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\(MN\)-convergence and \(\text{lim-inf}_M\)-convergence in partially ordered sets

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Abstract: In this paper, we first introduce the notion of \(MN\)-convergence in posets as an unified form of \(O\)-convergence and \(O_2\)-convergence. Then, by studying the fundamental properties of \(MN\)-topology which is determined by \(MN\)-convergence according to the standard topological approach, an equivalent characterization to the \(MN\)-convergence being topological is established. Finally, the \(\text{lim-inf}_M\)-convergence in posets is further investigated, and a sufficient and necessary condition for \(\text{lim-inf}_M\)-convergence to be topological is obtained.

Keywords: \(MN\)-convergence, \(MN\)-topology, \(\text{lim-inf}_M\)-convergence, \(M\)-topology

MSC: 54A20, 06A06

1 Introduction, Notations and Preliminaries

The concept of \(O\)-convergence in partially ordered sets (posets, for short) was introduced by Birkhoff [1], Frink [2] and Mcshane [3]. It is defined as follows: a net \((x_i)_{i \in I}\) in a poset \(P\) is said to \(O\)-converge to \(x \in P\) if there exist subsets \(D\) and \(F\) of \(P\) such that
1. \(D\) is directed and \(F\) is filtered;
2. \(\sup D = x = \inf F\);
3. for every \(d \in D\) and \(e \in F\), \(d \leq x_i \leq e\) holds eventually, i.e., there exists \(i_0 \in I\) such that \(d \leq x_i \leq e\) for all \(i \geq i_0\).

As what has been showed in [4], the \(O\)-convergence (Note: in [4], the \(O\)-convergence is called order-convergence) in a general poset \(P\) may not be topological, i.e., it is possible that \(P\) can not be endowed with a topology such that the \(O\)-convergence and the associated topological convergence are consistent. Hence, much work has been done to characterize those special posets in which the \(O\)-convergence is topological. The most recent result in [5] shows that the \(O\)-convergence in a poset which satisfies Condition \((\Delta)\) is topological if and only if the poset is \(O\)-doubly continuous. This means that for a special class of posets, a sufficient and necessary condition for \(O\)-convergence being topological is obtained.

As a direct generalization of \(O\)-convergence, \(O_2\)-convergence in posets has been discussed in [11] from the order-theoretical point of view. It is defined as follows: a net \((x_i)_{i \in I}\) in a poset \(P\) is said to \(O_2\)-converge to \(x \in P\) if there exist subsets \(A\) and \(B\) of \(P\) such that
(1) $\sup A = x = \inf B$;
(2) for every $a \in A$ and $b \in B$, $a \leq x_i \leq b$ holds eventually.

In fact, the $O_2$-convergence is also not topological generally. To clarify those special posets in which the $O_2$-convergence is topological, Zhao and Li [6] showed that for any poset $P$ satisfying Condition (*), $O_2$-convergence is topological if and only if $P$ is $a$-doubly continuous. As a further result, Li and Zou [7] proved that the $O_2$-convergence in a poset $P$ is topological if and only if $P$ is $O_2$-doubly continuous. This result demonstrates the equivalence between the $O_2$-convergence being topological and the $O_2$-double continuity of a given poset.

On the other hand, Zhou and Zhao [8] have defined the lim-inf$_{M^*}$-convergence in posets to generalize lim-inf$_{M}$-convergence and lim-inf$_{2}$-convergence to $M^*$-convergence. They also found that the lim-inf$_{M^*}$-convergence in a poset is topological if and only if the poset is $\alpha(M)$-continuous when additional conditions are satisfied (see [8], Theorem 3.1). This result clarified some special conditions of posets under which the lim-inf$_{M^*}$-convergence is topological. However, to the best of our knowledge, the equivalent characterization to the lim-inf$_{M^*}$-convergence in general posets being topological is still unknown.

One goal of this paper is to propose the notion of $M^N$-convergence in posets which can unify $O$-convergence and $O_2$-convergence and search the equivalent characterization to the $M^N$-convergence being topological. More precisely,

(G11) Given a general poset $P$, we hope to clarify the order-theoretical condition of $P$ which is sufficient and necessary for the $M^N$-convergence being topological.

(G12) Given a poset $P$ satisfying such condition, we hope to provide a topology on $P$ such that the $M^N$-convergence and the associated topological convergence agree.

Another goal is to look for the equivalent characterization to the lim-inf$_{M^*}$-convergence being topological. More precisely,

(G21) Given a general poset $P$, we expect to present a sufficient and necessary condition of $P$ which can precisely serve as an order-theoretical condition for the lim-inf$_{M^*}$-convergence being topological.

(G22) Given a poset $P$ satisfying such condition, we expect to give a topology on $P$ such that the lim-inf$_{M^*}$-convergence and the associated topological convergence are consistent.

To accomplish those goals, motivated by the ideal of introducing the $Z$-subsets system [9] for defining $Z$-continuous posets, we propose the notion of $M^N$-doubly continuous posets and define the $M^N$-topology on posets in Section 2. Based on the study of the basic properties of the $M^N$-topology, it is proved that the $M^N$-convergence in a poset $P$ is topological if and only if $P$ is an $M^N$-doubly continuous poset and if only if the $M^N$-convergence and the topological convergence with respect to $M$-topology are consistent. In Section 3, by introducing the notion of $\alpha'(M)$-continuous posets and presenting the fundamental properties of $M$-topology which is induced by the lim-inf$_{M^*}$-convergence, we show that the lim-inf$_{M^*}$-convergence in a poset $P$ is topological if and only if $P$ is an $\alpha'(M)$-continuous poset and if only if the lim-inf$_{M^*}$-convergence and the topological convergence with respect to $M$-topology are consistent.

Some conventional notations will be used in the paper. Given a set $X$, $F \subseteq X$ means that $F$ is a finite subset of $X$. Given a topological space $(X, T)$ and a net $(x_i)_{i \in I}$ in $X$, we take $(x_i)_{i \in I} \rightarrow x$ to mean the net $(x_i)_{i \in I}$ converges to $x \in P$ with respect to the topology $T$.

Let $P$ be a poset and $x \in P$. $\uparrow x$ and $\downarrow x$ are always used to denote the principal filter $\{y \in P : y \geq x\}$ and the principal ideal $\{z \in P : z \leq x\}$ of $P$, respectively. Given a poset $P$ and $A \subseteq P$, by writing sup $A$ we mean that the least upper bound of $A$ in $P$ exists and equals to sup $A \in P$; dually, by writing inf $A$ we mean that the greatest lower bound of $A$ in $P$ exists and equals to inf $A \in P$. And the set $A$ is called an upper set if $A = \uparrow A = \{b \in P : (\exists a \in A) a \leq b\}$, the lower set is defined dually.

For a poset $P$, we succinctly denote

- $\mathcal{P}(P) = \{A : A \subseteq P\}; \mathcal{P}_0(P) = \mathcal{P}(P)/\{\emptyset\}$;
- $\mathcal{D}(P) = \{D \in \mathcal{P}(P) : D$ is a directed subset of $P\}$;
To make this paper self-contained, we briefly review the following notions:

**Definition 1.1** ([5]). Let \( P \) be a poset and \( x, y, z \in P \). We say \( y \leq_{\mathcal{O}} x \) if for every net \( (x_i)_{i \in I} \) in \( P \) which \( \mathcal{O} \)-converges to \( x \in P \), \( x_i \geq y \) holds eventually; dually, we say \( z \geq_{\mathcal{O}} x \) if for every net \( (x_i)_{i \in I} \) in \( P \) which \( \mathcal{O} \)-converges to \( x \in P \), \( x_i \leq z \) holds eventually.

**Definition 1.2** ([5]). A poset \( P \) is said to be \( \mathcal{O} \)-doubly continuous if for every \( x \in P \), the set \( \{ a \in P : a \leq_{\mathcal{O}} x \} \) is directed, the set \( \{ b \in P : b \geq_{\mathcal{O}} x \} \) is filtered and \( \sup \{ a \in P : a \leq_{\mathcal{O}} x \} = x = \inf \{ b \in P : b \geq_{\mathcal{O}} x \} \).

**Condition \( (\Delta) \).** A poset \( P \) is said to satisfy Condition \( (\Delta) \) if

1. for any \( x, y, z \in P \), \( x \leq_{\mathcal{O}} y \subseteq z \) implies \( x \leq_{\mathcal{O}} z \);
2. for any \( w, s, t \in P \), \( w \geq_{\mathcal{O}} s \geq t \) implies \( w \geq_{\mathcal{O}} t \).

**Definition 1.3** ([6]). Let \( P \) be a poset and \( x, y, z \in P \). We say \( y \leq_{a} x \) if for every net \( (x_i)_{i \in I} \) in \( P \) which \( a \)-converges to \( x \in P \), \( x_i \geq y \) holds eventually; dually, we say \( z \geq_{a} x \) if for every net \( (x_i)_{i \in I} \) in \( P \) \( a \)-converges to \( x \in P \), \( x_i \leq z \) holds eventually.

**Definition 1.4** ([7]). A poset \( P \) is said to be \( \mathcal{O} \)-doubly continuous if for every \( x \in P \),

1. \( \sup \{ a \in P : a \leq_{a} x \} = x = \inf \{ b \in P : b \geq_{a} x \} \);
2. for any \( y, z \in P \) with \( y \leq_{a} x \) and \( z \geq_{a} x \), there exist \( A \supseteq \{ a \in P : a \leq_{a} x \} \) and \( B \supseteq \{ b \in P : b \geq_{a} x \} \) such that \( y \leq_{a} c \) and \( z \geq_{a} c \) for each \( c \in \bigcap \{ a \cap b : a \in A \land b \in B \} \).

## 2 MN-topology on posets

Based on the introduction of \( MN \)-convergence in posets, the \( MN \)-topology can be defined on posets. In this section, we first define the \( MN \)-double continuity for posets. Then, we show the equivalence between the \( MN \)-convergence being topological and the \( MN \)-double continuity of a given poset.

A \( PMN \)-space is a triplet \( (P, \mathcal{M}, \mathcal{N}) \) which consists of a poset \( P \) and two subfamily \( \mathcal{M}, \mathcal{N} \subseteq \mathcal{P}(P) \).

All \( PMN \)-spaces \( (P, \mathcal{M}, \mathcal{N}) \) considered in this section are assumed to satisfy the following conditions:

- \( (C1) \) If \( P \) has the least element \( \bot \), then \( \{ \bot \} \in \mathcal{M} \);
- \( (C2) \) If \( P \) has the greatest element \( \top \), then \( \{ \top \} \in \mathcal{N} \);
- \( (C3) \) \( \emptyset \notin \mathcal{M} \) and \( \emptyset \notin \mathcal{N} \).

**Definition 2.1.** Let \( (P, \mathcal{M}, \mathcal{N}) \) be a \( PMN \)-space. A net \( (x_i)_{i \in I} \) in \( P \) is said to \( MN \)-converge to \( x \in P \) if there exist \( M \in \mathcal{M} \) and \( N \in \mathcal{N} \) satisfying:

1. \( (MN1) \sup M = x = \inf N \);
2. \( (MN2) x_i \uparrow m \land n \) eventually for every \( m \in M \) and every \( n \in N \).

In this case, we will write \( (x_i)_{i \in I} \xrightarrow{MN} x \).

**Remark 2.2.** Let \( (P, \mathcal{M}, \mathcal{N}) \) be a \( PMN \)-space.

1. If \( \mathcal{M} = \mathcal{D}(P) \) and \( \mathcal{N} = \mathcal{F}(P) \), then a net \( (x_i)_{i \in I} \xrightarrow{MN} x \in P \) if and only if it \( O \)-converges to \( x \). That is to say, \( O \)-convergence is a particular case of \( MN \)-convergence.
2. If \( \mathcal{M} = \mathcal{N} = \mathcal{P}(P) \), then a net \( (x_i)_{i \in I} \xrightarrow{MN} x \in P \) if and only if it \( O_2 \)-converges to \( x \). That is to say, \( O_2 \)-convergence is a special case of \( MN \)-convergence.
3. If $\mathcal{M} = \mathcal{N} = \mathcal{L}_0(P)$, then a net $(x_i)_{i \in I} \rightarrow x \in P$ if and only if $x_1 = x$ holds eventually.

4. The $\mathcal{N}$-convergent point of a net $(x_i)_{i \in I}$ in $P$, if it exists, is unique.

Indeed, suppose that $(x_i)_{i \in I} \rightarrow x_1$ and $(x_i)_{i \in I} \rightarrow x_2$. Then there exist $A_k \in \mathcal{M}$ and $B_k \in \mathcal{N}$ such that $\sup A_k = x_k = \inf B_k$ and $a_k \leq x_1 \leq b_k$ holds eventually for every $a_k \in A_k$ and $b_k \in B_k$ ($k = 1, 2$). This implies that for any $a_1 \in A_1$, $a_2 \in A_2$, $b_1 \in B_1$ and $b_2 \in B_2$, there exists $i_0 \in I$ such that $a_1 \leq x_{i_0} \leq b_2$ and $a_2 \leq x_{i_0} \leq b_1$. Thus we have $A_1 = x_1 \leq \inf B_2 = x_2$ and $\sup A_2 = x_2 \leq \inf B_1 = x_1$. Therefore $x_1 = x_2$.

5. For any $A \in \mathcal{M}$ and $B \in \mathcal{N}$ with sup $A = \inf B = x \in P$, we denote $F^\mathcal{N}_{(A,B)} = \{ \bigcap \{ a \cap b : a \in A_0 \land b \in B_0 \} : A_0 \subseteq A \land B_0 \subseteq B \}$. Let $D^\mathcal{N}_{(A,B)} = \{ (d, D) \in \mathcal{P} \times P^\mathcal{N}_{(A,B)} : d \in D \}$, and let the preorder $\leq$ on $D^\mathcal{N}_{(A,B)}$ be defined by

$$(\forall (d_1, D_1), (d_2, D_2) \in D^\mathcal{N}_{(A,B)}) \quad (d_1, D_1) \leq (d_2, D_2) \iff D_2 \subseteq D_1.$$  

One can readily check that $(D^\mathcal{N}_{(A,B)}, \leq)$ is directed. Now if we take $x_{(d,D)} = d$ for every $(d, D) \in D^\mathcal{N}_{(A,B)}$, then the net $(x_{(d,D)})_{(d,D) \in D^\mathcal{N}_{(A,B)}} \rightarrow x$ because sup $A = \inf B = x$, and $a \leq x_{(d,D)} \leq b$ holds eventually for any $a \in A$ and $b \in B$.

6. Let $(x_{(d,D)})_{(d,D) \in D^\mathcal{N}_{(A,B)}}$ be the net defined in (5) for any $A \in \mathcal{M}$ and $B \in \mathcal{N}$ with sup $A = \inf B = x \in P$. If $(x_{(d,D)})_{(d,D) \in D^\mathcal{N}_{(A,B)}}$ converges to $p \in P$ with respect to some topology $\tau$ on the poset $P$, then for every open neighborhood $U_p$ of $p$, there exist $A_0 \subseteq A$ and $B_0 \subseteq B$ such that

$$\bigcap \{ \{ a \cap b : a \in A_0 \land b \in B_0 \} : a \in A_0 \land b \in B_0 \} \subseteq U_p.$$  

Indeed, suppose that $(x_{(d,D)})_{(d,D) \in D^\mathcal{N}_{(A,B)}} \rightarrow \tau p$. Then for every open neighborhood $U_p$ of $p$, there exists $(d_0, D_0) \in D^\mathcal{N}_{(A,B)}$ such that $x_{(d,D)} = d \in U_p$ for all $(d, D) \geq (d_0, D_0)$. Since $(d, D) \geq (d_0, D_0)$ for every $d \in D_0, x_{(d,D)} = d \in U_p$ for every $d \in D_0$. This shows $D_0 \subseteq U_p$. So, there exist $A_0 \subseteq A$ and $B_0 \subseteq B$ such that

$$D_0 = \bigcap \{ \{ a \cap b : a \in A_0 \land b \in B_0 \} : a \in A_0 \land b \in B_0 \} \subseteq U_p.$$  

Given a PMN-space $(P, \mathcal{M}, \mathcal{N})$, we can define two new approximate relations $\ll_{\mathcal{M}}$ and $\gg_{\mathcal{M}}$ on the poset $P$ in the following definition.

**Definition 2.3.** Let $(P, \mathcal{M}, \mathcal{N})$ be a PMN-space and $x, y, z \in P$.

1. We define $y \ll_{\mathcal{M}} x$ if for any $A \in \mathcal{M}$ and $B \in \mathcal{N}$ with sup $A = x = \inf B$, there exist $A_0 \subseteq A$ and $B_0 \subseteq B$ such that

$$\bigcap \{ \{ a \cap b : a \in A_0 \land b \in B_0 \} : a \in A_0 \land b \in B_0 \} \subseteq \uparrow y.$$  

2. Dually, we define $z \gg_{\mathcal{M}} x$ if for any $M \in \mathcal{M}$ and $N \in \mathcal{N}$ with sup $M = x = \inf N$, there exist $M_0 \subseteq M$ and $N_0 \subseteq N$ such that

$$\bigcap \{ \{ m \cap n : m \in M_0 \land n \in N_0 \} : m \in M_0 \land n \in N_0 \} \subseteq \downarrow z.$$  

For convenience, given a PMN-space $(P, \mathcal{M}, \mathcal{N})$ and $x \in P$, we will briefly denote

- $\nabla_{\mathcal{M}}^N x = \{ y \in P : y \ll_{\mathcal{M}} x \}$;
- $\nabla_{\mathcal{N}}^M x = \{ z \in P : x \ll_{\mathcal{N}} z \}$;
- $\Delta_{\mathcal{M}}^N x = \{ a \in P : x \gg_{\mathcal{M}} a \}$;
- $\Delta_{\mathcal{N}}^M x = \{ b \in P : b \gg_{\mathcal{N}} x \}$.

**Remark 2.4.** Let $(P, \mathcal{M}, \mathcal{N})$ be a PMN-space and $x, y, z \in P$.

1. If there is no $A \in \mathcal{M}$ such that sup $A = x$, then $p \ll_{\mathcal{M}} x$ and $p \gg_{\mathcal{N}} x$ for all $p \in P$; similarly, if there is no $B \in \mathcal{N}$ such that inf $B = x$, then $p \ll_{\mathcal{N}} x$ and $p \gg_{\mathcal{M}} x$ for all $p \in P$.

2. By Definition 2.3, one can easily check that if $p$ has the least element $\bot$, then $\bot \ll_{\mathcal{M}} p$ for every $p \in P$, and if $p$ has the greatest element $\top$, then $\top \gg_{\mathcal{N}} p$ for every $p \in P$.  

\[\text{From the logical point of view, we stipulate } \bigcap \{ \{ a \cap b : a \in A_0 \land b \in B_0 \} : a \in A_0 \land b \in B_0 \} = P \text{ if } A_0 = \emptyset \text{ or } B_0 = \emptyset.\]
(3) The implications \( y \lesssim_{MN}^N x \Rightarrow x \leq y \) and \( z \gtrsim_{MN}^N x \Rightarrow z \geq x \) are not true necessarily. See the following example:

Let \( \mathbb{R} \) be the set of all real numbers, in its ordinal order, and \( M = N = \{ n : n \in \mathbb{Z} \} \), where \( \mathbb{Z} \) is the set of all integers. Then, by (1), we have \( 1 \lesssim_{MN}^N 1/2 \) and \( 0 \gtrsim_{MN}^N 1/2 \). But \( 1 \lesssim 1/2 \) and \( 0 \gtrsim 1/2 \).

(4) Assume that \( \sup A_0 = x = \inf B_0 \) for some \( A_0 \in M \) and \( B_0 \in N \). Then it follows from Definition 2.3 that \( y \lesssim_{MN}^N x \implies y \leq x \) and \( z \gtrsim_{MN}^N x \implies z \geq x \). In particular, if \( S_0(P) \subseteq M, N \), then \( b \lesssim_{MN}^N a \) and \( c \gtrsim_{MN}^N a \) implies \( c \geq a \) for any \( a, b, c \in P \). More particularly, for any \( p_1, p_2, p_3 \in P \), we have \( p_1 \lesssim_{S_0} \iff p_1 \leq p_2 \) and \( p_3 \gtrsim_{S_0} \iff p_3 \geq p_2 \).

**Proposition 2.5.** Let \((P, M, N)\) be a PMN-space and \( x, y, z \in P \). Then

1. \( y \lesssim_{MN}^N x \) if and only if for every net \((x_i)_{i \in I} \) that \( MN \)-converges to \( x \), \( x_i \gtrsim y \) holds eventually.
2. \( z \gtrsim_{MN}^N x \) if and only if for every net \((x_i)_{i \in I} \) that \( MN \)-converges to \( x \), \( x_i \leq z \) holds eventually.

**Proof.** (1) Suppose \( y \lesssim_{MN}^N x \). If a net \((x_i)_{i \in I} \rightarrow x \), then there exist \( A \in M \) and \( B \in N \) such that \( \sup A = x = \inf B \), and for any \( a \in A \) and \( b \in B \), there exists \( i_0 \in I \) such that \( a \leq x_i \leq b \) for all \( i \geq i_0 \). According to Definition 2.3 (1), it follows that there exist \( A_0 = \{ a_1, a_2, ..., a_n \} \subseteq A \) and \( B_0 = \{ b_1, b_2, ..., b_m \} \subseteq B \) such that \( x \in \bigcap_{i \in I} \{ a_k \cap b_l : 1 \leq k \leq n \& 1 \leq l \leq m \} \subseteq y \). Take \( i_0 \in I \) with that \( i_0 \gtrsim_{MN}^N b_i \) for every \( i \in \{ 1, 2, ..., n \} \) and every \( j \in \{ 1, 2, ..., m \} \). Then \( x_i \in \bigcap_{i \in I} \{ a_k \cap b_l : 1 \leq k \leq n \& 1 \leq l \leq m \} \subseteq y \) for all \( i \geq i_0 \). This means \( x_i \gtrsim y \) holds eventually.

Conversely, suppose that for every net \((x_i)_{i \in I} \) that \( MN \)-converges to \( x \), \( x_i \gtrsim y \) holds eventually. For every \( A \in M \) and \( B \in N \) with \( \sup A = x = \inf B \), consider the net \((x_{(d,D)})_{(d,D) \in D_{(A,B)}} \) defined in Remark 2.2 (5). By Remark 2.2 (5), the net \((x_{(d,D)})_{(d,D) \in D_{(A,B)}} \rightarrow x \). So, there exists \( (d_0, D_0) \in D_{(A,B)} \) such that \( x_{(d,D)} = d \gtrsim y \) for all \( (d,D) \geq (d_0, D_0) \). Since \( (d,D) \geq (d_0, D_0) \) for all \( d \in D_0, x_{(d,D)} = d \gtrsim y \) for all \( d \in D_0 \). Thus, we have \( D_0 \subseteq y \).

It follows from the definition of \( D_{(A,B)} \) that there exist \( A_0 \subseteq A \) and \( B_0 \subseteq B \) such that \( D_0 = \bigcap_{i \in I} \{ a_k \cap b_l : a \in A_0 \& b \in B_0 \} \subseteq y \). This shows \( y \lesssim_{MN}^N x \).

The proof of (2) can be processed similarly. \( \square \)

**Remark 2.6.** Let \((P, M, N)\) be a PMN-space.

1. If \( M = \mathcal{D}(P) \) and \( N = \mathcal{I}(P) \), then \( \lesssim_{MN} = \lesssim_0 \) and \( \gtrsim_{MN} = \gtrsim_0 \).
2. If \( M = \mathcal{P}_0(P) \), then \( \lesssim_{P_0} = \lesssim_0 \) and \( \gtrsim_{P_0} = \gtrsim_0 \).

Given a PMN-space \((P, M, N)\), depending on the approximate relations \( \lesssim_{MN}^N \) and \( \gtrsim_{MN}^N \) on \( P \), we can define the \( MN \)-double continuity for the poset \( P \).

**Definition 2.7.** Let \((P, M, N)\) be a PMN-space. The poset \( P \) is called an \( MN \)-doubly continuous poset if for every \( x \in P \), there exist \( M_x \in M \) and \( N_x \in N \) such that

1. \( M_x \subseteq \nabla N_x \), \( N_x \subseteq \Delta M_x \) and \( M_x = x = \inf N_x \).
2. For any \( y \in \nabla M_x \) and \( z \in \Delta N_x \), \( n \cap m : m \in M_0 \& n \in N_0 \) \( \subseteq \Delta N_x \cap \nabla M_x \) for some \( M_0 \subseteq M_x \) and \( N_0 \subseteq N_x \).

By Remark 2.4 (4) and Definition 2.7, we have the following basic property about \( MN \)-doubly continuous posets:

**Proposition 2.8.** Let \((P, M, N)\) be a PMN-space and \( x, y, z \in P \). If the poset \( P \) is an \( MN \)-doubly continuous poset, then \( y \lesssim_{MN}^N x \) implies \( y \leq x \) and \( z \gtrsim_{MN}^N x \) implies \( z \geq x \).

**Example 2.9.** Let \((P, M, N)\) be a PMN-space.

1. If \( M = N = S_0(P) \), then by Remark 2.4 (4), we have \( \lesssim_{S_0} = \lesssim_0 \) and \( \gtrsim_{S_0} = \gtrsim_0 \). By Definition 2.7, one can easily check that \( P \) is an \( S_0 \)-doubly continuous poset.
2. If \( M = N = L_0(P) \), then by Definition 2.3, we have \( \lesssim_{L_0} = \lesssim_0 \) and \( \gtrsim_{L_0} = \gtrsim_0 \). It can be easily checked from Definition 2.7 that \( P \) is an \( L_0 \)-doubly continuous poset.
(3) Let \( M = \mathcal{D}(P) \) and \( N = \mathcal{F}(P) \). Then it is easy to check that if \( P \) is an \( \emptyset \)-doubly continuous poset which satisfies Condition (\( \delta \)), then it is a \( \mathcal{DF} \)-doubly continuous poset. Particularly, finite posets, chains and anti-chains, completely distributive lattices are all \( \mathcal{DF} \)-doubly continuous posets.

(4) Let \( M = N = \mathcal{P}_0(P) \). Then the poset \( P \) is \( \mathcal{P}_0\mathcal{P}_0 \)-double continuous if and only if it is \( O_2 \)-double continuous.

Thus, chains and finite posets are all \( \mathcal{P}_0\mathcal{P}_0 \)-doubly continuous posets.

Next, we are going to consider the \( MN \)-topology on posets, which is induced by the \( MN \)-convergence.

**Definition 2.10.** Given a PMN-space \( (P, M, N) \), a subset \( U \) of \( P \) is called an \( MN \)-open set if for every net \( (x_i)_{i \in I} \) with \( (x_i)_{i \in I} \to x \in U \), \( x_i \in U \) holds eventually.

Clearly, the family \( \mathcal{O}^M_N(P) \) consisting of all \( MN \)-open subsets of \( P \) forms a topology on \( P \). And this topology is called the \( MN \)-topology.

**Theorem 2.11.** Let \( (P, M, N) \) be a PMN-space. Then a subset \( U \) of \( P \) is an \( MN \)-open set if and only if for every \( M \in \mathcal{M} \) and \( N \in \mathcal{N} \) with sup \( M = x = \inf N \in U \), we have
\[
\bigcap\left\{ m \cap n : m \in M_0 \land n \in N_0 \right\} \subseteq U
\]
for some \( M_0 \subseteq M \) and \( N_0 \subseteq N \).

**Proof.** Suppose that \( U \) is a subset of \( P \) with the property that for any \( M \in \mathcal{M} \) and \( N \in \mathcal{N} \) with sup \( M = x = \inf N \in U \), let \( (x_{(d,D)})_{(d,D) \in \mathcal{D}(x_{d,D})} \) be the net defined in Remark 2.2 (5). Then the net \( (x_{(d,D)})_{(d,D) \in \mathcal{D}(x_{d,D})} \to x \). By the definition of \( MN \)-open set, the exists \( (d_0, D_0) \in \mathcal{D}_{(M,N)}^x \) such that \( x_{(d,D)} = d \in U \) for all \( (d, D) \in (d_0, D_0) \). Since \( (d, D) \geq (d_0, D_0) \) for all \( d \in D_0 \), \( x_{(d,D_0)} = d \in U \) for every \( d \in D_0 \), and thus \( D_0 \subseteq U \). It follows from the definition of the directed set \( \mathcal{D}_{(M,N)}^x \), that \( D_0 = \bigcap\left\{ m \cap n : m \in M_0 \land n \in N_0 \right\} \subseteq U \) for some \( M_0 \subseteq M \) and some \( N_0 \subseteq N \).

Conversely, assume that \( U \) is a subset of \( P \) with the property that for any \( M \in \mathcal{M} \) and \( N \in \mathcal{N} \) with sup \( M = x = \inf N \in U \), then exist \( M_0 = \{ m_1, m_2, ..., m_k \} \subseteq M \) and \( N_0 = \{ n_1, n_2, ..., n_l \} \subseteq N \) such that \( \bigcap\left\{ m_i \cap n_j : 1 \leq i \leq k \land 1 \leq j \leq l \right\} \subseteq U \). Let \( (x_i)_{i \in I} \) be a net that \( MN \)-converges to \( x \in U \). Then there exist \( M \in \mathcal{M} \) and \( N \in \mathcal{N} \) such that sup \( M = x = \inf N \in U \), and for every \( m \in M \) and \( n \in N \), \( m \cap n \subseteq \mathcal{U} \) holds eventually. This means that for every \( m_i \in M_0 \) and \( n_j \in N_0 \), there exist \( i, j, k \in I \) such that \( m_i \cap n_j \subseteq \mathcal{U} \) for all \( i \geq i_k \) and some \( j \). Then the \( x_k \in \bigcap\left\{ m_i \cap n_j : 1 \leq k \leq k \land 1 \leq j \leq l \right\} \subseteq U \) for all \( i \geq i_k \). Therefore, \( U \) is an \( MN \)-open subset of \( P \).

**Proposition 2.12.** Let \( (P, M, N) \) be a PMN-space in which \( P \) is an \( MN \)-doubly continuous poset, and \( y, z \in P \). Then \( \mathcal{A}_{\mathcal{M}N}^y \cap \mathcal{A}_{\mathcal{N}M}^z \in \mathcal{O}^N_M(P) \).

**Proof.** Suppose that \( M \in \mathcal{M} \) and \( N \in \mathcal{N} \) with sup \( M = x = \inf N \in \mathcal{A}_{\mathcal{M}N}^y \cap \mathcal{A}_{\mathcal{N}M}^z \). Since \( P \) is an \( MN \)-doubly continuous poset, there exist \( M_x \subseteq M \) and \( N_x \subseteq N \) satisfying condition (A1) and (A2) in Definition 2.7. This means that there exist \( M_x \subseteq M_x \subseteq \mathcal{A}_{\mathcal{M}N}^y \) and \( N_x \subseteq N_x \subseteq \mathcal{A}_{\mathcal{N}M}^z \) such that \( \bigcap\left\{ m_0 \cap n_0 : m_0 \in M_0 \land n_0 \in N_0 \right\} \subseteq \mathcal{A}_{\mathcal{M}N}^y \cap \mathcal{A}_{\mathcal{N}M}^z \). As \( M_x \subseteq M_x \subseteq \mathcal{A}_{\mathcal{M}N}^y \) and \( N_x \subseteq N_x \subseteq \mathcal{A}_{\mathcal{N}M}^z \), by Definition 2.3, there exist \( M_{M_0} \subseteq M \) and \( N_{N_0} \subseteq N \) such that \( \bigcap\left\{ m \cap n : m \in M_{M_0} \land n \in N_{N_0} \right\} \subseteq \mathcal{A}_{\mathcal{M}N}^y \cap \mathcal{A}_{\mathcal{N}M}^z \) for every \( m_0 \in M_{M_0} \) and \( n_0 \in N_{N_0} \). Take \( M_F = \bigcup\{ M_{M_0} : m_0 \in M_{M_0} \} \) and \( N_F = \bigcup\{ N_{M_0} : n_0 \in N_{M_0} \} \). Then it is easy to check that \( M_F \subseteq M \) and \( N_F \subseteq N \) and
\[
\bigcap\left\{ m \cap n : m \in M_F \land n \in N_F \right\} \subseteq \mathcal{A}_{\mathcal{M}N}^y \cap \mathcal{A}_{\mathcal{N}M}^z.
\]
So, it follows from Theorem 2.11 that \( \mathcal{A}_{\mathcal{M}N}^y \cap \mathcal{A}_{\mathcal{N}M}^z \in \mathcal{O}^N_M(P) \).

**Lemma 2.13.** Let \( (P, M, N) \) be a PMN-space in which \( P \) is an \( MN \)-doubly continuous poset. Then a net \( (x_i)_{i \in I} \to x \in P \iff (x_i)_{i \in I} \to x \).
Proof. From the definition of $\mathcal{O}_M^N(P)$, it is easy to see that a net
\[
(x_i)_{i \in I} \rightarrow x \in P \iff (x_i)_{i \in I} \rightarrow x.
\]

To prove the Lemma, it suffices to show that a net $(x_i)_{i \in I} \rightarrow x \in P$ implies $(x_i)_{i \in I} \rightarrow x$. Suppose a net $(x_i)_{i \in I} \rightarrow x$. Since $P$ is an $MN$-doubly continuous poset, there exist $M_x \in M$ and $N_x \in N$ such that $M_x \subseteq \Delta_M^N x$ and $\sup M_x = x = \inf N_x$. By Proposition 2.12, $x \in \Delta_M^N y \cap \Delta_M^N z \subseteq M_x$ for every $y \in M_x \subseteq \Delta_M^N x$ and every $z \in N_x \subseteq \Delta_M^N x$, and hence $x_i \in \Delta_M^N y \cap \Delta_M^N z$ holds eventually for every $y \in M_x \subseteq \Delta_M^N x$ and every $z \in N_x \subseteq \Delta_M^N x$. It follows from Proposition 2.8 that $y \leq x_i \leq z$ holds eventually for every $y \in M_x$ and $z \in N_x$. Thus $(x_i)_{i \in I} \rightarrow x$.

Lemma 2.14. Let $(P, M, N)$ be a PMN-space. If the MN-convergence in $P$ is topological, then $P$ is MN-doubly continuous.

Proof. Suppose that the MN-convergence in $P$ is topological. Then there exists a topology $\mathcal{T}$ on $P$ such that for every $x \in P$, a net $(x_i)_{i \in I} \rightarrow x$ if and only if $(x_i)_{i \in I} \rightarrow x$. Define $I_x = \{ (p, U) \in P \times N(x) : p \in U \}$, where $N(x)$ denotes the set of all open neighbourhoods of $x$ in the topological space $(P, \mathcal{T})$, i.e., $N(x) = \{ U \in \mathcal{T} : x \in U \}$. Define the preorder $\preceq$ on $I_x$ as follows:
\[
(\forall p_1, U_1, (p_2, U_2) \in I_x) (p_1, U_1) \preceq (p_2, U_2) \iff U_2 \subseteq U_1.
\]

Now one can easily see that $I_x$ is directed. Let $x_{(p, U)} = p$ for every $(p, U) \in I_x$. Then it is straightforward to check that the net $(x_{(p, U)})_{(p, U) \in I_x} \rightarrow x$, and thus $(x_{(p, U)})_{(p, U) \in I_x} \rightarrow x$. By Definition 2.1, there exist $M_x \in M$ and $N_x \in N$ such that $x = \sup M_x = \inf N_x$, and for every $m \in M_x$ and $n \in N_x$, there exists $(p^n_m, U^n_m) \in I_x$ such that $x_{(p^n_m)} = p \in \uparrow m \cap \downarrow n$ for all $(p, U) \supseteq (p^n_m, U^n_m)$. Since $(p, U^n_m) \supseteq (p^n_m, U^n_m)$ for every $p \in U^n_m$, $x_{(p, U^n_m)} = p \in \uparrow m \cap \downarrow n$ for every $p \in U^n_m$. This shows
\[
(\forall m \in M_x, n \in N_x) (\exists U^n_m \in N(x)) x \in U^n_m \subseteq \uparrow m \cap \downarrow n.
\]

For any $A \subseteq M$ and $B \subseteq N$ with sup $A = x = \inf B$, let $(x_{(d,D)})_{(d,D) \in D_{(A,B)}} \rightarrow x$ be the net defined as in Remark 2.2 (5). Then $(x_{(d,D)})_{(d,D) \in D_{(A,B)}} \rightarrow x$, and hence $(x_{(d,D)})_{(d,D) \in D_{(A,B)}} \rightarrow x$. This implies, by Remark 2.2 (6), that there exist $A_0 \subseteq A$ and $B_0 \subseteq B$ satisfying
\[
x \in \bigcap \{ \uparrow a \cap \downarrow b : a \in A_0 \& b \in B_0 \}
\subseteq U^n_m \subseteq \uparrow m \cap \downarrow n.
\]

Therefore, $m \in \Delta_M^N x$ and $n \in \Delta_M^N x$, and hence $M_x \subseteq \Delta_M^N x$ and $N_x \subseteq \Delta_M^N x$.

Let $y \in \Delta_M^N x$ and $z \in \Delta_M^N x$. Since $\sup M_x = x = \inf N_x$, by Definition 2.3, $\bigcap \{ \uparrow m \cap \downarrow n : m \in M_1 \& n \in N_1 \} \subseteq \uparrow y \cap \downarrow z$ for some $M_1 \subseteq M_x$ and $N_1 \subseteq N_x$. This concludes by Condition (*) and the finiteness of sets $M_1$ and $N_1$ that $\bigcap \{ U^n_m : m \in M_1 \& n \in N_1 \} \in N(x)$ and
\[
x \in \bigcap \{ U^n_m : m \in M_1 \& n \in N_1 \}
\subseteq \bigcap \{ \uparrow m \cap \downarrow n : m \in M_1 \& n \in N_1 \}
\subseteq \uparrow y \cap \downarrow z.
\]

Considering the net $(x_{(d,D)})_{(d,D) \in D_{(A,B)}} \rightarrow x$ defined in Remark 2.2 (5), we have $(x_{(d,D)})_{(d,D) \in D_{(A,B)}} \rightarrow x$. So, by Remark 2.2 (6), there exist $M_x \subseteq M_y$ and $N_x \subseteq N_y$ such that
\[
x \in \bigcap \{ \uparrow m \cap \downarrow n : m \in M_2 \& n \in N_2 \}
\subseteq \bigcap \{ U^n_m : m \in M_1 \& n \in N_1 \}
\subseteq \uparrow y \cap \downarrow z.
\]
Finally, we show $\cap\{m \cap n : m \in M_2 & n \in N_2\} \supseteq \bigwedge_{M} y \cap \bigvee_{N} z$. Let $(x_{(d), (d)})_{(d) \in D}^{M} \in D$ be the net defined in 2.2 (5) for any $M \in M$ and $N \in \mathcal{N}$ with $\sup M = \inf N = x \in \cap\{m \cap n : m \in M_2 & n \in N_2\}$. Then $(x_{(d), (d)})_{(d) \in D}^{M \cap N} \rightarrow x$, and thus $(x_{(d), (d)})_{(d) \in D}^{N} \rightarrow x$. This implies by Remark 2.2 (6) that there exist $M_0 \sqsubseteq M$ and $N_0 \sqsubseteq N$ satisfying

$$x' \in \cap\{m \cap n : m \in M_0 & n \in N_0\} \subseteq \cap\{U_m : m \in M_1 & n \in N_1\} \subseteq \uparrow y \cap \downarrow z.$$ 

Hence, we have $x' \in \bigwedge_{M} y \cap \bigvee_{N} z$ by Definition 2.3. This shows $\cap\{m \cap n : m \in M_2 & n \in N_2\} \subseteq \bigwedge_{M} y \cap \bigvee_{N} z$. Therefore, it follows from Definition 2.7 that $P$ is $M\mathcal{N}$-doubly continuous.

Combining Lemma 2.13 and Lemma 2.14, we obtain the following theorem.

**Theorem 2.15.** Let $(P, M, \mathcal{N})$ be a $PM\mathcal{N}$-space. Then the following statements are equivalent:

1. $P$ is an $M\mathcal{N}$-doubly continuous poset.
2. For any net $(x_i)_{i \in I} \in P$, $(x_i)_{i \in I} \rightarrow x$ if and only if $(x_i)_{i \in I} \rightarrow x$.
3. The $M\mathcal{N}$-convergence in $P$ is topological.

**Proof.** (1) $\Rightarrow$ (2): By Lemma 2.13.

(2) $\Rightarrow$ (3): It is clear.

(3) $\Rightarrow$ (1): By Lemma 2.14.

## 3 $M$-topology induced by lim-inf$_{M}$-convergence

In this section, the notion of lim-inf$_{M}$-convergence is reviewed and the $M$-topology on posets is defined. By exploring the fundamental properties of the $M$-topology, those posets under which the lim-inf$_{M}$-convergence is topological are precisely characterized.

By saying a $PM$-space, we mean a pair $(P, M)$ that contains a poset $P$ and a subfamily $M$ of $\mathcal{P}(P)$.

**Definition 3.1 ([8]).** Let $(P, M)$ be a PM-space. A net $(x_i)_{i \in I}$ in $P$ is said to lim-inf$_{M}$-converge to $x \in P$ if there exists $M \in M$ such that

(M1) $x \leq \sup M$;

(M2) for every $m \in M, x_i \geq m$ holds eventually.

In this case, we write $(x_i)_{i \in I} \rightarrow x$.

It is worth noting that both lim-inf-convergence and lim-inf$_{2}$-convergence [4] in posets are particular cases of lim-inf$_{M}$-convergence.

**Remark 3.2.** Let $(P, M)$ be a PM-space and $x, y \in P$.

1. Suppose that a net $(x_i)_{i \in I} \rightarrow x$ and $y \leq x$. Then $(x_i)_{i \in I} \rightarrow y$ by Definition 3.1. This concludes that the set of all lim-inf$_{M}$-convergent points of the net $(x_i)_{i \in I}$ in $P$ is a lower subset of $P$. Thus, the lim-inf$_{M}$-convergent points of the net $(x_i)_{i \in I}$ need not be unique.

2. If $P$ has the least element $\bot$ and $\emptyset \in M$, then we have $(x_i)_{i \in I} \rightarrow \bot$ for every net $(x_i)_{i \in I}$ in $P$. 


(3) For every $M \in \mathcal{M}$ with $\sup M \geq x$, we denote $F_M = \{ \{m : m \in M_0\} : M_0 \subseteq M\}$. Let $D_M = \{ (d, D) \in P \times F_M : d \in D \}$ be the preorder $\leq$ defined by

$$(\forall (d_1, D_1), (d_2, D_2) \in D_M) (d_1, D_1) \leq (d_2, D_2) \iff D_2 \subseteq D_1.$$ 

It is easy to see that the set $D_M$ is directed. Take $x_{(d,D)} = d$ for every $(d, D) \in D_M$. Then, by Definition 3.1, one can straightforwardly check that the net $x_{(d,D)} \to a$ for every $a \leq x$.

(4) If the net $(x_{(d,D)})(d,D) \in D_M$ defined in (3) converges to $p \in P$ with respect to some topology $\mathcal{T}$ on $P$, then for every open neighbourhood $U_p$ of $p$, there exists $M_0 \subseteq M$ such that $\bigcap\{ \{m : m \in M_0\} : M_0 \subseteq M\} \subseteq U_p$.

**Definition 3.3** ([8]). Let $(P, \mathcal{M})$ be a PM-space.

1. For $x, y \in P$, define $y \ll a(M)x$ if for every net $(x_i)_{i \in I}$ that lim-inf$_{M}$-converges to $x$, $x_i \geq y$ holds eventually.
2. The poset $P$ is said to be a(M)-continuous if for every $x \in P$, $x \ll a(M)x$ holds for every $a \in P$.

Given a PM-space $(P, \mathcal{M})$, the approximate relation $\ll a(M)$ on the poset $P$ can be equivalently characterized in the following proposition.

**Proposition 3.4.** Let $(P, \mathcal{M})$ be a PM-space and $x, y \in P$. Then $y \ll a(M)x$ if and only if for every $M \in \mathcal{M}$ with $\sup M \geq x$, there exists $M_0 \subseteq M$ such that

$$\bigcap\{ \{m : m \in M_0\} : M_0 \subseteq M\} \subseteq \uparrow y.$$ 

**Proof.** Suppose $y \ll a(M)x$. Let $(x_{(d,D)})_{(d,D) \in D_M}$ be the net defined in Remark 3.2 (3) for every $M \in \mathcal{M}$ with $\sup M = p \geq x$. Then the net $(x_{(d,D)})_{(d,D) \in D_M} \to x$. By Definition 3.3 (1), there exists $(d_0, D_0) \in D_M$ such that $x_{(d,D)} = d \geq y$ for all $(d, D) \geq (d_0, D_0)$. Since $(d, D) \geq (d_0, D_0)$ for every $d \in D_0$, $x_{(d,D)} = d \geq y$ for every $d \in D_0$. So $D_0 \subseteq \uparrow y$. This shows that there exists $M_0 \subseteq M$ such that $D_0 = \bigcap\{ \{m : m \in M_0\} : M_0 \subseteq M\} \subseteq \uparrow y$.

Conversely, suppose that for every $M \in \mathcal{M}$ with $\sup M \geq x$, there exists $M_0 \subseteq M$ such that $\bigcap\{ \{m : m \in M_0\} : M_0 \subseteq M\} \subseteq \uparrow y$. Let $x_{(i)}_{i \in I}$ be a net that lim-inf$_{M}$-converges to $x$. Then, by Definition 3.1, there exists $M \in \mathcal{M}$ such that $\sup M = p \geq x$, and for every $m \in M$, there exists $i_m \in I$ such that $x_i \geq m$ for all $i \geq i_m$. Take $i_0 \in I$ with that $i_0 \geq i_m$ for every $m \in M_0 \subseteq M$, we have that $x_i \in \bigcap\{ \{m : m \in M_0\} \subseteq \uparrow y$ for all $i \geq i_0$. This shows that $x_i \geq y$ holds eventually. Thus, by Definition 3.3 (1), we have $y \ll a(M)x$.

**Remark 3.5.** Let $(P, \mathcal{M})$ be a PM-space and $x, y \in P$.

1. If there is no $M \in \mathcal{M}$ such that $\sup M \geq x$, then $p \ll a(M)x$ for every $p \in P$. And, if the poset $P$ has the least element $\bot$, then $\bot \ll a(M)p$ for every $p \in P$.
2. The implication $y \ll a(M)x \implies y \leq x$ may not be true. For example, let $P = \{0, 1, 2, \ldots\}$ be in the discrete order $\leq$ defined by

$$(\forall i, j \in P) i \leq j \iff i = j.$$ 

And let $\mathcal{M} = \{\{2\}\}$. Then, it is easy to see from Remark 3.5 (1) that $0 \ll a(M)1$ and $0 \leq 1$.

3. Assume the PM-space $(P, \mathcal{M})$ has the property that for every $p \in P$, there exists $M_p \in \mathcal{M}$ such that $\sup M_p = p$. Then, by Proposition 3.4, we have

$$(\forall q, r \in P) q \ll a(M)r \Rightarrow q \leq r.$$ 

For more interpretations of the approximate relation $\ll a(M)$ on posets, the readers can refer to Example 3.2 and Remark 3.3 in [8].

For simplicity, given a PM-space $(P, \mathcal{M})$ and $x \in P$, we will denote

$$\nabla_M x = \{ y \in P : y \ll a(M)x \};$$

---

2 From the logical point of view, we stipulate $\bigcap\{ \{m : m \in M_0\} = P$ if $M_0 = \emptyset$. 

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Based on the approximate relation $\ll_{\alpha(M)}$ on posets, the $\alpha^*(\mathcal{M})$-continuity can be defined for posets in the following:

**Definition 3.6.** Let $(P, \mathcal{M})$ be a $PM$-space. The poset $P$ is called an $\alpha^*(\mathcal{M})$-continuous poset if for every $x \in P$, there exists $M_x \in \mathcal{M}$ such that

$(01) \sup M_x = x$ and $M_x \subseteq \nabla M_x$. And,

$(02)$ for every $y \in \nabla M_x$, there exists $F \subseteq M_x$ such that $\bigcap \{\{f : f \in F\} \subseteq \triangle M_x\} \subseteq y$.

Noticing Remark 3.5 (3), we have the following proposition about $\alpha^*(\mathcal{M})$-continuous posets.

**Proposition 3.7.** Let $(P, \mathcal{M})$ be a $PM$-space in which the poset $P$ is $\alpha^*(\mathcal{M})$-continuous. Then

$$(\forall x, y \in P) y \ll_{\alpha(M)} x \Rightarrow y \subseteq x.$$ 

The following examples of $\alpha^*(\mathcal{M})$-continuous posets can be formally checked by Definition 3.6.

**Example 3.8.** Let $(P, \mathcal{M})$ be a $PM$-space.

1. If $P$ is a finite poset, then $P$ is an $\alpha^*(\mathcal{M})$-continuous poset if and only if for every $x \in P$, there exists $M_x \in \mathcal{M}$ such that $\sup M_x = x$.

2. Let $\mathcal{M} = \mathcal{L}(P)$. Then $P$ is an $\alpha^*(\mathcal{L})$-continuous poset. This means that every poset is $\alpha^*(\mathcal{L})$-continuous.

3. Let $\mathcal{M} = \mathcal{D}(P)$. Then we have $\ll_{\alpha(\mathcal{D})} = \ll_{\alpha(\mathcal{D})}$ (see Example 3.2 (1) in [8]). The poset $P$ is a continuous poset if and only if it is an $\alpha^*(\mathcal{D})$-continuous poset. In particular, finite posets, chains, anti-chains and completely distributive lattices are all $\alpha^*(\mathcal{D})$-continuous.

4. Let $\mathcal{M} = \mathcal{D}(P)$. If $P$ is a finite poset (resp. chain, anti-chain), then $P$ is an $\alpha^*(\mathcal{D})$-continuous poset.

**Proposition 3.9.** Let $(P, \mathcal{M})$ be a $PM$-space. If $P$ is an $\alpha(\mathcal{M})$-continuous poset, and $\{y \in P : (\exists z \in P) y \ll_{\alpha(M)} z \ll_{\alpha(M)} a\} \in \mathcal{M}$ for every $a \in P$, then $P$ is an $\alpha^*(\mathcal{M})$-continuous poset.

**Proof.** Suppose that $P$ is an $\alpha(\mathcal{M})$-continuous poset, and $\{y \in P : (\exists z \in P) y \ll_{\alpha(M)} z \ll_{\alpha(M)} a\} \in \mathcal{M}$ for every $a \in P$. Take $M_a = \nabla M_a$. Then it is easy to see that $\sup M_a = a$ and $M_a \subseteq \nabla M_a$. By Remark 3.3 (4) in [8], we have $\sup \{y \in P : (\exists z \in P) y \ll_{\alpha(M)} z \ll_{\alpha(M)} a\} = a$. This implies, by Proposition 3.4 and Remark 3.5 (2), that for every $y \in \nabla M_a$, there exist $\{y_1, y_2, \ldots, y_n\}$, $\{z_1, z_2, \ldots, z_n\} \subseteq M_a = \nabla M_a$ such that

$$\bigcap \{\{z_i : i \in \{1, 2, \ldots, n\}\} \subseteq \nabla M_y \subseteq \nabla y.$$ 

and $y_i \ll_{\alpha(M)} z_i \ll_{\alpha(M)} a$ for every $i \in \{1, 2, \ldots, n\}$. Next, we show $\bigcap \{\{z_i : i \in \{1, 2, \ldots, n\}\} \subseteq \triangle M_y \subseteq \nabla M_y$. For every $M \in \mathcal{M}$ with $\sup M > b \in \bigcap \{\{z_i : i \in \{1, 2, \ldots, n\}\}$, by Proposition 3.4, there exists $M_i \subseteq M$ such that $\bigcap \{\{m : m \in M_i\} \subseteq \nabla y_i$ for every $i \in \{1, 2, \ldots, n\}$. Take $M_0 = \bigcup \{M_i : i \in \{1, 2, \ldots, n\}\}$. Then $M_0 \subseteq M$ and

$$\bigcap \{\{m : m \in M_0\} \subseteq \nabla y_i \subseteq \nabla y.$$ 

This shows $y \ll_{\alpha(M)} b$ for every $b \in \bigcap \{\{z_i : i \in \{1, 2, \ldots, n\}\}$. Hence, $\bigcap \{\{z_i : i \in \{1, 2, \ldots, n\}\} \subseteq \triangle M_y$. Thus $P$ is an $\alpha^*(\mathcal{M})$-continuous poset. 

The fact that an $\alpha^*(\mathcal{M})$-continuous poset $P$ in a $PM$-space $(P, \mathcal{M})$ may not be $\alpha(\mathcal{M})$-continuous can be demonstrated in the following example.
For every two conditions:

\[ \text{order} \text{ is an } R (\text{Theorem 3.12).} \]

De/f_inition 3.11. Let \( P \subseteq \mathbb{R} \) be a PM-space. A subset \( V \) of \( P \) is said to be \( M \)-open if for every net \( \{x_i\}_{i \in I} \rightarrow x \in V \), \( x_i \in V \) holds eventually.

Given a PM-space \(( P, M )\), one can formally verify that the set of all \( M \)-open subsets of \( P \) forms a topology on \( P \). This topology is called the \( M \)-topology, and denoted by \( O_M(P) \).

The following Theorem is an order-theoretical characterization of \( M \)-open sets.

**Theorem 3.12.** Let \( ( P, M ) \) be a PM-space. Then a subset \( V \) of \( P \) is \( M \)-open if and only if it satisfies the following two conditions:

\[ (V1) \forall V \in \mathcal{V} \text{, i.e., } V \text{ is an upper set.} \]

\[ (V2) \forall \text{every } M \in M \text{ with sup } M \in V \text{, there exists } M_0 \subseteq M \text{ such that } \bigcap \{ \uparrow m : m \in M_0 \} \subseteq V. \]

**Proof.** Suppose that \( V \) is an \( M \)-open subset of \( P \). By Remark 3.2 (1), it is easy to see that \( V \) is an upper set. Let \( \{x_{(d,D)} \}_{(d,D) \in D^+_M} \) be the net defined in Remark 3.2 (3) for every \( M \in M \) with sup \( M = x \). Then \( \{x_{(d,D)} \}_{(d,D) \in D^+_M} \rightarrow x \in V \). This implies, by De/f_inition 3.11, that there exists \( (d_0, D_0) \in D^+_M \) such that \( x_{(d,D)} = d \in V \) for all \( (d, D) \geq (d_0, D_0) \). Since \( (d, D) \geq (d_0, D_0) \) for all \( d \in D_0, x_{(d,D)} = d \in V \) for all \( d \in D_0 \). This shows \( D_0 \subseteq V \). Thus there exists \( M_0 \subseteq M \) such that \( D_0 = \bigcap \{ \uparrow m : m \in M_0 \} \subseteq V \).

Conversely, suppose \( V \) is a subset of \( P \) which satisfies Condition (V1) and (V2). Let \( \{x_i\}_{i \in I} \) be a net that \( \liminf_M \)-converges to \( x \in V \). Then there exists \( M \in M \) such that sup \( M = y \geq x \in V = \uparrow V \) (hence, \( y \in V \)), and for every \( m \in M \), there exists \( i_m \in I \) such that \( x_{i_m} \geq m \) for all \( i \geq i_m \). By Condition (V2), we have that \( \bigcap \{ \uparrow m : m \in M_0 \} \subseteq V \) for some \( M_0 \subseteq M \). Take \( i_0 \in I \) with that \( i_0 \geq i_m \) for all \( m \in M_0 \). Then \( x_i \in \bigcap \{ \uparrow m : m \in M_0 \} \subseteq V \) for all \( i \geq i_0 \). This shows that \( V \) is an \( M \)-open set. \( \square \)

Recall that given a topological space \(( X, \mathcal{T} )\) and a point \( x \in P \), a family \( \mathcal{B}(x) \) of open neighbourhoods of \( x \) is called a base for the topological space \(( X, \mathcal{T} )\) at the point \( x \) if for every neighbourhood \( V \) of \( x \) there exists an \( U \in \mathcal{B}(x) \) such that \( x \in U \subseteq V \).

If the poset \( P \) in a PM-space \(( P, M )\) is an \( a^*(M) \)-continuous poset, we provide a base for the topological space \(( P, O_M(P) )\) at a point \( x \in P \).

**Proposition 3.13.** Let \( ( P, M ) \) be a PM-space in which the poset \( P \) is \( a^*(M) \)-continuous. Then \( \uparrow_M x \in O_M(P) \) for every \( x \in P \).

**Proof.** One can readily see, by Proposition 3.4, that \( \uparrow_M x \) is an upper subset of \( P \) for every \( x \in P \). For every \( M \in M \) with sup \( M = y \in \uparrow_M x \), by De/f_inition 3.6 (O1) there exists \( M_y \subseteq M \) such that \( M_y \subseteq \mathbb{Y} \) and sup \( M_y = y \). Since \( x \leq_M y \), by De/f_inition 3.6 (O2), we have that \( \bigcap \{ \uparrow m_i : i \in \{1, 2, ..., n\} \} \subseteq \uparrow_M x \) for some \( \{m_1, m_2, ..., m_n\} \subseteq M_y \). Observing \( \{m_1, m_2, ..., m_n\} \subseteq M_y \subseteq \mathbb{Y} \), we can conclude that there exists \( M_i \subseteq M \) such that \( \bigcap \{ \uparrow a : a \in M_i \} \subseteq \uparrow m_i \) for every \( i \in \{1, 2, ..., n\} \). Let \( M_0 = \bigcup \{ M_i : i \in \{1, 2, ..., n\} \} \). Then \( M_0 \subseteq M \) and

\[
\bigcap \{ \uparrow m : m \in M_0 \} \\
\subseteq \bigcup \{ \uparrow m_i : i \in \{1, 2, ..., n\} \} \\
\subseteq \uparrow_M x.
\]

This shows, by Theorem 3.12, that \( \uparrow_M x \in O_M(P) \) for every \( x \in P \). \( \square \)
Lemma 3.15. Let \((P, M)\) be a PM-space in which the poset \(P\) is \(\alpha'(M)\)-continuous and \(x \in P\). Then \(\bigcap \{ \forall_{a} : a \in A \} : A \subseteq \bigvee_{x} M \) is a base for the topological space \((P, O_{M}(P))\) at the point \(x\).

Proof. Clearly, by Proposition 3.13, we have \(\bigcap \{ \forall_{a} : a \in A \} : A \subseteq \bigvee_{x} M \) for every \(A \subseteq \bigvee_{x} M\) and \(x \in U\). Since \(P\) is an \(\alpha'(M)\)-continuous poset, there exists \(M_{x} \in M\) such that \(M_{x} \subseteq \bigvee_{x} M\) and \(\sup M_{x} = x \in U\). By Theorem 3.12, it follows that \(\bigcap \{ \forall m : m \in M_{0} \} \subseteq U\) for some \(M_{0} \subseteq M\). So, from Proposition 3.7, we have

\[
x \in \bigcap \{ \forall_{a} m : m \in M_{0} \}
\]

\[
\subseteq \bigcap \{ \forall m : m \in M_{0} \} \subseteq U.
\]

Thus, \(\{ \forall_{a} : a \in A \} : A \subseteq \bigvee_{x} M\) is a base for the topological space \((P, O_{M}(P))\) at the point \(x\).

In the rest, we are going to establish a characterization theorem which demonstrates the equivalence between the lim-inf\(_{M}\)-convergence being topological and the \(\alpha'(M)\)-continuity of the poset in a given PM-space.

Lemma 3.16. Let \((P, M)\) be a PM-space. If \(P\) is an \(\alpha'(M)\)-continuous poset, then a net

\[
(x_{i})_{i \in I} \xrightarrow{M} x \iff (x_{i})_{i \in I} \xrightarrow{O_{M}(P)} x.
\]

Proof. By the definition of \(O_{M}(P)\), it is easy to see that a net

\[
(x_{i})_{i \in I} \xrightarrow{M} x \implies (x_{i})_{i \in I} \xrightarrow{O_{M}(P)} x.
\]

To prove the Lemma, we only need to show that a net \((x_{i})_{i \in I} \xrightarrow{O_{M}(P)} x \implies (x_{i})_{i \in I} \xrightarrow{M} x\) holds eventually. By Proposition 3.13, there exists \(M_{x} \in M\) such that \(M_{x} \subseteq \bigvee_{x} M\) and \(\sup M_{x} = x\). By Proposition 3.13, we have \(x \in \bigvee_{x} M\) for every \(y \in \bigvee_{x} M\). Hence, \(x_{i} \in \bigvee_{x} M\) holds eventually. This implies, by Proposition 3.7, that \(x_{i} \in \bigvee_{x} M\) holds eventually. By the definition of lim-inf\(_{M}\)-convergence, we have \((x_{i})_{i \in I} \xrightarrow{M} x\).

The converse direction is under the following Lemma.

Lemma 3.16. Let \((P, M)\) be a PM-space. If the lim-inf\(_{M}\)-convergence in \(P\) is topological, then \(P\) is an \(\alpha'(M)\)-continuous poset.

Proof. Suppose that the lim-inf\(_{M}\)-convergence in \(P\) is topological. Then there exists a topology \(\mathcal{T}\) such that for every \(x \in P\), a net

\[
(x_{i})_{i \in I} \xrightarrow{M} x \iff (x_{i})_{i \in I} \xrightarrow{\mathcal{T}} x.
\]

Define \(I_{x} = \{(p, V) \in P \times N(x) : p \in V\}\), where \(N(x)\) is the set of all open neighbourhoods of \(x\), namely, \(N(x) = \{ V \in \mathcal{T} : x \in V\}\). Define also the preorder \(\preceq\) on \(I_{x}\) as follows:

\[
(p_{1}, V_{1}), (p_{2}, V_{2}) \in I_{x} \iff (p_{1}, V_{1}) \preceq (p_{2}, V_{2}) \iff V_{2} \subseteq V_{1}.
\]

It is easy to see that \(I_{x}\) is directed. Now, let \(x_{(p, V)} = p\) for every \((p, V) \in I_{x}\). Then one can readily check that the net \((x_{(p, V)})_{(p, V) \in I_{x}} \xrightarrow{\mathcal{T}} x\), and hence \((x_{(p, V)})_{(p, V) \in I_{x}} \xrightarrow{M} x\). This means that there exists \(M_{x} \in M\) such that \(\sup M_{x} = x\), and for every \(m \in M_{x}\), there exists \((p_{m}, V_{m}) \in I_{x}\) with that \(x_{(p, V)} = p \geq m\) for all \((p, V) \geq (p_{m}, V_{m})\). Since \((p, V_{m}) \supseteq (p_{m}, V_{m})\) for all \(p \in V_{m}\), we have \(x_{(p, V_{m})} = p \geq m\) for all \(p \in V_{m}\). This shows

\[
(\forall m \in M_{x}) (\exists V_{m} \in N(x)) x \in V_{m} \subseteq \uparrow m.
\]

Next we prove \(M_{x} \subseteq \bigvee_{x} M\). For every \(m \in M_{x}\) and every \(M \in M\) with \(\sup M \geq x\), let \((x_{(d, D)})_{(d, D) \in D_{m}} \xrightarrow{M} x\) be the net defined in Remark 3.2 (3). Then the net \((x_{(d, D)})_{(d, D) \in D_{m}} \xrightarrow{\mathcal{T}} x\), and thus \((x_{(d, D)})_{(d, D) \in D_{m}} \xrightarrow{M} x\). It follows from
Remark 3.2 (4) that there exists $M_0 \subseteq M$ such that $x \in \bigcap\{a : a \in M_0\} \subseteq V_m$. By Condition (**), we have $x \in \bigcap\{a : a \in M_0\} \subseteq V_m \subseteq \uparrow m$. So, $m \ll_{\alpha(M)} x$. This shows $M_x \subseteq \nabla_M x$.

Let $y \in \nabla_M x$. Then there exists $\{m_1, m_2, ..., m_n\} \subseteq M_x$ such that $\bigcap\{m_i : i \in \{1, 2, ..., n\}\} \subseteq \uparrow y$ as $M_x \in \mathcal{M}$ and sup $M_x \geq x$. By Condition (**), it follows that $\bigcap\{V_m : i \in \{1, 2, ..., n\}\} \subseteq \bigcap\{m_i : i \in \{1, 2, ..., n\}\} \subseteq \uparrow y$. Considering the net $(x_{(d,D)})_{(d,D) \in D_M}$ defined in Remark 3.2 (3), we have $(x_{(d,D)})_{(d,D) \in D_M} \rightarrow x$, and hence $(x_{(d,D)})_{(d,D) \in D_M} \rightarrow x$. This implies, by Remark 3.2 (4), that

$$\bigcap\{\downarrow b : b \in M_0\} \subseteq \bigcap\{V_m : i \in \{1, 2, ..., n\}\} \subseteq \bigcap\{\downarrow m_i : i \in \{1, 2, ..., n\}\} \subseteq \uparrow y$$

for some $M_0 \subseteq M_x$. Finally, we show $\bigcap\{\downarrow b : b \in M_0\} \subseteq \Delta_M y$. For every $x' \in \bigcap\{\downarrow b : b \in M_0\}$ and every $M' \in \mathcal{M}$ with sup $M' \geq x'$, let $(x_{(d,D)})_{(d,D) \in D_M}$ be the net defined in Remark 3.2 (3). Then $(x_{(d,D)})_{(d,D) \in D_M} \rightarrow x'$, and thus $(x_{(d,D)})_{(d,D) \in D_M} \rightarrow x'$. It follows from Condition (***) and Remark 3.2 (4) that there exists $M_0 \subseteq M'$ such that

$$\bigcap\{\downarrow a' : a' \in M_0\} \subseteq \bigcap\{V_m : i \in \{1, 2, ..., n\}\} \subseteq \bigcap\{\downarrow m_i : i \in \{1, 2, ..., n\}\} \subseteq \uparrow y.$$ 

This shows $x' \in \Delta_M y$, and thus $\bigcap\{\downarrow b : b \in M_0\} \subseteq \Delta_M y$. Therefore, $P$ is an $\alpha^*(\mathcal{M})$-continuous poset.

Combining Lemma 3.15 and Lemma 3.16, we deduce the following result.

**Theorem 3.17.** Let $(P, \mathcal{M})$ be a PM-space. The following statements are equivalent:

1. $P$ is an $\alpha^*(\mathcal{M})$-continuous poset.
2. For any net $(x_i)_{i \in I}$ in $P$, $(x_i)_{i \in I} \rightarrow x \in P$ if and only if $(x_i)_{i \in I} \rightarrow_{\mathcal{M}} x$.
3. The lim-inf$^\mathcal{M}$-convergence in $P$ is topological.

**Proof.** (1) $\Rightarrow$ (2): By Lemma 3.15.

(2) $\Rightarrow$ (3): Clear.

(3) $\Rightarrow$ (1): By Lemma 3.16. 

**Corollary 3.18** ([8]). Let $(P, \mathcal{M})$ be a PM-space with $S_0(P) \subseteq \mathcal{M} \subseteq \mathcal{P}(P)$. Suppose $\nabla_M a \in \mathcal{M}$ and $\{y \in P : (\exists z \in P) y \ll_{\alpha(M)} z \ll_{\alpha(M)} a\} \in \mathcal{M}$ holds for every $a \in P$. Then the lim-inf$^\mathcal{M}$-convergence in $P$ is topological if and only if $P$ is an $\alpha(M)$-continuous poset.

**Proof.** ($\Rightarrow$): To show the $\alpha(M)$-continuity of $P$, it suffices to prove sup $\nabla_M a = a$ for every $a \in P$. Since the lim-inf$^\mathcal{M}$-convergence in $P$ is topological, by Theorem 3.17, $P$ is an $\alpha^*(\mathcal{M})$-continuous poset. This implies that there exists $M_a \in \mathcal{M}$ such that sup $M_a \subseteq \nabla_M a$ and sup $M_a = a$ for every $a \in P$. By Proposition 3.7, we have $\nabla_M a \subseteq \downarrow a$. So sup $\nabla_M a = a$.

($\Leftarrow$): By Proposition 3.9 and Theorem 3.17.

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