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**MN-convergence and lim-inf\(_M\)-convergence in partially ordered sets**

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**Abstract:** In this paper, we first introduce the notion of **MN**-convergence in posets as an unified form of **O**-convergence and **O**\(_2\)-convergence. Then, by studying the fundamental properties of **MN**-topology which is determined by **MN**-convergence according to the standard topological approach, an equivalent characterization to the **MN**-convergence being topological is established. Finally, the lim-inf\(_M\)-convergence in posets is further investigated, and a sufficient and necessary condition for lim-inf\(_M\)-convergence to be topological is obtained.

**Keywords:** **MN**-convergence, **MN**-topology, lim-inf\(_M\)-convergence, **M**-topology

**MSC:** 54A20, 06A06

1 Introduction, Notations and Preliminaries

The concept of **O**-convergence in partially ordered sets (posets, for short) was introduced by Birkhoff [1], Frink [2] and Mcshane [3]. It is defined as follows: a net \((x_i)_{i \in I}\) in a poset \(P\) is said to **O**-converge to \(x \in P\) if there exist subsets \(D\) and \(F\) of \(P\) such that

1. \(D\) is directed and \(F\) is filtered;
2. \(\sup D = x = \inf F\);
3. for every \(d \in D\) and \(e \in F\), \(d \leq x_i \leq e\) holds eventually, i.e., there exists \(i_0 \in I\) such that \(d \leq x_i \leq e\) for all \(i \geq i_0\).

As what has been showed in [4], the **O**-convergence (Note: in [4], the **O**-convergence is called order-convergence) in a general poset \(P\) may not be topological, i.e., it is possible that \(P\) can not be endowed with a topology such that the **O**-convergence and the associated topological convergence are consistent. Hence, much work has been done to characterize those special posets in which the **O**-convergence is topological. The most recent result in [5] shows that the **O**-convergence in a poset which satisfies Condition \((\triangle)\) is topological if and only if the poset is **O**-doubly continuous. This means that for a special class of posets, a sufficient and necessary condition for **O**-convergence being topological is obtained.

As a direct generalization of **O**-convergence, **O**\(_2\)-convergence in posets has been discussed in [11] from the order-theoretical point of view. It is defined as follows: a net \((x_i)_{i \in I}\) in a poset \(P\) is said to **O**\(_2\)-converge to \(x \in P\) if there exist subsets \(A\) and \(B\) of \(P\) such that

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(1) \( \sup A = x = \inf B; \)
(2) for every \( a \in A \) and \( b \in B \), \( a \leq x \leq b \) holds eventually.

In fact, the \( O_2 \)-convergence is also not topological generally. To clarify those special posets in which the \( O_2 \)-convergence is topological, Zhao and Li [6] showed that for any poset \( P \) satisfying Condition \((*)\), \( O_2 \)-convergence is topological if and only if \( P \) is \( a \)-doubly continuous. As a further result, Li and Zou [7] proved that the \( O_2 \)-convergence in a poset \( P \) is topological if and only if \( P \) is \( O_2 \)-doubly continuous. This result demonstrates the equivalence between the \( O_2 \)-convergence being topological and the \( O_2 \)-double continuity of a given poset.

On the other hand, Zhou and Zhao [8] have defined the lim-inf\(_M\)-convergence in posets to generalize lim-inf-convergence and lim-inf\(_2\)-convergence [4]. They also found that the lim-inf\(_M\)-convergence in a poset is topological if and only if the poset is \( \alpha(\mathcal{M}) \)-continuous when some additional conditions are satisfied (see [8], Theorem 3.1). This result clarified some special conditions of posets under which the lim-inf\(_M\)-convergence is topological. However, to the best of our knowledge, the equivalent characterization to the lim-inf\(_M\)-convergence in general posets being topological is still unknown.

One goal of this paper is to propose the notion of\( \mathcal{M}\mathcal{N}\)-convergence in posets which can unify \( \mathcal{O}\)-convergence and \( O_2 \)-convergence and search the equivalent characterization to the \( \mathcal{M}\mathcal{N}\)-convergence being topological. More precisely,

\begin{enumerate}
\item[(G11)] Given a general poset \( P \), we hope to clarify the order-theoretical condition of \( P \) which is sufficient and necessary for the \( \mathcal{M}\mathcal{N}\)-convergence being topological.
\item[(G12)] Given a poset \( P \) satisfying such condition, we hope to provide a topology that can be equipped on \( P \) such that the \( \mathcal{M}\mathcal{N}\)-convergence and the associated topological convergence agree.
\end{enumerate}

Another goal is to look for the equivalent characterization to the lim-inf\(_M\)-convergence being topological. More precisely,

\begin{enumerate}
\item[(G21)] Given a general poset \( P \), we expect to present a sufficient and necessary condition of \( P \) which can precisely serve as an order-theoretical condition for the lim-inf\(_M\)-convergence being topological.
\item[(G22)] Given a poset \( P \) satisfying such condition, we expect to give a topology on \( P \) such that the lim-inf\(_M\)-convergence and the associated topological convergence are consistent.
\end{enumerate}

To accomplish those goals, motivated by the ideal of introducing the \( \mathcal{Z}\)-subsets system [9] for defining \( \mathcal{Z}\)-continuous posets, we propose the notion of \( \mathcal{M}\mathcal{N}\)-doubly continuous posets and define the \( \mathcal{M}\mathcal{N}\)-topology on posets in Section 2. Based on the study of the basic properties of the \( \mathcal{M}\mathcal{N}\)-topology, it is proved that the \( \mathcal{M}\mathcal{N}\)-convergence in a poset \( P \) is topological if and only if \( P \) is an \( \mathcal{M}\mathcal{N}\)-doubly continuous poset if and only if the \( \mathcal{M}\mathcal{N}\)-convergence and the topological convergence with respect to \( \mathcal{M}\mathcal{N}\)-topology are consistent. In Section 3, by introducing the notion of \( \alpha'(\mathcal{M}) \)-continuous posets and presenting the fundamental properties of \( \mathcal{M}\)-topology which is induced by the lim-inf\(_M\)-convergence, we show that the lim-inf\(_M\)-convergence in a poset \( P \) is topological if and only if \( P \) is an \( \alpha'(\mathcal{M}) \)-continuous poset if and only if the lim-inf\(_M\)-convergence and the topological convergence with respect to \( \mathcal{M}\)-topology are consistent.

Some conventional notations will be used in the paper. Given a set \( X \), \( F \subseteq X \) means that \( F \) is a finite subset of \( X \). Given a topological space \((X, \mathcal{T})\) and a net \((x_i)_{i \in I} \) in \( X \), we take \( (x_i)_{i \in I} \rightarrow x \) to mean the net \((x_i)_{i \in I} \) converges to \( x \in P \) with respect to the topology \( \mathcal{T} \).

Let \( P \) be a poset and \( x \in P \). \( \uparrow x \) and \( \downarrow x \) are always used to denote the principal filter \( \{y \in P : y \geq x\} \) and the principal ideal \( \{z \in P : z \leq x\} \) of \( P \), respectively. Given a poset \( P \) and \( A \subseteq P \), by writing \( \sup A \) we mean that the least upper bound of \( A \) in \( P \) exists and equals to \( \sup A \in P \); dually, by writing \( \inf A \) we mean that the greatest lower bound of \( A \) in \( P \) exists and equals to \( \inf A \in P \). And the set \( A \) is called an upper set if \( A = \uparrow A = \{b \in P : (\exists a \in A) a \leq b\} \), the lower set is defined dually.

For a poset \( P \), we succinctly denote

\begin{align*}
\powerset(P) & = \{ A : A \subseteq P \}; \powerset_0(P) = \powerset(P) / \{\emptyset\}; \\
\downset(P) & = \{ D \in \powerset(P) : D \text{ is a directed subset of } P \};
\end{align*}
- $\mathcal{T}(P) = \{ F \in \mathcal{P}(P) : F$ is a filtered subset of $P\};$
- $\mathcal{L}(P) = \{ L \in \mathcal{P}(P) : L \subseteq P \}; \mathcal{L}_0(P) = \mathcal{L}(P) / \{ \emptyset \};$
- $\mathcal{S}_0(P) = \{ \{ x \} : x \in P \}.$

To make this paper self-contained, we briefly review the following notions:

**Definition 1.1** ([5]). Let $P$ be a poset and $x, y, z \in P.$ We say $y \leq \hat{o} x$ if for every net $(x_i)_{i \in I}$ in $P$ which $O$-converges to $x \in P,$ $x_i \geq y$ holds eventually; dually, we say $z \hat{o} x$ if for every net $(x_i)_{i \in I}$ in $P$ which $O$-converges to $x \in P,$ $x_i \leq z$ holds eventually.

**Definition 1.2** ([5]). A poset $P$ is said to be $O$-doubly continuous if for every $x \in P,$ the set $\{ a \in P : a \leq \hat{o} x \}$ is directed, the set $\{ b \in P : b \geq \hat{o} x \}$ is filtered and sup$\{ a \in P : a \leq \hat{o} x \} = x = \inf \{ b \in P : b \geq \hat{o} x \}.$

**Condition ($\triangle$).** A poset $P$ is said to satisfy Condition($\triangle$) if
1. for any $x, y, z \in P,$ $x \leq \hat{o} y \leq z$ implies $x \leq \hat{o} z;$
2. for any $w, s, t \in P,$ $w \geq \hat{o} s \geq t$ implies $w \geq \hat{o} t.$

**Definition 1.3** ([6]). Let $P$ be a poset and $x, y, z \in P.$ We say $y \leq a x$ if for every net $(x_i)_{i \in I}$ in $P$ which $O$-converges to $x \in P,$ $x_i \geq y$ holds eventually; dually, we say $z \geq a x$ if for every net $(x_i)_{i \in I}$ in $P$ which $O$-converges to $x \in P,$ $x_i \leq z$ holds eventually.

**Definition 1.4** ([7]). A poset $P$ is said to be $O_2$-doubly continuous if for every $x \in P,$
1. sup$\{ a \in P : a \leq a x \} = x = \inf \{ b \in P : b \geq a x \};$
2. for any $y, z \in P$ with $y \leq a x$ and $z \geq a x,$ there exist $A \subseteq \{ a \in P : a \leq a x \}$ and $B \subseteq \{ b \in P : b \geq a x \}$ such that $y \leq a c$ and $z \geq a c$ for each $c \in \bigcap \{ a \cap b : a \in A \& b \in B \}.$

## 2 MN-topology on posets

Based on the introduction of $\mathcal{M}\mathcal{N}$-convergence in posets, the $\mathcal{M}\mathcal{N}$-topology can be defined on posets. In this section, we first define the $\mathcal{M}\mathcal{N}$-double continuity for posets. Then, we show the equivalence between the $\mathcal{M}\mathcal{N}$-convergence being topological and the $\mathcal{M}\mathcal{N}$-double continuity of a given poset.

A PMN-space is a triplet $(P, \mathcal{M}, \mathcal{N})$ which consists of a poset $P$ and two subfamily $\mathcal{M}, \mathcal{N} \subseteq \mathcal{P}(P).$

All PMN-spaces $(P, \mathcal{M}, \mathcal{N})$ considered in this section are assumed to satisfy the following conditions:

(C1) If $P$ has the least element $\|,$ then $\{ \| \} \in \mathcal{M};$
(C2) If $P$ has the greatest element $\top,$ then $\{ \top \} \in \mathcal{N};$
(C3) $\emptyset \notin \mathcal{M}$ and $\emptyset \notin \mathcal{N}.$

**Definition 2.1.** Let $(P, \mathcal{M}, \mathcal{N})$ be a PMN-space. A net $(x_i)_{i \in I}$ in $P$ is said to $\mathcal{M}\mathcal{N}$-converge to $x \in P$ if there exist $M \in \mathcal{M}$ and $N \in \mathcal{N}$ satisfying:

$\mathcal{M}\mathcal{N}$sup$M = x = \inf N;$

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In this case, we will write $(x_i)_{i \in I} \stackrel{\mathcal{M}\mathcal{N}}{\rightarrow} x.$

**Remark 2.2.** Let $(P, \mathcal{M}, \mathcal{N})$ be a PMN-space.

1. If $\mathcal{M} = \mathcal{D}(P)$ and $\mathcal{N} = \mathcal{F}(P),$ then a net $(x_i)_{i \in I} \stackrel{\mathcal{M}\mathcal{N}}{\rightarrow} x \in P$ if and only if it $O$-converges to $x.$ That is to say, $O$-convergence is a particular case of $\mathcal{M}\mathcal{N}$-convergence.

2. If $\mathcal{M} = \mathcal{N} = \mathcal{F}(P),$ then a net $(x_i)_{i \in I} \stackrel{\mathcal{M}\mathcal{N}}{\rightarrow} x \in P$ if and only if it $O_2$-converges to $x.$ That is to say, $O_2$-convergence is a special case of $\mathcal{M}\mathcal{N}$-convergence.
If $M = N = \mathcal{L}_0(P)$, then a net $(x_i)_{i \in I} \to x \in P$ if and only if $x_1 = x$ holds eventually.

(4) The $\mathcal{MN}$-convergent point of a net $(x_i)_{i \in I}$ in $P$, if it exists, is unique.

Indeed, suppose that $(x_i)_{i \in I} \to x_1$ and $(x_i)_{i \in I} \to x_2$. Then there exist $A_k \in M$ and $B_k \in N$ such that

$$\sup A_k = x_k = \inf B_k$$

and $a_k \leq x_i \leq b_k$ holds eventually for every $a_k \in A_k$ and $b_k \in B_k$ ($k = 1, 2$). This implies that for any $a_1 \in A_1$, $a_2 \in A_2$, $b_1 \in B_1$ and $b_2 \in B_2$, there exists $i_0 \in I$ such that $a_1 \leq x_{i_0} \leq b_2$ and $a_2 \leq x_{i_0} \leq b_1$. Thus we have $x_1 = x_{i_0} \leq \inf B_2 = x_2$ and $x_2 \leq x_{i_0} = \inf B_1$. Therefore $x_1 = x_2$.

(5) For any $A \in M$ and $B \in N$ with $\sup A = \inf B = x \in P$, we denote $F^\mathcal{MN}_{(A,B)} = \{\bigcap{\{\sup a \cap \inf b : a \in A_0 \& b \in B_0\}} : A_0 \subseteq A \& B_0 \subseteq B\}$. Let $D^\mathcal{MN}_{(A,B)} = \{(d, D) \in \mathcal{P} \times \sup P_{(A,B)} : d \in D\}$ and let the preorder $\preceq$ on $D^\mathcal{MN}_{(A,B)}$ be defined by

$$(\forall(d_1, D_1), (d_2, D_2) \in D^\mathcal{MN}_{(A,B)})(d_1, D_1) \preceq (d_2, D_2) \iff D_2 \subseteq D_1.$$ 

One can readily check that $(D^\mathcal{MN}_{(A,B)}, \preceq)$ is directed. Now if we take $x_{(d,D), (d', D')} \to x$ for every $(d, D) \in D^\mathcal{MN}_{(A,B)}$, then the net $(x_{(d,D), (d', D')})_{d,D} \to x$ because $\sup A = \inf B = x$, and $a \leq x_{(d,D), (d', D')}$ holds eventually for any $a \in A$ and $b \in B$.

(6) Let $(x_{(d,D), (d', D')})_{d,D} \in D^\mathcal{MN}_{(A,B)}$ be the net defined in (5) for any $A \in M$ and $B \in N$ with $\sup A = \inf B = x \in P$. If

$$x_{(d,D), (d', D')} \to x_{(d', D')}$$

converges to $x \in P$ with respect to some topology $\tau$ on the poset $P$, then for every open neighborhood $U_p$ of $p$, there exist $A_0 \subseteq A$ and $B_0 \subseteq B$ such that

$$\bigcap{\{\sup a \cap \inf b : a \in A_0 \& b \in B_0\}} \subseteq U_p.$$ 

Indeed, suppose that $(x_{(d,D), (d', D')})_{d,D} \to p$. Then for every open neighborhood $U_p$ of $p$, there exists $(d_0, D_0) \in D^\mathcal{MN}_{(A,B)}$ such that $x_{(d,D), (d', D')} = d \in U_p$ for all $(d, D) \supseteq (d_0, D_0)$. Since $(d, D) \supseteq (d_0, D_0)$ for every $d \in D_0$, $x_{(d,D), (d', D')} = d \in U_p$ for every $d \in D_0$. This shows $D_0 \subseteq U_p$. So, there exist $A_0 \subseteq A$ and $B_0 \subseteq B$ such that

$$D_0 = \bigcap{\{\sup a \cap \inf b : a \in A_0 \& b \in B_0\}} \subseteq U_p.$$ 

Given a PMN-space $(P, M, N)$, we can define two new approximate relations $\ll^M$ and $\gg^M$ on the poset $P$ in the following definition.

**Definition 2.3.** Let $(P, M, N)$ be a PMN-space and $x, y, z \in P$.

(1) We define $y \ll^M x$ if for any $A \in M$ and $B \in N$ with $\sup A = x = \inf B$, there exist $A_0 \subseteq A$ and $B_0 \subseteq B$ such that

$$\bigcap{\{\sup a \cap \inf b : a \in A_0 \& b \in B_0\}} \subseteq U_y.$$ 

(2) Dually, we define $y \gg^M x$ if for any $M \in M$ and $N \in N$ with $\sup M = x = \inf N$, there exist $M_0 \subseteq M$ and $N_0 \subseteq N$ such that

$$\bigcap{\{\sup m \cap \inf n : m \in M_0 \& n \in N_0\}} \subseteq U_y.$$ 

For convenience, given a PMN-space $(P, M, N)$ and $x \in P$, we will briefly denote

- $\ll^M x = \{y \in P : y \ll^M x\};$
- $\gg^M x = \{z \in P : x \gg^M z\};$
- $\ll^N x = \{a \in P : x \ll^N a\};$
- $\gg^N x = \{b \in P : b \gg^N x\}.$

**Remark 2.4.** Let $(P, M, N)$ be a PMN-space and $x, y, z \in P$.

(1) If there is no $A \in M$ such that $\sup A = x$, then $p \ll^M x$ and $p \gg^M x$ for all $p \in P$; similarly, if there is no $B \in N$ such that $\inf B = x$, then $p \ll^N x$ and $p \gg^N x$ for all $p \in P$.

(2) By Definition 2.3, one can easily check that if $P$ has the least element $\bot$, then $\bot \ll^M p$ for every $p \in P$, and if $P$ has the greatest element $\top$, then $\top \gg^N p$ for every $p \in P$.

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1 From the logical point of view, we stipulate $\bigcap{\{\sup a \cap \inf b : a \in A_0 \& b \in B_0\}} = P$ if $A_0 = \emptyset$ or $B_0 = \emptyset$. 

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(3) The implications $y \llp_M x \Rightarrow x \leq y$ and $z \ggp_M x \Rightarrow z \preceq x$ are not true necessarily. See the following example: let $\mathbb{R}$ be the set of all real numbers, in its ordinal order, and $\mathbb{M} = \mathbb{N} = \{n : n \in \mathbb{Z}\}$, where $\mathbb{Z}$ is the set of all integers. Then, by (1), we have $1 \llp_M 1/2$ and $0 \ggp_M 1/2$. But $1 \leq 1/2$ and $0 \geq 1/2$.

(4) Assume that $\sup A_0 = x = \inf B_0$ for some $A_0 \in \mathbb{M}$ and $B_0 \in \mathbb{N}$. Then it follows from Definition 2.3 that $y \llp_M x$ implies $y \leq x$ and $z \ggp_M x$ implies $z \preceq x$. In particular, if $\mathcal{S}_0(P) \subseteq \mathbb{M} \cap \mathbb{N}$, then $b \llp_M a$ and $c \ggp_M a$ implies $b \leq a$ and $c \preceq a$ for any $a, b, c \in P$. More particularly, for any $p_1, p_2, p_3 \in P$, we have $p_1 \llp_{\mathcal{S}_0} p_2 \iff p_1 \leq p_2$ and $p_3 \ggp_{\mathcal{S}_0} p_2 \iff p_3 \preceq p_2$.

**Proposition 2.25.** Let $(P, \mathbb{M}, \mathbb{N})$ be a PMN-space and $x, y, z \in P$. Then

(1) $y \llp_M x$ if and only if for every net $(x_i)_{i \in I}$ that MN-converges to $x$, $x_i \geq y$ holds eventually.

(2) $z \ggp_M x$ if and only if for every net $(x_i)_{i \in I}$ that MN-converges to $x$, $x_i \leq z$ holds eventually.

**Proof.** (1) Suppose $y \llp_M x$. If a net $(x_i)_{i \in I} \rightarrow x$, then there exist $A \in \mathbb{M}$ and $B \in \mathbb{N}$ such that $\sup A = x = \inf B$, and for any $a \in A$ and $b \in B$, there exists $i_0 \in I$ such that $a \leq x_i \leq b$ for all $i \geq i_0$. According to Definition 2.3 (1), it follows that there exist $A_0 = \{a_1, a_2, \ldots, a_n\} \subseteq A$ and $B_0 = \{b_1, b_2, \ldots, b_m\} \subseteq B$, such that $x \in \bigcap \{a_k \cap b_j : 1 \leq k \leq n \text{ and } 1 \leq j \leq m\} \subseteq \uparrow y$. Take $i_0 \in I$ with $i_0 \geq i_0$, and every $j \in \{1, 2, \ldots, n\}$, such that $x_i \in \bigcap \{a_k \cap b_j : 1 \leq k \leq n \text{ and } 1 \leq j \leq m\} \subseteq \downarrow y$ for all $i \geq i_0$. This means $x_i \geq y$ holds eventually.

Conversely, suppose that for every net $(x_i)_{i \in I}$ that MN-converges to $x$, $x_i \geq y$ holds eventually. For every $A \in \mathbb{M}$ and $B \in \mathbb{N}$ with $\sup A = x = \inf B$, consider the net $(x_{(d,b)})_{(d,b) \in \mathcal{D}(A,B)}$ defined in Remark 2.2 (5). By Remark 2.2 (5), the net $(x_{(d,b)})_{(d,b) \in \mathcal{D}(A,B)} \rightarrow x$. So, there exists $(d_0, D_0) \in \mathcal{D}(A,B)$ such that $x_{(d_0, D_0)} = d \geq y$ for all $(d, D) \geq (d_0, D_0)$. Since $(d, D) \geq (d_0, D_0)$ for all $d \in D_0$, $x_{(d, D_0)} = d \geq y$ for all $d \in D_0$. Thus, we have $D_0 \subseteq \downarrow y$. It follows from the definition of $\mathcal{D}(A,B)$ that there exist $A_0 \subseteq A$ and $B_0 \subseteq B$ such that $D_0 = \bigcap \{a \cap b : a \in A_0 \text{ and } b \in B_0\} \subseteq \downarrow y$. This shows $y \llp_M x$.

The proof of (2) can be processed similarly.

**Remark 2.6.** Let $(P, \mathbb{M}, \mathbb{N})$ be a PMN-space.

(1) If $\mathbb{M} = \mathcal{D}(P)$ and $\mathbb{N} = \mathcal{F}(P)$, then $x \llp_{\mathcal{D}} = x \llp_\mathcal{F}$ and $x \ggp_{\mathcal{D}} = x \ggp_\mathcal{F}$.

(2) If $\mathbb{M} = \mathcal{P}_0(P)$, then $x \llp_{\mathcal{P}_0} = x \llp_\mathcal{P}$ and $x \ggp_{\mathcal{P}_0} = x \ggp_\mathcal{P}$.

Given a PMN-space $(P, \mathbb{M}, \mathbb{N})$, depending on the approximate relations $\llp_{\mathbb{M}}$ and $\ggp_{\mathbb{M}}$ on $P$, we can define the MN-double continuity for the poset $P$.

**Definition 2.7.** Let $(P, \mathbb{M}, \mathbb{N})$ be a PMN-space. The poset $P$ is called an MN-doubly continuous poset if for every $x \in P$, there exist $M_x \in \mathbb{M}$ and $N_x \in \mathbb{N}$ such that

(A1) $M_x \subseteq \bigwedge_{\mathbb{M}} x$ and $N_x \subseteq \bigwedge_{\mathbb{N}} x$ and $\sup M_x = x = \inf N_x$.

(A2) For any $y \in \bigwedge_{\mathbb{M}} x$ and $z \in \bigwedge_{\mathbb{N}} x$, $\bigcap \{m \cap n : m \in M_x \text{ and } n \in N_x\} \subseteq \bigwedge_{\mathbb{M}} y \cap \bigwedge_{\mathbb{N}} z$ for some $M_0 \subseteq M_x$ and $N_0 \subseteq N_x$.

By Remark 2.4 (4) and Definition 2.7, we have the following basic property about MN-doubly continuous posets:

**Proposition 2.8.** Let $(P, \mathbb{M}, \mathbb{N})$ be a PMN-space and $x, y, z \in P$. If the poset $P$ is an MN-doubly continuous poset, then $y \llp_{\mathbb{M}} x$ implies $y \leq x$ and $z \ggp_{\mathbb{M}} x$ implies $z \geq x$.

**Example 2.9.** Let $(P, \mathbb{M}, \mathbb{N})$ be a PMN-space.

(1) If $\mathbb{M} = \mathbb{N} = \mathcal{S}_0(P)$, then by Remark 2.4 (4), we have $y \llp_{\mathcal{S}_0} x \iff y \llp_{\mathcal{S}_0} x = y \ggp_{\mathcal{S}_0} x = z \ggp_{\mathcal{S}_0} y$. By Definition 2.7, one can easily check that $P$ is an $\mathcal{S}_0\mathcal{S}_0$-doubly continuous poset.

(2) If $\mathbb{M} = \mathbb{N} = \mathcal{L}_0(P)$, then by Definition 2.3, we have $y \llp_{\mathcal{L}_0} x \iff y \llp_{\mathcal{L}_0} x = \% \ggp_{\mathcal{L}_0} y = \%$. It can be easily checked from Definition 2.7 that $P$ is an $\mathcal{L}_0\mathcal{L}_0$-doubly continuous poset.
(3) Let $M = \mathcal{D}(P)$ and $N = \mathcal{I}(P)$. Then it is easy to check that if $P$ is an $\emptyset$-doubly continuous poset which satisfies Condition (5), then it is a $\emptyset$-doubly continuous poset. Particularly, finite posets, chains and anti-chains, completely distributive lattices are all $\emptyset$-doubly continuous posets.

(4) Let $M = N = \mathcal{P}_0(P)$. Then the poset $P$ is a $\mathcal{P}_0\mathcal{P}_0$-double continuous if and only if it is $O_2$-double continuous. Thus, chains and finite posets are all $\mathcal{P}_0\mathcal{P}_0$-double continuous posets.

Next, we are going to consider the $M\cap N$-topology on posets, which is induced by the $M\cap N$-convergence.

**Definition 2.10.** Given a PMN-space $(P, M, N)$, a subset $U$ of $P$ is called an $M\cap N$-open set if for every net $(x_i)_{i \in I}$ with that $(x_i)_{i \in I} \rightarrow x \in U$, $x_i \in U$ holds eventually. Clearly, the family $\mathcal{O}_M^N(P)$ consisting of all $M\cap N$-open subsets of $P$ forms a topology on $P$. And this topology is called the $M\cap N$-topology.

**Theorem 2.11.** Let $(P, M, N)$ be a PMN-space. Then a subset $U$ of $P$ is an $M\cap N$-open set if and only if for every $M \subseteq M$ and $N \subseteq N$ with supremum $M = x = \inf N \in U$, we have

$$\bigcap \{\uparrow m \cap \downarrow n : m \in M_0 \cap N_0 \subseteq U \}$$

for some $M_0 \subseteq M$ and $N_0 \subseteq N$.

**Proof.** Suppose that $U$ is a $M\cap N$-open subset of $P$. For every $M \subseteq M$ and $N \subseteq N$ with supremum $M = x = \inf N \in U$, let $(X_{(d,D),d,D})_{d,D} \subseteq D_{(d,D)}$ be the net defined in Remark 2.2 (5). Then the net $(X_{(d,D),d,D} \subseteq D_{(d,D))} \rightarrow x$. By the definition of $M\cap N$-open set, the $(d_0, D_0) \subseteq D_{(d,D)}$ such that $x_{(d,D)} = d \in U$ for all $(d, D) \subseteq (d_0, D_0)$. Since $(d, D_0) \subseteq (d_0, D_0)$ for all $d \in D_0$, $(X_{(d,D),d,D}) \subseteq D_{(d,D)} \subseteq U$, and thus $D_0 \subseteq U$. It follows from the definition of the directed set $D_{(d,D)}$ that $D_0 = \bigcap \{\uparrow m \cap \downarrow n : m \in M_0 \cap N_0 \subseteq U \}$ for some $M_0 \subseteq M$ and some $N_0 \subseteq N$.

Conversely, assume that $U$ is a subset of $P$ with the property that for any $M \subseteq M$ and $N \subseteq N$ with supremum $M = x = \inf N \in U$, there exist $M_0 = \{m_1, m_2, \ldots, m_k\} \subseteq M$ and $N_0 = \{n_1, n_2, \ldots, n_l\} \subseteq N$ such that $\bigcap \{\uparrow m_i \cap \downarrow n_j : 1 \leq h \leq k \cap 1 \leq j \leq l \} \subseteq U$. Then there exists $M \subseteq M$ and $N \subseteq N$ such that $\inf N \in U$, and for every $m \in M$ and $n \in N$, $m \leq x \subseteq n$ holds eventually. This means that for every $m_i \in M_0$ and $n_j \in N_0$, there exists $i_{h,j,k} \subseteq I$ such that $m_i \leq x_i \subseteq n_i$ for all $i \in i_{h,j,k}$. Take $i_0 \subseteq I$ such that $i_0 \leq i_{h,j,k}$ for all $h \in \{1, 2, \ldots, k\}$ and $j \in \{1, 2, \ldots, l\}$. Then $x_i \subseteq \bigcap \{\uparrow m_i \cap \downarrow n_j : 1 \leq h \leq k \cap 1 \leq j \leq l \} \subseteq U$ for all $i \in i_0$. Therefore, $U$ is an $M\cap N$-open subset of $P$.

**Proposition 2.12.** Let $(P, M, N)$ be a PMN-space in which $P$ is an $M\cap N$-doubly continuous poset, and $y, z \in P$. Then $\uparrow^M_N y \cap \downarrow^M_N z \subseteq \mathcal{O}_M^N(P)$.

**Proof.** Suppose that $M \subseteq M$ and $N \subseteq N$ with supremum $M = \inf N = x \subseteq \uparrow^M_N y \cap \downarrow^M_N z$. Since $P$ is an $M\cap N$-doubly continuous poset, there exist $M_0 \subseteq M$ and $N_0 \subseteq N$ satisfying condition (A1) and (A2) in Definition 2.7. This means that there exist $M_0 \subseteq M_0 \subseteq \uparrow^M_N x$ and $N_0 \subseteq N_0 \subseteq \downarrow^M_N x$ such that $\bigcap \{\uparrow m_0 \cap \downarrow n_0 : m_0 \in M_0 \cap N_0 \} \subseteq \uparrow^M_N y \cap \downarrow^M_N z$. As $M_0 \subseteq M_0 \subseteq \uparrow^M_N x$ and $N_0 \subseteq N_0 \subseteq \downarrow^M_N x$, by Definition 2.3, there exist $M_{n_0} \subseteq M \cap N_0$ and $N_{n_0} \subseteq N$ such that $\bigcap \{\uparrow m_0 \cap \downarrow n_0 : m_0 \in M_{n_0} \cap N_{n_0} \} \subseteq \uparrow m_0 \cap \downarrow n_0$ for every $m_0 \in M_0$ and $n_0 \in N_0$. Take $M_0 = \bigcup \{M_{n_0} : m_0 \in M_0 \cap N_0 \}$ and then it is easy to check that $M_0 \subseteq M$ and $N_0 \subseteq N$ and

$$x \subseteq \bigcap \{\uparrow m_0 \cap \downarrow n_0 : m_0 \in M_0 \cap N_0 \} \subseteq \uparrow^M_N y \cap \downarrow^M_N z.$$

So, it follows from Theorem 2.11 that $\uparrow^M_N y \cap \downarrow^M_N z \subseteq \mathcal{O}_M^N(P)$.

**Lemma 2.13.** Let $(P, M, N)$ be a PMN-space in which $P$ is an $M\cap N$-doubly continuous poset. Then a net

$$(x_i)_{i \in I} \rightarrow x \in P \iff (x_i)_{i \in I} \rightarrow x.$$
Proof. From the definition of $O^N_M(P)$, it is easy to see that a net 
\[ (x_i)_{i \in I} \rightarrow x \in P \Rightarrow (x_i)_{i \in I} \rightarrow x. \]

To prove the Lemma, it suffices to show that a net $(x_i)_{i \in I} \rightarrow x$ implies $(x_i)_{i \in I} \rightarrow x$. Suppose a net $(x_i)_{i \in I} \rightarrow x$. Since $P$ is an $MN$-doubly continuous poset, there exist $M_x \in M$ and $N_x \in N$ such that $M_x \subseteq \bigwedge^N_M x$ and $\sup M_x = x = \inf N_x$. By Proposition 2.12, $x \in \bigwedge^N_M y \cap \bigwedge^N_M z \in O^N_M(P)$ for every $y \in M_x \subseteq \bigwedge^N_M x$ and every $z \in N_x \subseteq \Delta^N_M x$, and hence $x_i \in \bigwedge^N_M y \cap \bigwedge^N_M z$ holds eventually for every $y \in M_x \subseteq \bigwedge^N_M x$ and every $z \in N_x \subseteq \Delta^N_M x$. It follows from Proposition 2.8 that $y \subseteq x_i \subseteq z$ holds eventually for every $y \in M_x$ and $z \in N_x$. Thus $(x_i)_{i \in I} \rightarrow x$.

Lemma 14.4. Let $(P, M, N)$ be a PMN-space. If the $MN$-convergence in $P$ is topological, then $P$ is $MN$-doubly continuous.

Proof. Suppose that the $MN$-convergence in $P$ is topological. Then there exists a topology $\tau$ on $P$ such that for every $x \in P$, a net $(x_i)_{i \in I} \rightarrow x$ if and only if $(x_i)_{i \in I} \rightarrow x$. Define $I_x = \{ (p, U) \in P \times N(x) : p \in U \}$, where $N(x)$ denotes the set of all open neighbourhoods of $x$ in the topological space $(P, \tau)$, i.e., $N(x) = \{ U \in \tau : x \in U \}$. Define the preorder $\preceq$ on $I_x$ as follows:

\[ ((p_1, U_1), (p_2, U_2)) \in I_x \text{ if and only if } U_2 \subseteq U_1. \]

Now one can easily see that $I_x$ is directed. Let $x_{(p, U)} = p$ for every $(p, U) \in I_x$. Then it is straightforward to check that the net $(x_{(p, U)})_{(p, U) \in I_x} \rightarrow x$, and thus $(x_{(p, U)})_{(p, U) \in I_x} \rightarrow x$. By Definition 2.1, there exist $M_x \in M$ and $N_x \in N$ such that $\sup M_x = x = \inf N_x$, and for every $m \in M_x$ and $n \in N_x$, there exists some $p \in U_m$ such that $x_{(p, U)} = p \in \uparrow m \cap \downarrow n$ for all $(p, U) \in I_x$. Since $p \in U_m$, $x_{(p, U)} = p \in \uparrow m \cap \downarrow n$ for every $p \in P$. This shows

\[ (\forall m \in M_x, n \in N_x)(\exists U_m \in N(x)) x \in U_m \subseteq \uparrow m \cap \downarrow n. \]

For any $A \subseteq M$ and $B \subseteq N$ with $\sup A = x = \inf B$, let $(x_{(d, D)})_{(d, D) \in D_{(A, B)}} \rightarrow x$, and hence $(x_{(d, D)})_{(d, D) \in D_{(A, B)}} \rightarrow x$. This implies, by Remark 2.2 (6), that there exist $A_0 \subseteq A$ and $B_0 \subseteq B$ satisfying

\[ x \in \bigcap \{ \uparrow a \cap \downarrow b : a \in A_0 \text{ and } b \in B_0 \} \subseteq U_m \subseteq \uparrow m \cap \downarrow n. \]

Therefore, $m \in \bigwedge^N_M x$ and $n \in \Delta^N_M x$, and hence $M_x \subseteq \bigwedge^N_M x$ and $N_x \subseteq \Delta^N_M x$.

Let $y \in \bigwedge^N_M x$ and $z \in \Delta^N_M x$. Since $\sup M_x = x = \inf N_x$, by Definition 2.3, $\bigcap \{ \uparrow m \cap \downarrow n : m \in M_1 \text{ and } n \in N_1 \} \subseteq \uparrow y \cap \downarrow z$ for some $M_1 \subseteq M_x$ and $N_1 \subseteq N_x$. This concludes by Condition (*) and the finiteness of sets $M_1$ and $N_1$ that $\bigcap \{ U_m : m \in M_1 \text{ and } n \in N_1 \} \in N(x)$ and

\[ x \in \bigcap \{ U_m : m \in M_1 \text{ and } n \in N_1 \} \subseteq \bigcap \{ \uparrow m \cap \downarrow n : m \in M_1 \text{ and } n \in N_1 \} \subseteq \uparrow y \cap \downarrow z. \]

Considering the net $(x_{(d, D)})_{(d, D) \in D_{(A, B)}}$, defined as in Remark 2.2 (5), we have $(x_{(d, D)})_{(d, D) \in D_{(A, B)}} \rightarrow x$, and hence $(x_{(d, D)})_{(d, D) \in D_{(A, B)}} \rightarrow x$. So, by Remark 2.2 (6), there exist $M_2 \subseteq M_x$ and $N_2 \subseteq N_x$ such that

\[ x \in \bigcap \{ \uparrow m \cap \downarrow n : m \in M_2 \text{ and } n \in N_2 \} \subseteq \bigcap \{ U_m : m \in M_2 \text{ and } n \in N_2 \} \subseteq \uparrow y \cap \downarrow z. \]
Finally, we show \( \bigcap \{ \uparrow m \cap \downarrow n : m \in M_2 \& n \in N_2 \} \subseteq \bigwedge^\infty_M y \cap \bigvee^\infty_M z \). Let \( (x_{(a,b)})_{(a,b) \in D^\infty_{(M,N)}} \) be the net defined in 2.2 (5) for any \( M \in \mathcal{M} \) and \( N \in \mathcal{N} \) with \( \sup M = \inf N \neq x \in \bigcap \{ \uparrow m \cap \downarrow n : m \in M_2 \& n \in N_2 \} \). Then \( (x_{(a,b)})_{(a,b) \in D^\infty_{(M,N)}} \to x \), and thus \( (x_{(a,b)})_{(a,b) \in D^\infty_{(M,N)}} \to x' \). This implies by Remark 2.2 (6) that there exist \( M_0 \subseteq M \) and \( N_0 \subseteq N \) satisfying
\[
x' \in \bigcap \{ \uparrow m' \cap \downarrow n' : m \in M_0 \& n \in N_0 \}
\subseteq \bigcap \{ U_m^n : m \in M_1 \& n \in N_1 \}
\subseteq \uparrow y \cap \downarrow z.
\]
Hence, we have \( x' \in \bigwedge^\infty_M y \cap \bigvee^\infty_M z \) by Definition 2.3. This shows \( \bigcap \{ \uparrow m \cap \downarrow n : m \in M_2 \& n \in N_2 \} \subseteq \bigwedge^\infty_M y \cap \bigvee^\infty_M z \). Therefore, it follows from Definition 2.7 that \( P \) is \( \mathcal{MN} \)-doubly continuous.

Combining Lemma 2.13 and Lemma 2.14, we obtain the following theorem.

**Theorem 2.15.** Let \( (P, \mathcal{M}, \mathcal{N}) \) be a \( \mathcal{MN} \)-space. Then the following statements are equivalent:

1. \( P \) is an \( \mathcal{MN} \)-doubly continuous poset.

2. For any net \( (x_i)_{i \in I} \) in \( P \), \( (x_i)_{i \in I} \to x \) if and only if \( (x_i)_{i \in I} \to x \).

3. The \( \mathcal{MN} \)-convergence in \( P \) is topological.

**Proof.** (1) \( \Rightarrow \) (2): By Lemma 2.13.

(2) \( \Rightarrow \) (3): It is clear.

(3) \( \Rightarrow \) (1): By Lemma 2.14.

\[ \square \]

### 3 \( \mathcal{M} \)-topology induced by \( \text{lim-inf}_{\mathcal{M}} \)-convergence

In this section, the notion of \( \text{lim-inf}_{\mathcal{M}} \)-convergence is reviewed and the \( \mathcal{M} \)-topology on posets is defined. By exploring the fundamental properties of the \( \mathcal{M} \)-topology, those posets under which the \( \text{lim-inf}_{\mathcal{M}} \)-convergence is topological are precisely characterized.

By saying a \( \mathcal{PM} \)-space, we mean a pair \( (P, \mathcal{M}) \) that contains a poset \( P \) and a subfamily \( \mathcal{M} \) of \( \mathcal{P}(P) \).

**Definition 3.1 [(8)].** Let \( (P, \mathcal{M}) \) be a \( \mathcal{PM} \)-space. A net \( (x_i)_{i \in I} \) in \( P \) is said to \( \text{lim-inf}_{\mathcal{M}} \)-converge to \( x \in P \) if there exists \( M \in \mathcal{M} \) such that

1. (M1) \( x \leq \text{sup} M; \)
2. (M2) for every \( m \in M \), \( x_i \geq m \) holds eventually.

In this case, we write \( (x_i)_{i \in I} \to x \).

It is worth noting that both \( \text{lim-inf}_{\mathcal{M}} \)-convergence and \( \text{lim-inf}_{\mathcal{2}} \)-convergence are particular cases of \( \text{lim-inf}_{\mathcal{M}} \)-convergence.

**Remark 3.2.** Let \( (P, \mathcal{M}) \) be a \( \mathcal{PM} \)-space and \( x, y \in P \).

1. Suppose that a net \( (x_i)_{i \in I} \to x \) and \( y \leq x \). Then \( (x_i)_{i \in I} \to y \) by Definition 3.1. This concludes that the set of all \( \text{lim-inf}_{\mathcal{M}} \)-convergent points of the net \( (x_i)_{i \in I} \) in \( P \) is a lower subset of \( P \). Thus, the \( \text{lim-inf}_{\mathcal{M}} \)-convergent points of the net \( (x_i)_{i \in I} \) need not be unique.

2. If \( P \) has the least element \( \bot \) and \( \emptyset \in \mathcal{M} \), then we have \( (x_i)_{i \in I} \to \bot \) for every net \( (x_i)_{i \in I} \) in \( P \).
(3) For every $M \in \mathcal{M}$ with $\sup M \geq x$, we denote $F^x_M = \{ \{ \uparrow m : m \in M_0 \} : M_0 \subseteq M \}$. Let $D^x_M = \{ (d, D) \in P \times F^x_M : d \in D \}$ be the preorder $\leq$ defined by

$$(\forall (d_1, D_1), (d_2, D_2) \in D^x_M) (d_1, D_1) \leq (d_2, D_2) \iff D_2 \subseteq D_1.$$  

It is easy to see that the set $D^x_M$ is directed. Take $x_{(d, D)} = d$ for every $(d, D) \in D^x_M$. Then, by Definition 3.1, one can straightforwardly check that the net $(x_{(d, D)})_{(d, D) \in D^x_M} \xrightarrow{\mathcal{M}} a$ for every $a \leq x$.

(4) If the net $(x_{(d, D)})_{(d, D) \in D^x_M}$ defined in (3) converges to $p \in P$ with respect to some topology $\mathcal{T}$ on $P$, then for every open neighbourhood $U_p$ of $p$, there exists $M_0 \subseteq M$ such that $\bigcap \{ \{ \uparrow m : m \in M_0 \} \leq \uparrow y \}$.

**Definition 3.3** ([8]). Let $(P, \mathcal{M})$ be a PM-space.

(1) For $x, y \in P$, define $y \ll_{a(\mathcal{M})} x$ if for every net $(x_i)_{i \in I}$ that lim-inf$_{a}$ converges to $x$, $x_i \geq y$ holds eventually.

(2) The poset $P$ is said to be $a(\mathcal{M})$-continuous if the set $\{ y \in P : x \ll_{a(\mathcal{M})} a \}$ holds eventually for every $a \in P$.

Given a PM-space $(P, \mathcal{M})$, the approximate relation $\ll_{a(\mathcal{M})}$ on the poset $P$ can be equivalently characterized in the following proposition.

**Proposition 3.4.** Let $(P, \mathcal{M})$ be a PM-space and $x, y \in P$. Then $y \ll_{a(\mathcal{M})} x$ if and only if for every $M \in \mathcal{M}$ with $\sup M \geq x$, there exists $M_0 \subseteq M$ such that

$$\bigcap \{ \{ \uparrow m : m \in M_0 \} \leq \uparrow y \}.$$  

**Proof.** Suppose $y \ll_{a(\mathcal{M})} x$. Let $(x_{(d, D)})_{(d, D) \in D^x_M}$ be the net defined in Remark 3.2 (3) for every $M \in \mathcal{M}$ with $\sup M = p \geq x$. Then the net $(x_{(d, D)})_{(d, D) \in D^x_M} \xrightarrow{\mathcal{M}} x$. By Definition 3.3 (1), there exists $(d_0, D_0) \in D^x_M$ such that $x_{(d, D)} = d \geq y$ for all $(d, D) \geq (d_0, D_0)$. Since $(d, D_0) \geq (d_0, D_0)$ for every $d \in D_0$, $x_{(d, D)} = d \geq y$ for every $d \in D_0$. So $D_0 \subseteq \uparrow y$. This shows that there exists $M_0 \subseteq M$ such that $D_0 = \bigcap \{ \{ \uparrow m : m \in M_0 \} \leq \uparrow y \}$.

Conversely, suppose that for every $M \in \mathcal{M}$ with $\sup M \geq x$, there exists $M_0 \subseteq M$ such that $\bigcap \{ \{ \uparrow m : m \in M_0 \} \leq \uparrow y \}$. Let $(x_i)_{i \in I}$ be a net that lim-inf$_{a}$ converges to $x$. Then, by Definition 3.1, there exists $M \in \mathcal{M}$ such that $\sup M = p \geq x$, and for every $m \in M$, there exists $i_m \in I$ such that $x_i \geq m$ for all $i \geq i_m$. Take $i_0 \in I$ with that $i_0 \geq i_m$ for every $m \in M_0 \subseteq M$, we have that $x_i \in \bigcap \{ \{ \uparrow m : m \in M_0 \} \leq \uparrow y \}$ for all $i \geq i_0$. This shows that $x_i \geq y$ holds eventually. Thus, by Definition 3.3 (1), we have $y \ll_{a(\mathcal{M})} x$. \qed

**Remark 3.5.** Let $(P, \mathcal{M})$ be a PM-space and $x, y \in P$.

(1) If there is no $M \in \mathcal{M}$ such that $\sup M \geq x$, then $p \ll_{a(\mathcal{M})} x$ for every $p \in P$. And, if the poset $P$ has the least element $\bot$, then $\bot \ll_{a(\mathcal{M})} p$ for every $p \in P$.

(2) The implication $y \ll_{a(\mathcal{M})} x \Rightarrow y \leq x$ may not be true. For example, let $P = \{ 0, 1, 2, \ldots \}$ be in the discrete order $\leq$ defined by

$$(\forall i, j \in P) i \leq j \iff i = j.$$  

And let $\mathcal{M} = \{ \{ 2 \} \}$. Then, it is easy to see from Remark 3.5 (1) that $0 \ll_{a(\mathcal{M})} 1$ and $\mathcal{M} \leq 1$.

(3) Assume the PM-space $(P, \mathcal{M})$ has the property that for every $p \in P$, there exists $M_p \in \mathcal{M}$ such that $\sup M_p = p$. Then, by Proposition 3.4, we have

$$\forall q, r \in P) q \ll_{a(\mathcal{M})} r \Rightarrow q \leq r.$$  

For more interpretations of the approximate relation $\ll_{a(\mathcal{M})}$ on posets, the readers can refer to Example 3.2 and Remark 3.3 in [8].

For simplicity, given a PM-space $(P, \mathcal{M})$ and $x \in P$, we will denote

$$(\forall M \in \mathcal{M}) x = \{ y \in P : y \ll_{a(\mathcal{M})} x \};$$  

\footnote{From the logical point of view, we stipulate $\bigcap \{ \{ \uparrow m : m \in M_0 \} = P \text{ if } M_0 = \emptyset$.}
\[ N \times A = \{ z \in P : x \leq_{at(M)} z \} \]

Based on the approximate relation \( \leq_{at(M)} \) on posets, the \( \alpha^*(M) \)-continuity can be defined for posets in the following:

**Definition 3.6.** Let \( (P, M) \) be a PM-space. The poset \( P \) is called an \( \alpha^*(M) \)-continuous poset if for every \( x \in P \), there exists \( M_x \in M \) such that

\[ \text{(O1)} \sup M_x = x \text{ and } M_x \subseteq M \times N \times A. \]

And,

\[ \text{(O2)} \text{for every } y \in M \times N \times A, \text{ there exists } F \subseteq M_x \text{ such that } \bigcap\{ \uparrow f : f \in F \} \subseteq \Delta_{N \times A} y. \]

Noticing Remark 3.5 (3), we have the following proposition about \( \alpha^*(M) \)-continuous posets.

**Proposition 3.7.** Let \( (P, M) \) be a PM-space in which the poset \( P \) is \( \alpha^*(M) \)-continuous. Then

\[ (\forall x, y \in P) \ y \leq_{at(M)} x \implies y \leq x. \]

The following examples of \( \alpha^*(M) \)-continuous posets can be formally checked by Definition 3.6.

**Example 3.8.** Let \( (P, M) \) be a PM-space.

1. If \( P \) is a finite poset, then \( P \) is an \( \alpha^*(M) \)-continuous poset if and only if for every \( x \in P \), there exists \( M_x \in M \) such that \( \sup M_x = x \).
2. Let \( M = \mathcal{L}(P) \). Then \( P \) is an \( \alpha^*(\mathcal{L}) \)-continuous poset. This means that every poset is \( \alpha^*(\mathcal{L}) \)-continuous.
3. Let \( M = \mathcal{D}(P) \). Then we have \( \ll = \ll_{\mathcal{D}(P)} \) (see Example 3.2 (1) in [8]). The poset \( P \) is a continuous poset if and only if it is an \( \alpha^*(\mathcal{D}) \)-continuous poset. In particular, finite posets, chains, anti-chains and completely distributive lattices are all \( \alpha^*(\mathcal{D}) \)-continuous.
4. Let \( M = \mathcal{D}(P) \). If \( P \) is a finite poset (resp. chain, anti-chain), then \( P \) is an \( \alpha^*(\mathcal{D}) \)-continuous poset.

**Proposition 3.9.** Let \( (P, M) \) be a PM-space. If \( P \) is an \( \alpha(M) \)-continuous poset, and \( \{ y \in P : (\exists z \in P) \ y \leq_{at(M)} z \leq_{at(M)} a \} \in M \) for every \( a \in P \), then \( P \) is an \( \alpha^*(M) \)-continuous poset.

**Proof.** Suppose that \( P \) is an \( \alpha(M) \)-continuous poset, and \( \{ y \in P : (\exists z \in P) \ y \leq_{at(M)} z \leq_{at(M)} a \} \in M \) for every \( a \in P \). Take \( M_x = \bigwedge_{M} a \). Then it is easy to see that \( \sup M_x = a \) and \( M_x \subseteq M \times A \). By Remark 3.3 (4) in [8], we have \( \sup \{ y \in P : (\exists z \in P) \ y \leq_{at(M)} z \leq_{at(M)} a \} = a \). This implies, by Proposition 3A and Remark 3.5 (2), that for every \( y \in \bigwedge_{M} a \), there exist \( \{ y_1, y_2, ..., y_n \}, \{ z_1, z_2, ..., z_n \} \subseteq M_a = \bigwedge_{M} a \) such that

\[ \bigcap\{ \uparrow z_i : i \in \{1, 2, ..., n\} \} \subseteq \bigcap\{ \uparrow y_i : i \in \{1, 2, ..., n\} \} \subseteq \Delta \]

and \( y_i \leq_{at(M)} z_i \leq_{at(M)} a \) for every \( i \in \{1, 2, ..., n\} \). Next, we show \( \bigcap\{ \uparrow z_i : i \in \{1, 2, ..., n\} \} \subseteq \Delta \). For every \( M \in M \) with \( \sup P > b \in \bigcap\{ \uparrow z_i : i \in \{1, 2, ..., n\} \} \), by Proposition 3A, there exists \( M_i \subseteq M \) such that \( \bigcap\{ \uparrow m : m \in M_i \} \subseteq \uparrow y_i \) for every \( i \in \{1, 2, ..., n\} \). Take \( P_0 = \bigcup\{ M_i : i \in \{1, 2, ..., n\} \} \). Then \( M_0 \subseteq M \) and

\[ \bigcap\{ \uparrow m : m \in M_0 \} \subseteq \bigcap\{ \uparrow y_i : i \in \{1, 2, ..., n\} \} \subseteq \uparrow y. \]

This shows \( y \leq_{at(M)} b \) for every \( b \in \bigcap\{ \uparrow z_i : i \in \{1, 2, ..., n\} \} \). Hence, \( \bigcap\{ \uparrow z_i : i \in \{1, 2, ..., n\} \} \subseteq \Delta \). Thus \( P \) is an \( \alpha^*(M) \)-continuous poset.

The fact that an \( \alpha^*(M) \)-continuous poset \( P \) in a PM-space \( (P, M) \) may not be \( \alpha(M) \)-continuous can be demonstrated in the following example.
Example 3.10. Let \((P, \mathcal{M})\) be the PM-space in which the poset \(P = \mathbb{R}\) is the set of all real number with its usual order \(\leq\) and \(\mathcal{M} = \mathcal{S}_0(\mathbb{R})\). Then we have \(\ll_{a(\delta_0)} = \ll\) by Proposition 3.4. It is easy to check, by Definition 3.6, that \(\mathbb{R}\) is an \(a^*(\delta_0)\)-continuous poset. But \(\mathbb{R}\) is not an \(a(\delta_0)\)-continuous poset because \(\nabla_{\delta_0} x = \downarrow x \in \mathcal{S}_0(P)\) for every \(x \in \mathbb{R}\).

We turn to consider the topology induced by the \(\liminf_{\mathcal{M}}\)-convergence in posets.

Definition 3.11. Let \((P, \mathcal{M})\) be a PM-space. A subset \(V\) of \(P\) is said to be \(\mathcal{M}\)-open if for every net \((x_i)_{i \in I} \rightarrow x \in V\), \(x_i \in V\) holds eventually.

Given a PM-space \((P, \mathcal{M})\), one can formally verify that the set of all \(\mathcal{M}\)-open subsets of \(P\) forms a topology on \(P\). This topology is called the \(\mathcal{M}\)-topology, and denoted by \(\mathcal{O}_{\mathcal{M}}(P)\).

The following Theorem is an order-theoretical characterization of \(\mathcal{M}\)-open sets.

Theorem 3.12. Let \((P, \mathcal{M})\) be a PM-space. Then a subset \(V\) of \(P\) is \(\mathcal{M}\)-open if and only if it satisfies the following two conditions:

\[(V1)\forall x \in V, \text{ i.e., } V \text{ is an upper set.}\]
\[(V2)\text{For every } M \in \mathcal{M} \text{ with } \sup M \in V, \text{ there exists } M_0 \subseteq M \text{ such that } \bigcap \{\uparrow m : m \in M_0\} \subseteq V.\]

Proof. Suppose that \(V\) is an \(\mathcal{M}\)-open subset of \(P\). By Remark 3.2 (1), it is easy to see that \(V\) is an upper set. Let \((x_{(d,D)})_{(d,D)\in D_M^P}^M\) be the net defined in Remark 3.2 (3) for every \(M \in \mathcal{M}\) with \(\sup M = x \in V\). Then \((x_{(d,D)})_{(d,D)\in D_M^P}^M\) \(\rightarrow x \in V\). This implies, by Definition 3.11, that there exists \((d_0, D_0) \in D_M^P\) such that \(x_{(d,D)} = d \in V\) for all \((d, D) \geq (d_0, D_0)\). Since \((d, D) \geq (d_0, D_0)\) for all \(d \in D_0\), \(x_{(d,D)} = d \in V\) for all \(d \in D_0\). This shows \(D_0 \subseteq V\). Thus there exists \(M_0 \subseteq M\) such that \(D_0 = \bigcap \{\uparrow m : m \in M_0\} \subseteq V\).

Conversely, suppose \(V\) is a subset of \(P\) which satisfies Condition (V1) and (V2). Let \((x_i)_{i \in I}\) be a net that \(\liminf_{\mathcal{M}}\)-converges to \(x \in V\). Then there exists \(M \in \mathcal{M}\) such that \(\sup M = y \geq x \in V = \uparrow V\) (hence, \(y \in V\), and for every \(m \in M\), there exists \(i_m \in I\) such that \(x_i \geq m\) for all \(i \geq i_m\). By Condition (V2), we have that \(\bigcap \{\uparrow m : m \in M_0\} \subseteq V\) for some \(M_0 \subseteq M\). Take \(i_0 \in I\) with that \(i_0 \geq i_m\) for all \(m \in M_0\). Then \(x_i \in \bigcap \{\uparrow m : m \in M_0\} \subseteq V\) for all \(i \geq i_0\). This shows that \(V\) is an \(\mathcal{M}\)-open set.

Recall that given a topological space \((X, \mathcal{T})\) and a point \(x \in P\), a family \(\mathcal{B}(x)\) of open neighbourhoods of \(x\) is called a base for the topological space \((X, \mathcal{T})\) at the point \(x\) if for every neighbourhood \(V\) of \(x\) there exists an \(U \in \mathcal{B}(x)\) such that \(x \in U \subseteq V\).

If the poset \(P\) in a PM-space \((P, \mathcal{M})\) is an \(a^*(\mathcal{M})\)-continuous poset, we provide a base for the topological space \((P, \mathcal{O}_{\mathcal{M}}(P))\) at a point \(x \in P\).

Proposition 3.13. Let \((P, \mathcal{M})\) be a PM-space in which the poset \(P\) is an \(a^*(\mathcal{M})\)-continuous. Then \(\bigtriangleup_{\mathcal{M}} x \in \mathcal{O}_{\mathcal{M}}(P)\) for every \(x \in P\).

Proof. One can readily see, by Proposition 3.4, that \(\bigtriangleup_{\mathcal{M}} x\) is an upper subset of \(P\) for every \(x \in P\). For every \(M \in \mathcal{M}\) with \(\sup M = y \in \bigtriangleup_{\mathcal{M}} x\), by Definition 3.6 (01) there exists \(M_y \in \mathcal{M}\) such that \(M_y \subseteq \bigtriangleup_{\mathcal{M}} x\) and \(\sup M_y = y\). Since \(x \ll_{a(\mathcal{M})} y\), by Definition 3.6 (02), we have \(\bigcap \{\uparrow m_i : i \in \{1, 2, ..., n\}\} \subseteq \bigtriangleup_{\mathcal{M}} x\) for some \(\{m_1, m_2, ..., m_n\} \subseteq M_y\). Observing \(\{m_1, m_2, ..., m_n\} \subseteq M_y \subseteq \bigtriangleup_{\mathcal{M}} x\), we can conclude that there exists \(M_i \subseteq M\) such that \(\bigcap \{\uparrow a : a \in M_i\} \subseteq \uparrow m_i\) for every \(i \in \{1, 2, ..., n\}\). Let \(M_0 = \bigcup \{M_i : i \in \{1, 2, ..., n\}\}\). Then \(M_0 \subseteq M\) and
\[
\bigcap \{\uparrow m : m \in M_0\} \\
\subseteq \bigcap \{\uparrow m_i : i \in \{1, 2, ..., n\}\} \\
\subseteq \bigtriangleup_{\mathcal{M}} x.
\]
This shows, by Theorem 3.12, that \(\bigtriangleup_{\mathcal{M}} x \in \mathcal{O}_{\mathcal{M}}(P)\) for every \(x \in P\).
Proposition 3.14. Let \((P, \mathcal{M})\) be a PM-space in which the poset \(P\) is \(\alpha'(\mathcal{M})\)-continuous and \(x \in P\). Then \(\bigcap \{\langle a \in \mathcal{M} : a \in A \rangle : A \subseteq \mathcal{M} x \}\) is a base for the topological space \((P, \mathcal{O}_{\mathcal{M}}(P))\) at the point \(x\).

Proof. Clearly, by Proposition 3.13, we have \(\bigcap \{\langle a \in \mathcal{M} : a \in A \rangle : A \subseteq \mathcal{M} x \} \subseteq \mathcal{O}_{\mathcal{M}}(P)\) for every \(A \subseteq \mathcal{M} x\) and \(x \in U\). Since \(P\) is an \(\alpha'(\mathcal{M})\)-continuous poset, there exists \(M_x \in \mathcal{M}\) such that \(M_x \subseteq \mathcal{M} x\) and \(\sup M_x = x \in U\). By Theorem 3.12, it follows that \(\bigcap \{\langle m : m \in M_0 \rangle : A \subseteq \mathcal{M} x \} \subseteq \mathcal{O}_{\mathcal{M}}(P)\) for some \(M_0 \subseteq \mathcal{M} x\). So, from Proposition 3.7, we have

\[x \in \bigcap \{\langle a \in \mathcal{M} : a \in A \rangle : A \subseteq \mathcal{M} x \} \subseteq \bigcap \{\langle m : m \in M_0 \rangle : A \subseteq \mathcal{M} x \} \subseteq U.\]

Thus, \(\bigcap \{\langle a \in \mathcal{M} : a \in A \rangle : A \subseteq \mathcal{M} x \}\) is a base for the topological space \((P, \mathcal{O}_{\mathcal{M}}(P))\) at the point \(x\).

In the rest, we are going to establish a characterization theorem which demonstrates the equivalence between the \(\text{lim-}\inf_{\mathcal{M}}\)-convergence being topological and the \(\alpha'(\mathcal{M})\)-continuity of the poset in a given PM-space.

Lemma 3.15. Let \((P, \mathcal{M})\) be a PM-space. If \(P\) is an \(\alpha'(\mathcal{M})\)-continuous poset, then any net \((x_i)_{i \in I} \in \mathcal{M}(P)\) such that \(x_i \in \mathcal{M} x \) for every \(x \in P\). Hence, \(x_i \in \mathcal{M} y \) holds eventually. By the definition of \(\text{lim-}\inf_{\mathcal{M}}\)-convergence, we have \((x_i)_{i \in I} \in \mathcal{M}(P)\).

In the converse direction, we have the following Lemma.

Lemma 3.16. Let \((P, \mathcal{M})\) be a PM-space. If the \(\text{lim-}\inf_{\mathcal{M}}\)-convergence in \(P\) is topological, then \(P\) is an \(\alpha'(\mathcal{M})\)-continuous poset.

Proof. Suppose that the \(\text{lim-}\inf_{\mathcal{M}}\)-convergence in \(P\) is topological. Then there exists a topology \(\mathcal{T}\) such that for every \(x \in P\), a net \((x_i)_{i \in I} \in \mathcal{M}(P)\) implies \((x_i)_{i \in I} \in \mathcal{M}(P)\). Suppose \((x_i)_{i \in I} \rightarrow x\). As \(P\) is an \(\alpha'(\mathcal{M})\)-continuous poset, there exists \(M_x \in \mathcal{M}\) such that \(M_x \subseteq \mathcal{M} x\) and \(\sup M_x = x\). By Proposition 3.13, we have \(x \in \mathcal{M} y \) for every \(y \in M_x \subseteq \mathcal{M} x\). Hence, \(x_i \in \mathcal{M} y\) holds eventually. This implies, by Proposition 3.7, that \(x_i \in \mathcal{M} y \subseteq \mathcal{M} x\) holds eventually. By the definition of \(\text{lim-}\inf_{\mathcal{M}}\)-convergence, we have \((x_i)_{i \in I} \in \mathcal{M}(P)\).

Next we prove \(M_x \subseteq \mathcal{M} x\). For every \(m \in M_x\) and every \(M \in M\) with \(\sup M \geq x\), let \((x_{(d, D)})_{(d, D) \in D^d_m}\) be the net defined in Remark 3.2 (3). Then the net \((x_{(d, D)})_{(d, D) \in D^d_m} \in \mathcal{M}(P)\), and thus \((x_{(d, D)})_{(d, D) \in D^d_m} \rightarrow x\). It follows from
Remark 3.2 (4) that there exists $M_0 \sqsubseteq M$ such that $x \in \bigcap \{ \uparrow a : a \in M_0 \} \subseteq V_m$. By Condition (**), we have $x \in \bigcap \{ \uparrow a : a \in M_0 \} \subseteq V_m \subseteq \uparrow m$. So, $m \ll a(M)x$. This shows $M_x \subseteq \nabla_M x$.

Let $y \in \nabla_M x$. Then there exists $\{ m_1, m_2, \ldots, m_n \} \subseteq M_x$ such that $\bigcap \{ \uparrow m_i : i \in \{ 1, 2, \ldots, n \} \} \subseteq \uparrow y$ as $M_x \in M$ and sup $M_x \geq x$. By Condition (**), it follows that $\bigcap \{ \uparrow V_m : i \in \{ 1, 2, \ldots, n \} \} \subseteq \bigcap \{ \uparrow m_i : i \in \{ 1, 2, \ldots, n \} \} \subseteq \uparrow y$. Considering the net $(x_{(d,D)})_{(d,D) \in D_t}$ defined in Remark 3.2 (3), we have $(x_{(d,D)})_{(d,D) \in D_t} \xrightarrow{\uparrow} x$, and hence $(x_{(d,D)})_{(d,D) \in D_t} \xrightarrow{\uparrow} x$. This implies, by Remark 3.2 (4), that

\[
\bigcap \{ \uparrow b : b \in M_{00} \} \subseteq \bigcap \{ \uparrow V_m : i \in \{ 1, 2, \ldots, n \} \} \subseteq \bigcap \{ \uparrow m_i : i \in \{ 1, 2, \ldots, n \} \} \subseteq \uparrow y
\]

for some $M_{00} \sqsubseteq M_x$. Finally, we show $\bigcap \{ \uparrow b : b \in M_{00} \} \subseteq M_y$. For every $x' \in \bigcap \{ \uparrow b : b \in M_{00} \}$ and every $M' \in M$ with sup $M' \geq x'$, let $(x_{(d,D)})_{(d,D) \in D_{t'}}$ be the net defined in Remark 3.2 (3). Then $(x_{(d,D)})_{(d,D) \in D_{t'}} \xrightarrow[M_{t'}]{\uparrow} x'$, and thus $(x_{(d,D)})_{(d,D) \in D_{t'}} \xrightarrow[\uparrow]{} x'$. It follows from Condition (***) and Remark 3.2 (4) that there exists $M_0 \sqsubseteq M'$ such that

\[
\bigcap \{ \uparrow a' : a' \in M_0 \} \subseteq \bigcap \{ \uparrow V_m : i \in \{ 1, 2, \ldots, n \} \} \subseteq \bigcap \{ \uparrow m_i : i \in \{ 1, 2, \ldots, n \} \} \subseteq \uparrow y.
\]

This shows $x' \in M_y$, and thus $\bigcap \{ \uparrow b : b \in M_{00} \} \subseteq M_y$. Therefore, $P$ is an $\alpha' (M)$-continuous poset.

Combining Lemma 3.15 and Lemma 3.16, we deduce the following result.

**Theorem 3.17.** Let $(P, M)$ be a PM-space. The following statements are equivalent:

1. $P$ is an $\alpha' (M)$-continuous poset.
2. For any net $(x_i)_{i \in I}$ in $P$, $(x_i)_{i \in I} \xrightarrow[M_{t'}]{\uparrow} x \in P \iff (x_i)_{i \in I} \xrightarrow[\uparrow]{} x$.
3. The lim-inf$_M$-convergence in $P$ is topological.

**Proof.** (1) $\Rightarrow$ (2): By Lemma 3.15.

(2) $\Rightarrow$ (3): Clear.

(3) $\Rightarrow$ (1): By Lemma 3.16.

**Corollary 3.18** ([8]). Let $(P, M)$ be a PM-space with $S_0(P) \subseteq M \subseteq P(P)$. Suppose $\nabla_M a \in M$ and $\{ y \in P : (\exists z \in P) y \ll a(M)z \ll a(M)a \} \in M$ holds for every $a \in P$. Then the lim-inf$_M$-convergence in $P$ is topological if and only if $P$ is an $\alpha' (M)$-continuous poset.

**Proof.** ($\Rightarrow$): To show the $\alpha' (M)$-continuity of $P$, it suffices to prove sup $\nabla_M a = a$ for every $a \in P$. Since the lim-inf$_M$-convergence in $P$ is topological, by Theorem 3.17, $P$ is an $\alpha' (M)$-continuous poset. This implies that there exists $M_a \in M$ such that sup $M_a \subseteq \nabla_M a$ and sup $M_a = a$ for every $a \in P$. By Proposition 3.7, we have $\nabla_M a \subseteq \downarrow a$. So sup $\nabla_M a = a$.

($\Leftarrow$): By Proposition 3.9 and Theorem 3.17.

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References

[7] Li Q.G., Zou Z.Z., A result for $O_2$-convergence to be topological in posets, Open Math., 2016, 14, 205-211.