.614 × 10^{−18} | 5.819 × 10^{−18} | 7.877 × 10^{−18} |

3.2 Three lemmas

As is well known, analytic inequality \([40–42]\) is playing a very important role in different branches of modern mathematics. To further compare \( W_1(x) \), \( W_{c1}(x) \), \( W_{01}(x) \), \( W_{01}(x) \), \( W_{11}(x) \), we first give the following inequality.

**Lemma 3.1.** The inequality

\[
\psi' \left( x + 1, \frac{1}{2} \right) \leq -\frac{1}{x} \frac{x^4 + 67 \, x^2 + 256}{18 \, x^2 + 407} \tag{23}
\]

holds for all \( x > 0 \).

**Proof.** Let

\[
g(x) = x \psi \left( x + 1, \frac{1}{2} \right) - \frac{1}{x} \frac{x^4 + 67 \, x^2 + 256}{18 \, x^2 + 407}.
\]

Then we have

\[
g(x + 1) - g(x) = x \psi \left( x + 1, \frac{1}{2} \right) + \frac{1}{x} \frac{x^4 + 67 \, x^2 + 256}{18 \, x^2 + 407} - \psi \left( x, \frac{1}{2} \right) - \frac{1}{x} \frac{x^4 + 67 \, x^2 + 256}{18 \, x^2 + 407}
\]

\[
= \frac{921 \, 600}{x(x + 1)(2x + 1)^2(1008x^4 + 1960x^2 + 407)(1008x^4 + 4032x^3 + 8008x^2 + 7952x + 3375)} > 0.
\]

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Lemma 3.3. Let
\[ f(x) < f(x+1) < \cdots < \lim_{n \to \infty} f(x+n) = 0, \]
which proves (23), and the proof is done.

The second lemma offers a simple criterion to determine the sign of a class of special polynomial on a given interval contained in \((0, \infty)\) without using Descartes’ Rule of Signs, which plays an important role in the study of certain special functions, see for example [43, 44]. A series version can be found in [45].

Lemma 3.2 ([43, Lemma 7]). Let \( n \in \mathbb{N} \) and \( m \in \mathbb{N} \cup \{0\} \) with \( n > m \) and let \( P_n(t) \) be an \( n \) degrees polynomial defined by
\[
P_n(t) = \sum_{i=m+1}^{n} a_i t^i - \sum_{i=0}^{m} a_i t^i, \tag{24}
\]
where \( a_n, a_m > 0, a_1 \geq 0 \) for \( 0 \leq i \leq n-1 \) with \( i \neq m \). Then, there is a unique number \( t_{m+1} \in (0, \infty) \) to satisfy \( P_n(t) = 0 \) such that \( P_n(t) < 0 \) for \( t \in (0, t_{m+1}) \) and \( P_n(t) > 0 \) for \( t \in (t_{m+1}, \infty) \).

Consequently, for given \( t_0 > 0 \), if \( P_n(t_0) > 0 \) then \( P_n(t) > 0 \) for \( t \in (t_0, \infty) \) and if \( P_n(t_0) < 0 \) then \( P_n(t) < 0 \) for \( t \in (t_0, 0) \).

Lemma 3.3. Let \( W_{01}(x), W_{01}^*(x), W_1(x), W_{c1}(x) \) and \( W_{11}(x) \) be defined by (19), (20), (6), (21) and (22), respectively. Then we have
\[
W_1(x) < W_{c1}(x) < W_{01}(x) < W_{01}^*(x) < W_{11}(x)
\]
for all \( x \geq 1 \).

Proof. (i) The first inequality \( W_1(x) < W_{c1}(x) \) is equivalent to
\[
h_1(t) = \ln \left( \frac{\sinh t}{t} + \frac{1}{810} t^6 \right) - \left( 1 + \frac{1}{135} t^4 \right) \ln \left( \frac{\sinh t}{t} \right) < 0
\]
for \( t = 1/x \in (0, 1] \). We have
\[
\frac{d}{dy} \left( \ln \left( y + \frac{1}{810} t^6 \right) - \left( 1 + \frac{1}{135} t^4 \right) \ln y \right) = -\frac{1}{135} t^4 \frac{810 y + 135 t^2 + t^6}{y (810y + t^6)} < 0
\]
for \( y > 1 \), which together with
\[
y = \frac{\sinh t}{t} > 1 + \frac{1}{6} t^2
\]
yields
\[
h_1(t) < \ln \left( 1 + \frac{1}{6} t^2 + \frac{1}{810} t^6 \right) - \left( 1 + \frac{1}{135} t^4 \right) \ln \left( 1 + \frac{1}{6} t^2 \right) := h_{11}(t).
\]
Differentiation leads us to
\[
\frac{135}{2t^3} h_{11}'(t) = -2 \ln \left( \frac{1}{6} t^2 + 1 \right) - t^2 \frac{t^6 - 135 t^2 - 1620}{(t^2 + 6)(t^6 + 135 t^2 + 810)} := h_{12}'(t),
\]
\[
h_{12}'(t) = -4 \left( t^{12} + 9 t^{10} + 540 t^8 + 8505 t^6 + 47385 t^4 + 328050 t^2 + 1312200 \right) \left( t^2 + 6 \right)^2 \left( t^6 + 135 t^2 + 810 \right)^2 < 0
\]
for \( t > 0 \). Therefore, we obtain \( h_{12}(t) < h_{12}(0) = 0 \), and so \( h_{11}(t) < h_{11}(0) = 0 \), which implies \( h_1(t) < 0 \) for \( t > 0 \).

(ii) The second inequality \( W_{c1}(x) < W_{01}^*(x) \) is equivalent to
\[
\frac{x}{2} \left( 1 + \frac{1}{135 x^2} \right) \ln \left( x \sinh \frac{1}{x} \right) < \frac{x}{2} \ln \left( x \sinh \frac{1}{x} \right) + \ln \left( 1 + \frac{1}{1620 x^3} \right),
\]
or equivalently,
\[
h_2(t) = \frac{1}{270} t^3 \ln \left( \frac{\sinh t}{t} \right) - \ln \left( 1 + \frac{1}{1620} t^5 \right) < 0
\]
for \( t = 1/x \in (0, 1] \). Taking \( n = 2 \) in the inequalities (13) gives

\[
\ln \left( \frac{\sinh t}{t} \right) < \frac{1}{6} t^2 - \frac{1}{180} t^6 + \frac{1}{2835} t^6,
\]

which is applied to the expression of \( h_2 (t) \):

\[
h_2 (t) < \frac{1}{270} t^3 \left( \frac{1}{6} t^2 - \frac{1}{180} t^6 + \frac{1}{2835} t^6 \right) - \ln \left( 1 + \frac{1}{1620} t^3 \right) := h_{21} (t).
\]

Differentiation yields

\[
h'_{21} (t) = \frac{t^6}{340200} \left( 4t^2 - 49t^5 + 1050t^3 + 6480t^2 - 79380 \right) < 0
\]

for \( t \in (0, 1] \), which proves \( h_2 (t) < 0 \) for \( t \in (0, 1] \).

(iii) The third inequality \( W_{01}^* (x) < W_{01} (x) \) is equivalent to

\[
1 + \frac{1}{1620x^5} < \exp \left( \frac{1}{1620x^5} \right)
\]

which follows by a simple inequality \( 1 + y < e^y \) for \( y \neq 0 \).

(iv) The fourth inequality \( W_{01} (x) < W_{11} (x) \) is equivalent to

\[
x \frac{1}{2} \ln \left( x \sinh \left( \frac{1}{x} + \frac{1}{810x^2} \right) \right) > x \frac{1}{2} \ln \left( x \sinh \frac{1}{x} \right) + \frac{1}{1620x^5},
\]

or equivalently,

\[
h_3 (t) = \ln \sinh \left( t + \frac{1}{810} t^7 \right) - \ln \sinh t - \frac{1}{810} t^6 > 0
\]

for \( t = 1/x > 0 \). Denote \( h_{30} (t) = \ln \sinh t \). Then by Taylor formula we have

\[
h_3 (t) = h_{30} \left( t + \frac{1}{810} t^7 \right) - h_{30} (t) - \frac{1}{810} t^6
\]

\[
= \frac{t^7}{810} h'_{30} (t) + \frac{1}{2} \frac{t^{14}}{810^2} h''_{30} (t) + \frac{1}{3!} \frac{t^{21}}{810^3} h'''_{30} (\xi) - \frac{1}{810} t^6,
\]

where \( t < \xi < t + t^7/810 \). Since \( h'''_{30} (t) = 2 (\cosh t)/\sinh^3 t > 0 \), we get

\[
h_3 (t) > \frac{1}{810} \frac{t^7 \cosh t}{\sinh t} - \frac{t^{14}}{2 \times 810^2} \frac{1}{\sinh^3 t} - \frac{1}{810} t^6 := \frac{t^6 \times h_{31} (t)}{2 \times 810^2 \sinh^2 t},
\]

where

\[
h_{31} (t) = 810 t \sinh 2t - 810 \cosh 2t + 810 - t^8.
\]

Due to

\[
h_{31} (t) = 540t^4 + 144t^6 + \frac{101}{7} t^8 + 810 \sum_{n=5}^{\infty} \frac{(n-1) 2^{2n}}{(2n)!} t^{2n} > 0,
\]

we conclude that \( h_3 (t) > 0 \) for \( t > 0 \), which completes the proof.

\[
\square
\]

### 3.3 The comparison theorem for \( W_1 (x), W_{c1} (x), W_{01}^* (x), W_{01} (x) \) and \( W_{11} (x) \)

**Theorem 3.4.** (i) The function

\[
f_1 (x) = \ln \Gamma(x + 1) - \ln \sqrt{2\pi} - \left( x + \frac{1}{2} \right) \ln x + x - \frac{x}{2} \ln \left( x \sinh \frac{1}{x} + \frac{1}{810x^6} \right)
\]

is strictly increasing and concave on \([1, \infty)\).
(ii) For $x \geq 1$, we have

$$\beta_0 \left( x \sinh \frac{1}{x} + \frac{1}{810x^6} \right)^{x/2} < \frac{\Gamma (x + 1)}{\sqrt{2\pi x} (x/e)^x} < \left( x \sinh \frac{1}{x} + \frac{1}{810x^6} \right)^{x/2}$$

$$< \left( x \sinh \frac{1}{x} \right)^{x/2} \left( 1 + \frac{1}{1620x^5} \right)$$

$$< \left( x \sinh \frac{1}{x} \right)^{x/2} \exp \left( \frac{1}{1620x^5} \right) < \left( \sinh \frac{1}{x} \right)^{x/2} \left( x \sinh \left( \frac{1}{x} + \frac{1}{810x^7} \right) \right)^{x/2}$$

with the best constant

$$\beta_0 = \frac{e}{\sqrt{2\pi \sinh 1 + \pi/405}} \approx 0.99981.$$

Proof. (i) Differentiation yields

$$f_1' (x) = \psi (x + 1) - \ln x - \frac{1}{2x} - \frac{1}{2} \ln \left( x \sinh \frac{1}{x} + \frac{1}{810x^6} \right) + 3 \frac{135x^6 \cosh \frac{1}{x} - 135x^7 \sinh \frac{1}{x} + 1}{810x^7 \sinh \frac{1}{x} + 1},$$

$$f_1'' (x) = \psi' (x + 1) - \frac{1}{x} + \frac{1}{2x^2}$$

$$- \frac{3 \left( 109350x^{14} \sinh^2 \frac{1}{x} + 5940x^7 \sinh \frac{1}{x} + 135x^5 \sinh \frac{1}{x} - 1890x^6 \cosh \frac{1}{x} - 109350x^{12} - 1 \right)}{x \left( 810x^7 \sinh \frac{1}{x} + 1 \right)^2}.$$ 

Replacing $x$ by $x + 1/2$ in inequality (23) yields

$$\psi' (x + 1) < \frac{1}{30} \frac{13780x^4 + 7560x^3 + 12705x^2 + 8925x + 3019}{(2x + 1) (63x^3 + 126x^2 + 217x^2 + 154x + 60)}$$

for $x > -1/2$, and applying the above inequality combined with the change of variable $x = 1/t \in (0, 1]$ yield

$$f_1'' (x) < \frac{1}{30} \frac{t (3019t^4 + 8925t^3 + 12705t^2 + 7560t + 3780)}{(t + 2) (60t^4 + 154t^3 + 217t^2 + 126t + 63)} - t + \frac{1}{2} t^2$$

$$- \frac{3t \left( 109350 \sinh^2 t - 1890t^8 \cosh t + 5940t^7 \sinh t + 135t^5 \sinh t - 109350t^2 - t^{14} \right)}{(810 \sinh t + t^2)^2}$$

$$\frac{810^2 t \times f_1(t)}{(t + 2) (126t + 217t^2 + 154t^3 + 60t^4 + 63)} (810 \sinh t + t^2)^2,$$

where

$$f_{11} (t) = p_6 (t) \sinh^2 t + p_{13} (t) \cosh t - p_{14} (t) \sinh t + p_{20} (t),$$

$$p_6 (t) = 30^6 + 47t^4 - \frac{718}{15} t^4 - 210^3 - 259t^2 - \frac{315}{2} t - 63,$$

$$p_{13} (t) = \frac{7}{810} t^6 (t + 2) \left( 60t^4 + 154t^3 + 217t^2 + 126t + 63 \right),$$

$$p_{14} (t) = t^7 \left( \frac{1}{27} t^2 + \frac{77}{810} t^6 + \frac{2857}{1620} t^8 + \frac{45973}{6075} t^{14} + \frac{1547}{108} t^3 + \frac{12341}{810} t^2 + \frac{77}{9} t + \frac{154}{45} \right),$$

$$p_{20} (t) = \frac{1}{21870} t^{20} + \frac{257}{656100} t^{19} + \frac{13667}{9841500} t^{18} + \frac{217}{87480} t^{17} + \frac{7}{2700} t^{16} + \frac{7}{4860} t^{15} + \frac{7}{12150} t^{14} + 30t^7 + 137t^6 + \frac{525}{2} t^5 + 280t^4 + \frac{315}{2} t^3 + 63t^2.$$
To prove \( f_{11} ( t ) < 0 \) for \( t \in ( 0, 1 ] \), we use formula \( \sinh^2 t = \cosh^2 t - 1 \) to write \( f_{11} ( t ) \) as
\[
 f_{11} ( t ) = [ p_6 ( t ) \cosh t + p_{13} ( t ) ] \cosh t - p_{14} ( t ) \sinh t + p_{20} ( t ) - p_6 ( t ).
\]
Since the coefficients of polynomial \( -p_6 ( t ) \) satisfy those conditions of Lemma 3.2, and \(-p_6 ( 1 ) = 19811/30 > 0\), we see that \(-p_6 ( t ) > 0\) for \( t \in ( 0, 1 ] \). It then follows from that
\[
p_6 ( t ) \cosh t + p_{13} ( t ) < p_6 ( t ) + p_{13} ( t )
\]
\[
= \frac{14}{27} t^{13} + 959/405 t^{12} + 245/54 t^{11} + 392/81 t^{10} + 49/18 t^9 + 49/45 t^8 + 30 t^6
\]
\[+ 47 t^5 - 718/15 t^4 - 210 t^3 - 259 t^2 - 315/2 t - 63 := p_{13} ( t ).
\]
Application of Lemma 3.2 again with \(-p_{13} ( 1 ) = 173959/270 > 0\) yields \(-p_{13} ( t ) > 0\) for \( t \in ( 0, 1 ] \), and so
\[
p_6 ( t ) \cosh t + p_{13} ( t ) < 0 \text{ for } t \in ( 0, 1 ] .
\]
Since \( p_{16} ( t ) > 0 \) for \( t > 0 \), using the inequalities
\[
\cosh t > \frac{2n}{(2n-1)!} \frac{t^{2n}}{2n} + 1 \frac{t^{2n}}{720} + \frac{t^{2n}}{12} t^2 + 1,
\]
\[
\sinh t > \frac{2n-1}{(2n-1)!} \frac{t^{2n-1}}{2n-1} + 1 \frac{t^{2n-1}}{5040} + \frac{t^{2n-1}}{6} t^2 + t,
\]
we have
\[
f_{11} ( t ) = [ p_6 ( t ) \cosh t + p_{13} ( t ) ] \cosh t - p_{14} ( t ) \sinh t + p_{20} ( t ) - p_6 ( t )
\]
\[
< \left[ p_6 ( t ) \sum_{n=0}^{\frac{t^n}{(2n)!}} + p_{13} ( t ) \sum_{n=1}^{\frac{t^n}{(2n)!}} \right] - p_{14} ( t ) \sum_{n=1}^{\frac{t^n}{(2n-1)!}} + p_{20} ( t ) - p_6 ( t )
\]
\[
= \frac{1}{54} 190080 t^{22} + \frac{22}{1625} 7024000 t^{21} + \frac{21}{10973} 912000 t^{20} + \frac{19}{9405} 849600 t^{18}
\]
\[+ \frac{18}{739} 363013 t^{17} + \frac{17}{1287} 226500 t^{16} + \frac{16}{6287} 199900 t^{15} + \frac{15}{4733} 64800 t^{14}
\]
\[- \frac{14}{3732} 480 t^{13} - \frac{13}{193} 536 t^{12} - \frac{12}{87} 912 t^{11} - \frac{11}{34} 560 t^{10} =: p_{12} ( t ).
\]
From Lemma 3.2 and \(-p_{12} ( 1 ) = 67766507 802 179/3950456 832 000 > 0\) it follows that \(-p_{12} ( t ) > 0\) for \( t \in ( 0, 1 ] \), and so \( f_{11} ( t ) < 0 \) for \( t \in ( 0, 1 ] \), which implies \( f_1'' ( x ) < 0 \) for \( x \geq 1 \).

(ii) Using the increasing property of \( f_1 \) and noting that
\[
f_1 ( 1 ) = \ln \frac{e}{\sqrt{2 \pi} \sinh 1 + \pi/405} \quad \text{and} \quad \lim_{x \to \infty} f_1 ( x ) = 0,
\]
we have
\[
\ln \frac{e}{\sqrt{2 \pi} \sinh 1 + \pi/405} < \ln \frac{\Gamma ( x + 1 )}{\Gamma ( x )} = \ln \left( x \sinh \frac{1}{x} + \frac{1}{810 x^2} \right)^{x/2} < 0,
\]
which imply the first and second inequalities of (25). The other ones of (25) follow from Lemma 3.3, which completes the proof.

\[\square\]

4 Conclusions

In this paper, by a little known power series expansion of \( \ln \left( t^{-1} \sinh t \right) \), that is, (12), we establish an asymptotic expansion (10) for the gamma function related to Windschitl’s formula, in which its coefficients have a closed-form expression (14). Moreover, we give an estimate of the remainder in the asymptotic expansion
(10) by means of inequalities (3) and (13). Due to (12), we also give other two asymptotic expansions (11) and (18), but their coefficients formulas are of recursive form. Despite all that, the latter improves Chen’s result [34, Theorem 2]

Furthermore, we compare the accuracy of all five approximation formulas for the gamma function generated by truncating five asymptotic series (10), (11), (9), (18) and (7) by numeric computations and some inequalities. These show that the approximation formula (6) is the best. Some general properties of truncated series (truncated polynomials) can refer to [46].

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