Research Article

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New topology in residuated lattices

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Abstract: In this paper, by using the notion of upsets in residuated lattices and defining the operator $D_a(X)$, for an upset $X$ of a residuated lattice $L$ we construct a new topology denoted by $\tau_a$ and $(L, \tau_a)$ becomes a topological space. We obtain some of the topological aspects of these structures such as connectivity and compactness. We study the properties of upsets in residuated lattices and we establish the relationship between them and filters. O. Zahiri and R. A. Borzooei studied upsets in the case of BL-algebras, their results become particular cases of our theory, many of them work in residuated lattices and for that we offer complete proofs. Moreover, we investigate some properties of the quotient topology on residuated lattices and some classes of semitopological residuated lattices. We give the relationship between two types of quotient topologies $\tau_{a/F}$ and $\tau_a$. Finally, we study the uniform topology $\tau_{\Lambda}$ and we obtain some conditions under which $(L/J, \tau_J)$ is a Hausdorff space, a discrete space or a regular space relative to the uniform topology. We discuss briefly the applications of our results on classes of residuated lattices such as divisible residuated lattices, MV-algebras and involutive residuated lattices and we find that any of this subclasses of residuated lattices with respect to these topologies form semitopological algebras.

Keywords: Residuated lattice, divisible residuated lattice, Semitopological algebra

MSC: 03G10, 03G25, 06D05, 06D35, 08A72

1 Introduction

Residuation is a fundamental concept of ordered structures and categories. The origin of residuated lattices is in Mathematical Logic without contraction. The general definition of a residuated lattice was given by Galatos et al. (2007)[1]. They first developed the structural theory of this kind of algebra about residuated lattices.

Hajek (1998)[2] introduced the notion of BL-algebras and the concepts of filters and prime filters in BL-algebras in order to provide an algebraic proof of the completeness theorem of basic logic (BL, for short), arising from the continuous triangular norms, familiar in the fuzzy logic frame-work. Using prime filters in BL-algebras, he proved the completeness of Basic Logic. Soon after, Turunen (1999)[3] published a study on BL-algebras and their deductive systems.

A weaker logic than BL called Monoidal t-norm based logic (MTL, for short) was defined by Esteva and Godo (2001) [4] and proved by Jenei and Montagna (2002)[5] to be the logic of left continuous t-norms and their residua. The algebraic counterpart of this logic is MTL-algebra, also introduced by Esteva and Godo (2001)[4]. In Esteva and Godo (2001)[4] a residuated lattice $L$ is called MTL-algebra if the prelinearity property holds in $L$. 

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Recently, Turunen and Mertanen (2008)[6] and D. Buşneag et al. (2013)[7] defined the notion of semidivisible residuated lattice and investigated their properties. Also, D. Buşneag et al. (2015) [8] investigated the notion of Stonean residuated lattices and they discussed it from the view of ideal theory. Semitopological and topological BL-algebras were defined by R. A. Borzooei et al. (2011)[9], and they establish the relationships between them. A. Borumand Saeid and S. Motamed (2009)[10] introduced the set notion of Stonean residuated lattices and they discussed it from the view of ideal theory.

In this section, we recall some basic notions relevant to residuated lattices which will need in the sequel.

**Definition 2.1** ([1]). A residuated lattice is an algebra \((L, \vee, \wedge, \circ, \rightarrow, 0, 1)\) of type \((2, 2, 2, 2, 0, 0)\) such that

- \((L_1)\) \((L, \vee, \wedge, 0, 1)\) is a bounded lattice;
- \((L_2)\) \((L, \circ, 1)\) is a commutative monoid;
- \((L_3)\) \(\circ\) and \(\rightarrow\) form an adjoint pair, i.e., \(a \circ x \leq b\) iff \(x \leq a \rightarrow b\).

We call them simply residuated lattices. For examples of residuated lattices see [12–17].

In what follows (unless otherwise specified) we denote by \(L\) a residuated lattice. If \(L\) is totally ordered, then \(L\) is called a chain.

For \(x \in L\) and \(n \geq 1\) we define \(x^n = x \rightarrow 0\), \(x^{**} = (x^*)^*\), \(x^0 = 1\) and \(x^n = x^{n-1} \circ x\).

We refer to [3–21] for detailed proofs of these well-known results.

**Lemma 2.2.** Let \(L\) be a residuated lattice. Then for every \(x, y, z \in L\), we have:

- \((r_1)\) \(x \rightarrow x = 1\), \(x \rightarrow 1 = 1\), \(1 \rightarrow x = x\);
- \((r_2)\) \(x \leq y\) iff \(x \rightarrow y = 1\);
- \((r_3)\) If \(x \vee y = 1\), then \(x \circ y = x \wedge y\);\n- \((r_4)\) If \(x \leq y\), then \(z \circ x \leq z \circ y\), \(z \rightarrow x \leq z \rightarrow y\), \(y \rightarrow z \leq x \rightarrow z\);
- \((r_5)\) \(x \rightarrow (y \rightarrow z) = (x \circ y) \rightarrow z = y \rightarrow (x \rightarrow z)\);
Following the above mentioned literature, we consider the identities:

(i) \( x \land y = x \land (x \rightarrow y) \) (divisibility);
(ii) \( (x \land y)^* = [x^* \land (x' \rightarrow y')]^* \) (semi-divisibility);
(iii) \( x \land y \lor (y \rightarrow x) = 1 \) (pre-linearity);
(iv) \( x^* \lor x^{**} = 1; \)
(v) \( x^2 = x; \)
(vi) \( x = x^{**}. \)

Then the residuated lattice \( L \) is called:

(i) Divisible if \( L \) verifies (i);
(ii) Semi-divisible if \( L \) verifies (ii);
(iii) MTL-algebra if \( L \) verifies (iii);
(iv) BL-algebra if \( L \) verifies (i) and (iii);
(v) Stonean if \( L \) verifies (iv);
(vi) Involutive if \( L \) verifies (vi).

An MV-algebra is an algebra \( L = (L, \oplus, *, 0) \) of type \( (2, 1, 0) \) satisfying the following equations:

\[
\begin{align*}
(mv_1) \quad x \oplus (y \oplus z) &= (x \oplus y) \oplus z; \\
(mv_2) \quad x \oplus y &= y \oplus x; \\
(mv_3) \quad x \oplus 0 &= x; \\
(mv_4) \quad x^* &= x; \\
(mv_5) \quad x \oplus 0^* &= 0^*; \\
(mv_6) \quad (x^* \oplus y)^* \oplus z &= (y^* \oplus x)^* \oplus z, \text{ for all } x, y, z \in L.
\end{align*}
\]

Note that axioms \((mv_1) - (mv_3)\) state that \((L, \oplus, *, 0)\) is a commutative monoid.

Every BL-algebra \( L \) with \( x^{**} = x \) for all \( x \in L \), is an MV-algebra.

By Lemma 2.2, in the case of divisible residuated lattices we have:

**Corollary 2.3** ([7]). If \( L \) is a divisible residuated lattice, then for every \( x, y \in L \) we have:

\[
\begin{align*}
(r_{21}) \quad x^{**} \rightarrow x^* &= 0; \\
(r_{22}) \quad (x \rightarrow y)^{**} &= x^{**} \rightarrow y^{**}; \\
(r_{23}) \quad (x \land y)^{**} &= x^{**} \land y^{**}; \\
(r_{24}) \quad x \oplus (y \land z) &= (x \oplus y) \land (x \oplus z); \\
(r_{25}) \quad x \land (y \lor z) &= (x \land y) \lor (x \land z).
\end{align*}
\]
On any residuated lattice \( L ([6, 8]) \) we may define an operator \( \oplus \) by setting for all \( x, y \in L \),

\[
x \oplus y = (x^* \odot y^*)^*.
\]

(1)

By \((r_5)\), the identity (1) is equivalent with

\[
x \oplus y = x^* \rightarrow y'^* = y'^* \rightarrow x'^*, \text{ for all } x, y \in L.
\]

(2)

Lemma 2.4 ([8]). Let \( L \) be a residuated lattice and \( x, y, z \in L \). Then:

\[
\begin{align*}
(r_{26}) & \quad x \oplus 0 = x'^*, \ x \oplus 1 = 1, x \oplus x^* = 1; \\
(r_{27}) & \quad x \oplus y = y \oplus x, x, y \leq x \oplus y; \\
(r_{28}) & \quad x \oplus (y \oplus z) = (x \oplus y) \oplus z; \\
(r_{29}) & \quad \text{If } x \leq y, \text{ then } x \oplus z \leq y \oplus z.
\end{align*}
\]

In what follows we will establish other necessary properties of operator \( \oplus \) in a residuated lattice \( L \).

Proposition 2.5. Let \( L \) be a residuated lattice and \( x, y, z \in L \). Then:

\[
\begin{align*}
(r_{30}) & \quad (x \oplus y)^* = x \oplus y = x'^* \odot y'^*; \\
(r_{31}) & \quad \text{If } x \lor y = 1, \text{ then } x \oplus y = 1; \\
(r_{32}) & \quad x \oplus (y \land z)^* = (x \oplus y)^* \land (x \oplus z)^*; \\
(r_{33}) & \quad x \oplus (y' \land z')^* = (x \oplus y) \land (x \oplus z).
\end{align*}
\]

Proof. \((r_{30})\). We obtain successively \((x \oplus y)^* = (x^* \odot y^*)^* (r_{35}) (x^* \odot y^*)^* = x \oplus y (r_{35}) [(x^*^*)^* \odot (y'^*)^*]^* = x'^* \odot y'^*\).

\((r_{31})\). Since \( x \leq x'^* \), \( y \leq y'^* \) and \( x'^*, y'^* \leq x \oplus y \) implies \( x \lor y \leq x \oplus y \). Since \( 1 = x \lor y \leq x \oplus y \), then \( x \oplus y = 1 \).

\((r_{32})\). By \((r_9), (r_{12}), (r_{15})\) and (2) we obtain successively \( x \oplus (y \lor z)^* \odot (2) x^* \rightarrow (y \lor z)^* (r_{12}) x^* \rightarrow (y^* \land z^*)^* (r_{15}) (x^* \rightarrow y^*) \land (x^* \rightarrow z^*)^* (r_{12}) (y^* \land z^* \rightarrow x^*^*)^* (r_{15}) x^* \rightarrow (y^* \land z^*) (r_{35}) (x^* \rightarrow y^*) \land (x^* \rightarrow z^*)^* (r_{15}) x^* \rightarrow (y^* \land z^*) (r_{15}) x^* \rightarrow (y^* \land z^*) \).

\((r_{33})\). By \((r_9), (r_{12}), (r_{15})\) and (2) we obtain successively \( x \oplus (y \lor z)^* \odot (2) x^* \rightarrow (y \lor z)^* (r_{12}) x^* \rightarrow (y^* \land z^*)^* (r_{15}) x^* \rightarrow (y^* \land z^*) (r_{35}) (x^* \rightarrow y^*) \land (x^* \rightarrow z^*)^* (r_{15}) x^* \rightarrow (y^* \land z^*) \).

Corollary 2.6. Let \( L \) be a residuated lattice and \( x, y \in L \). Then:

\[
(r_{34}) \quad (x^* \odot y^*)^* \odot x = (y^* \rightarrow x'^*)^* \rightarrow x'^*.
\]

Proof. \((r_{34})\). Since \((x^* \odot y^*)^* \odot y \odot (2) (x^* \rightarrow y'^*)^* \rightarrow y'^* \) and \((y^* \rightarrow x'^*^*)^* \rightarrow x'\). Thus, our claim holds.

Deductive systems correspond to subsets closed with respect to Modus Ponens, their are also called filters, too. Apart from their logical interest, filters have important algebraic properties and they have been intensively studied from an algebraic point of view. Filter theory plays an important role in studying logical algebras.

We consider \( L \) as a residuated lattice.

Definition 2.7 ([2]). An implicative filter (filter, for short) is a nonempty subset \( F \) of \( L \) such that

\[
\begin{align*}
(F_1) & \quad \text{If } x \leq y \text{ and } x \in F, \text{ then } y \in F; \\
(F_2) & \quad \text{If } x, y \in F, \text{ then } x \oplus y \in F.
\end{align*}
\]

We notice that:

1. \( F \) is an implicative filter ([13]) of \( L \) iff \( 1 \in F \) and \( x, x \rightarrow y \in F \), then \( y \in F \) (that is, \( F \) is a deductive system of \( L \).

2. Every filter is a latticial filter in the lattice \((L, \land, \lor)\), but the converse does not hold ([3, 14]).

Hence, if we denote by \( \mathcal{F}(L) \) \((\mathcal{F}(L)) \) the set of all latticial filters (implicative filters) of \( L \), then \( \mathcal{F}(L) \subseteq \mathcal{F}(L) \).
We have \( ([13]) \mathcal{F}_i(L) = \mathcal{F}(L) \) iff \( x \circ y = x \land y \) for every \( x, y \in L \).

We recall \((3, 14)\) that for a nonempty subset \( D \) of \( L \) we denote by \( (D) \) the filter generated by \( D \), and \( (\bar{D}) = \{ x \in L : d_1 \circ \ldots \circ d_n \leq x, \) for some \( d_1, \ldots, d_n \in D \) \). If \( a \in L \), the filter generated by \( \{ a \} \) will be denoted by \( (a) \), and \( (a) = \{ x \in L : a^n \leq x \) for some \( n \geq 1 \) \). If \( F \in \mathcal{F}_i(L) \) and \( a \in L \setminus F \), then \( (F \cup \{ a \}) \) will be denoted by \( F(a) \), and \( F(a) = \{ x \in L : a^n \rightarrow x \in F \) for some \( n \geq 1 \) \). If \( F, G \in \mathcal{F}_i(L) \), then \( F \lor G = F \lor F_i(L) \) \( G = (F \cup G) = \{ x \in L : b \circ c \leq x \) for some \( b \in F, c \in G \) \).

We say that \( P \in \mathcal{F}_i(L) \), \( P \neq L \) is a prime filter \((14)\) if for every \( x, y \in L \) and \( x \vee y \in P \), then \( x \in P \) or \( y \in P \). We denote by \( \text{Spec}_i(L) \) the set of all prime filters of \( L \).

We recall that a filter \( M \) of \( L \) is called maximal if \( M \neq L \) and \( M \) is not strictly contained in any proper filter of \( L \).

Every maximal filter \( M \) of \( L \) is obvious prime because, if there exist two proper filters \( N, P \in \mathcal{F}_i(L) \) such that \( M = N \cap P \), then \( M \leq N \) and \( M \leq P \), by the maximality of \( M \) we deduce that \( M = N = P \), that is, \( M \) is an inf-irreducible, so prime element in the lattice of filters \( (\mathcal{F}_i(L), \subseteq) \) of \( L \) (by the distributivity of the lattice of filters \( (\mathcal{F}_i(L), \subseteq) \) of \( L \)).

So, if we denote by \( \text{Max}_i(L) \) the set of all maximal filters of \( L \), then \( \text{Max}_i(L) \subseteq \text{Spec}_i(L) \).

**Proposition 2.8** \((1, 3, 14)\). For \( M \in \mathcal{F}_i(L) \), \( M \neq L \), the following are equivalent:

(i) \( M \) is maximal;

(ii) If \( x \in M \), then there exists \( n \geq 1 \) such that \( (x^n)^* \in M \).

For \( F \in \mathcal{F}_i(L) \) we define a relation \( \equiv_F \) on \( L \) by \( x \equiv_F y \) iff \( x \rightarrow y, y \rightarrow x \in F \), for all \( x, y \in L \) iff \( x \rightarrow y \equiv (y \rightarrow x) \in F \).

Then \((3, 14) \equiv_F \) is a congruence relation on \( L \). For \( x \in L \) we denote by \([x] = x/F \) the class of congruence of \( x \) modulo \( \equiv_F \) and \( L/F = \{ x/F : x \in L \} \).

Define the binary operations \( \lor, \land, \circ \) and \( \rightarrow \) on \( L/F \) by \( (x/F) \lor (y/F) = (x \lor y)/F, (x/F) \land (y/F) = (x \land y)/F, (x/F) \circ (y/F) = (x \circ y)/F \) and \( (x/F) \rightarrow (y/F) = (x \rightarrow y)/F \) for all \( x, y \in L \).

Then \((L/F, \lor, \land, \circ, \rightarrow, 0, 1) \) is a residuated lattice, which is called the quotient residuated lattice of \( L \) with respect to \( F \), where \( 0 = 0/F \) and \( 1 = 1/F \).

The relation of order on \( L/F \) is defined by \( (x/F) \leq (y/F) \) iff \( x \rightarrow y \in F \).

For a nonempty subset \( S \) of \( L \) we denote by \( S/F = \{ x/F : x \in S \} \). Clearly, for \( x \in L \), \( x/F = 0 \) iff \( x^* \in F \) and \( x/F = 1 \) iff \( x \in F \).

We denote by \( \pi : L \rightarrow L/F \) the canonical epimorphism defined by \( \pi(x) = [x] \).

**Definition 2.9.** If \( L, L' \) are residuated lattices, a map \( f : L \rightarrow L' \) is called morphism of residuated lattices if \( f \) is a morphism of bounded lattices, \( f(x \circ y) = f(x) \circ f(y) \) and \( f(x \rightarrow y) = f(x) \rightarrow f(y) \) for every \( x, y \in L \).

If \( f \) is a bijective map (one-to-one and onto), then we say that \( L \) and \( L' \) are isomorphic and we write \( L \cong L' \).

**Remark 2.10.** If \( F \in \mathcal{F}_i(L') \), then \( f^{-1}(F) \in \mathcal{F}_i(L) \). In particular \( f^{-1}(\{1\}) = \{ x \in L : f(x) = 1 \} \) is denoted by \( \text{Ker}(f) \) and \( \text{Ker}(f) \in \mathcal{F}_i(L) \).

Clearly, \( f \) is one-to-one iff \( \text{Ker}(f) = \{1\} \). Also, \( f(L) \) is a subalgebra of \( L' \) denoted by \( \text{Im}(f) \).

### 3 Construction of some topologies on residuated lattices

In general, the concept of topology represents the study of topological spaces. Important topological properties include connectedness and compactness.

A topology tells how elements of a set relate spatially to each other. The same set can have different topologies. For instance, the real line, the complex plane, and the Cantor set can be thought of as the same set with different topologies.

Let \( X \) be a set and let \( \tau \) be a family of subsets of \( X \). We denote by \( \mathcal{P}(X) \) the family of all subsets of \( X \). Then \( \tau \) is a topology on \( X \) if:

1. \( T_1 \). Both the empty set \( \emptyset \) and \( X \) are elements of \( \tau \);
T2. Any union of elements of \( \tau \) is an element of \( \tau \);

T3. Any intersection of finitely many elements of \( \tau \) is an element of \( \tau \).

If \( \tau \) is a topology on \( X \), then the pair \((X, \tau)\) is called a topological space. The notation \( X_{\tau} \) is used to denote a set \( X \) endowed with the particular topology \( \tau \).

The members of \( \tau \) are called open sets in \( X \). A subset of \( X \) is said to be closed if its complement is in \( \tau \) (i.e., its complement is open). A subset of \( X \) may be open, closed, both (clopen set), or neither. The empty set \( \varnothing \) and \( X \) itself are always both closed and open. An open set containing a point \( x \) is called a neighborhood of \( x \).

A set with a topology is called a topological space. A topological space \((X, \tau)\) is called connected if \( \{ \varnothing, X \} \) is the set of all closed and open subsets of \( X \).

A base (or basis) \( \beta \) for a topological space \( X \) with topology \( \tau \) is a collection of open sets in \( \tau \) such that every open set in \( \tau \) can be written as a union of elements of \( \beta \). We say that the base generates the topology \( \tau \).

**Definition 3.1** ([11]). Let \((X, \leq)\) be an ordered set. Then we define \( \uparrow: \mathcal{P}(X) \to \mathcal{P}(X) \), by \( \uparrow S = \{ x \in X | a \leq x, \text{ for some } a \in S \} \), for any subset \( S \) of \( X \). A subset \( F \) of \( X \) is called an upset if \( \uparrow F = F \). We denote by \( U(X) \) the set of all upsets of \( X \). An upset \( F \) is called finitely generated if there exists \( n \in N \) such that \( F = \uparrow \{ x_1, x_2, \ldots, x_n \} \), for some \( x_1, x_2, \ldots, x_n \in X \).

**Remark 3.2.** Clearly, if \( F \) is a filter of \( L \), then \( F \) is an upset of \( \mathcal{L} \), but the converse does not hold. Indeed, we consider \( L = \{0, n, a, b, c, d, e, f, m, 1\} \) with \( 0 < n < a < c < m < 1, 0 < n < a < e < m < 1, 0 < n < e < c < m < 1, 0 < n < b < e < m < 1, 0 < n < d < e < m < 1 \) and elements \( \{a, b\}, \{a, d\}, \{b, d\}, \{a, f\}, \{c, d\}, \{b, e\}, \{c, e\}, \{c, f\} \) and \( \{e, f\} \) are pairwise incomparable.

Then ([12]) \( L \) becomes a residuated lattice relative to the following operations:

\[
\begin{align*}
\rightarrow & \quad \begin{array}{cccccccccc}
0 & n & a & b & c & d & e & f & m & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
n & m & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
a & f & f & f & f & f & f & f & f & f \\
b & e & e & e & e & e & e & e & e & e \\
c & d & d & d & d & f & f & f & f & f \\
d & c & c & c & c & c & c & c & c & c \\
e & b & b & b & b & b & b & b & b & b \\
f & a & a & a & a & a & a & a & a & a \\
m & n & n & n & n & n & n & n & n & n \\
1 & 0 & n & a & b & c & d & e & f & m \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array} \\
\odot & \quad \begin{array}{cccccccccc}
0 & n & a & b & c & d & e & f & m & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a & 0 & 0 & 0 & a & 0 & a & 0 & a & a \\
b & 0 & 0 & 0 & b & b & 0 & 0 & b & b \\
c & 0 & 0 & a & b & c & 0 & a & b & c \\
d & 0 & 0 & c & c & c & c & c & c & c \\
e & 0 & 0 & b & b & b & b & b & b & b \\
f & 0 & 0 & a & a & a & a & a & a & a \\
m & 0 & 0 & a & b & c & d & e & f & m \\
1 & 1 & 0 & n & a & b & c & d & e & f \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array} \\
\uparrow & \quad \begin{array}{cccccccccc}
0 & n & a & b & c & d & e & f & m & 1 \\
0 & n & a & b & c & d & e & f & m & 1 \\
\end{array} \\
\end{align*}
\]

Then \( \uparrow n = \{n, a, b, c, d, e, f, m, 1\} \) is an upset, but \( n \odot n = 0 \), that is \( \uparrow n \) is not a filter.

In [11], O. Zahiri and R. A. Borzooei construct a topology on BL-algebras considering the notion of upsets, and inspired by their work we construct a topology in the case of residuated lattices and some of their results will become particular cases. We offer complete proofs for the results available in residuated lattices.
Definition 3.3 ([11]). Let \( \tau \) and \( \tau' \) be two topologies on a given set \( X \). If \( \tau' \subseteq \tau \), then we say that \( \tau' \) is finer than \( \tau \). Let \( (X, \tau) \) and \( (Y, \tau') \) be two topological spaces. A map \( f : X \to Y \) is called continuous if the inverse image of each open set of \( Y \) is open in \( X \). A homeomorphism is a continuous function, bijective and has a continuous inverse.

Lemma 3.4 ([11]). Let \( \beta \) and \( \beta' \) be two bases for topologies \( \tau \) and \( \tau' \), respectively on \( X \). Then the following are equivalent:

(i) \( \tau' \) is finer than \( \tau \);

(ii) For any \( x \in X \) and each basis element \( B \in \beta \) containing \( x \), there is a basis element \( B' \in \beta' \) such that \( x \in B \subseteq B' \).

Definition 3.5 ([11]). Let \( (A, *) \) be an algebra of type 2 and \( \tau \) be a topology on \( A \). Then \( (A, *, \tau) \) is called:

(i) Right (left) topological algebra, if for all \( a \in A \) the map \( * : A \to A \) defined by \( x \mapsto a * x \) (\( x \mapsto x * a \)) is continuous;

(ii) Semitopological algebra if \( A \) is a right and left topological algebra.

If \( (A, *) \) is a commutative algebra, then right and left topological algebras are equivalent.

Definition 3.6 ([11]). Let \( A \) be a nonempty set, \( \{ *_i \}_{i \in I} \) be a family of binary operations on \( A \) and \( \tau \) be a topology on \( A \). Then:

(i) \( (A, \{ *_i \}_{i \in I}, \tau) \) is a right (left) topological algebra, if for \( i \in I \), \( (A, *_i, \tau) \) is a right (left) topological algebra;

(ii) \( (A, \{ *_i \}_{i \in I}, \tau) \) is a right (left) semitopological algebra, if for \( i \in I \), \( (A, *_i, \tau) \) is a right (left) semitopological algebra.

On any residuated lattice \( L \) we define an operator \( \ominus \) by setting for all \( x, y \in L \),

\[
x \ominus y = x^* \ominus y.
\]

By (rs), the identity (3) is equivalent with

\[
x \ominus y = x^* \ominus y = x^{**} \xrightarrow{\text{id}} y^{**} = y^\tau \xrightarrow{\text{id}} x^\tau, \text{ for all } x, y \in L.
\]

Definition 3.7. Consider \( L \) a residuated lattice and \( a \in L \). For any nonempty upset \( X \) of \( L \) we define the set

\[
D_a(X) = \{ x \in L | a^n \ominus x \in X, \text{ for some } n \in N \},
\]

where \( a^n \ominus x = a \ominus (a^{n-1} \ominus x) \), for any \( n \in \{2, 3, 4, \ldots \} \).

Lemma 3.8. Let \( L \) be a residuated lattice and \( a, x \in L \). Then:

\[ (r_{35}) \quad a^n \ominus x = (a^{**})^n \to x^{**}; \]

\[ (r_{36}) \quad (a^n \ominus x)^{**} = a^n \ominus x, \text{ for any } n \in N. \]

Proof. (\( r_{35} \)). Mathematical induction relative to \( n \).

If \( n = 2 \), by \( (r_s) \), \( (r_{30}) \) and identity (4) we obtain successively \( a^2 \ominus x = a \ominus (a \ominus x) \xrightarrow{id, (4)} a^{**} \to (a^* \ominus x)^{**} \xrightarrow{\text{id, (4)}} a^{***} \to (a^{**} \ominus x) \xrightarrow{id, (4)} a^{***} \to x^{**} = (a^{**})^2 \to x^{**}. \)

Suppose \( a^{n-1} \ominus x = (a^{**})^{n-1} \to x^{**}. \) By \( (r_s) \) we obtain successively \( a^n \ominus x = a \ominus (a^{n-1} \ominus x) = a^{**} \to ((a^{**})^{n-1} \to x^{**}) \xrightarrow{\text{id, (4)}} (a^{**} \ominus (a^{**})^{n-1}) \to x^{**} = (a^{**})^n \to x^{**}. \) Therefore, \( a^n \ominus x = (a^{**})^n \to x^{**}, \) for any \( n \in N. \)

\[ (r_{36}) \quad \text{By \( (r_{30}) \) we have } (a^n \ominus x)^{**} = [a \ominus (a^{n-1} \ominus x)]^{**} \xrightarrow{id, (4)} [a^* \ominus (a^{n-1} \ominus x)]^{**} \xrightarrow{\text{rs}} a^* \ominus (a^{n-1} \ominus x) \xrightarrow{id, (4)} a \ominus (a^{n-1} \ominus x) = a^n \ominus x, \text{ for any } n \in N. \]

A directly consequence of \( (r_{35}) \) is the following result:

Proposition 3.9. Let \( L \) be a residuated lattice and \( a \in L \). For any nonempty upset \( X \) of \( L \) the set \( D_a(X) = \{ x \in L : (a^{**})^n \to x^{**} \in X, \text{ for some } n \in N \}. \)
Remark 3.10. If \( a = 1 \), then \( D_a(X) = D_1(X) = \{ x \in L : x^{**} \in X \} \) is the set of double complemented elements of \( X \).

Corollary 3.11. If \( L \) is a residuated lattice and \( a, x \in L \), then:

\[(37)\ a \odot x \leq a \land x \leq a \lor x \leq a^{**} \lor x^{**} \leq a \odot x;\]

\[(38)\ x^{**} \leq a \odot x \leq a^n \odot x, \text{ for any } n \in \mathbb{N}.\]

Proof. \((37)\). Indeed, \( a \odot x \leq a \land x \leq a \lor x \leq a^{**} \lor x^{**} \). By \((r_{27})\) and identity \((2)\) we obtain successively

\[ a \odot x = a^{id(2)} \rightarrow x^{**} \geq x^{**} \text{ and } a \odot x = a^{id(2)} \rightarrow x \geq a^{**} \geq a^{**}. \]

\[(38)\). Since \( a^n \odot x = (a^{**})^n \rightarrow x^{**} \geq a^{**} \rightarrow x^{**} = a \odot x \geq x^{**}, \text{ hence } x^{**} \leq a \odot x \leq a^n \odot x, \text{ for any } n \in \mathbb{N}. \]

The following properties hold for any residuated lattice:

\[(39)\ z \rightarrow y \leq (x \rightarrow z) \rightarrow (x \rightarrow y) \text{ and } z \rightarrow y \leq (y \rightarrow x) \rightarrow (z \rightarrow x);\]

\[(40)\ (x \rightarrow y) \odot (y \rightarrow z) \leq x \odot z.\]

In the next result, we investigate some properties of the operator \( D_a(X) \) for nonempty upsets. Clearly, \( D_a(\emptyset) = \emptyset \).

In Proposition 3.3, \([11]\), there are the properties of operator \( D_a \) for BL-algebras, we offer complete proofs for them in the case of residuated lattices:

Theorem 3.12. If \( L \) is a residuated lattice and \( a, x \in L \). Consider \( X, Y \) two nonempty upsets of \( L \). Then:

\[(i)\ D_a(X) \text{ is an upset of } L;\]

\[(ii)\ 1 \in D_a(X), a \in D_a(X) \text{ and } X \subseteq D_a(X);\]

\[(iii)\ a^m \odot (a^n \odot x) = a^{m+n} \odot x, \text{ for any } m, n \in \mathbb{N};\]

\[(iv)\ \text{if } X \subseteq Y, \text{ then } D_a(X) \subseteq D_a(Y);\]

\[(v)\ D_a(D_a(X)) = D_a(X);\]

\[(vi)\ \text{if } F \text{ is a filter of } L, \text{ then } D_a(F) \text{ is a filter of } L;\]

\[(vii)\ \text{if } a \leq s, \text{ then } D_a(X) \subseteq D_a(X);\]

\[(viii)\ \text{if } \{ X_\alpha | \alpha \in I \} \text{ is a family of upsets of } L, \text{ then } D_a(\cup \{ X_\alpha | \alpha \in I \}) = \cup \{ D_a(\{ X_\alpha | \alpha \in I \}) \};\]

\[(ix)\ \text{if } \{ X_1, X_2, \ldots, X_n \} \text{ is a finite set of upsets of } L, \text{ then } D_a(\cap \{ X_i | i = 1, 2, \ldots, n \}) = \cap \{ D_a(X_i) | i = 1, 2, \ldots, n \};\]

\[(x)\ \text{if } D_a(X), \text{ then } D_a(X) \cap D_a(Y) \subseteq D_a(X \cap Y) \text{ and } D_a(X) \cup D_a(Y) \subseteq D_a(X \cup Y);\]

\[(xi)\ \text{if } X_1, X_2, X_3 \text{ are upsets of } L, \text{ then } D_a(X_1) \cap D_a(X_2) \cup D_a(X_3) = [D_a(X_1) \cap D_a(X_2)] \cup [D_a(X_1) \cap D_a(X_3)] \text{ and } D_a(X_1) \cap D_a(X_2) \cup D_a(X_3) = [D_a(X_1) \cup D_a(X_2)] \cap [D_a(X_1) \cup D_a(X_3)].\]

Proof. \((i)\). Clearly, \( D_a(X) \subseteq D_a(X) \). We consider \( s \in D_a(X) \), then there exists \( x \in D_a(X) \) such that \( x \leq s \).

Since \( x \in D_a(X) \) we have \( a^n \odot x \in X \), by \((r_{35})\) we deduce that \((a^{**})^n \rightarrow x^{**} \in X \), for some \( n \in \mathbb{N} \). By \((r_a)\) and \((r_{16})\), since \( x \leq s \) we obtain successively \( x^{**} \leq s^{**}, (a^{**})^n \rightarrow x^{**} \leq (a^{**})^n \rightarrow s^{**} \). Since \( X \) is an upset of \( L \) and \((a^{**})^n \rightarrow s^{**} \in X \), then \((a^{**})^n \rightarrow s^{**} \in X \), hence \( s \in D_a(X) \) and so \( D_a(X) \subseteq D_a(X) \). We deduce that \( D_a(X) \) is an upset of \( L \).

\((ii)\). Since \( 1 \in X \), by \((r_{35})\) we have \( a^n \odot 1 = (a^{**})^n \rightarrow 1^{**} = (a^{**})^n \rightarrow 1 = 1 \in X \), for any \( n \in \mathbb{N} \). Hence \( 1 \in D_a(X) \).

Since \( 1 \in X \), by \((r_{18})\) and \((r_{35})\) we have \( a^n \odot a = (a^{**})^n \rightarrow a^{**} = 1 \in X \), for any \( n \in \mathbb{N} \). Hence \( a \in D_a(X) \).

Now, we consider \( x \in X \). Then by \((r_{16})\) and \((r_{38})\) we have \( x \leq x^{**} \leq a^n \odot x \in X \), hence \( x \in D_a(X) \). We deduce that \( X \subseteq D_a(X) \).

\((iii)\). Consider \( m, n \in \mathbb{N} \) and \( a, x \in L \). By \((r_5)\) and \((r_{36})\) we obtain successively \( a^m \odot (a^n \odot x) = (a^{**})^m \rightarrow (a^{**})^n \rightarrow a^{**} \in X \).

\((iv)\). Consider \( X \subseteq Y \) and \( x \in D_a(X) \). Then there exists \( n \in \mathbb{N} \) such that \( a^n \odot x \in X \subseteq Y \) and so \( x \in D_a(Y) \). Hence \( D_a(X) \subseteq D_a(Y) \).
(v). Since $X \subseteq D_\alpha(X)$, by (iv) we have $D_\alpha(X) \subseteq D_\alpha(D_\alpha(X))$. Consider $x \in D_\alpha(D_\alpha(X))$. Then there exists $m \in \mathbb{N}$ such that $a^m \triangledown x \in D_\alpha(X)$ and so $a^n \triangledown (a^m \triangledown x) \in X$, for some $n \in \mathbb{N}$. By (iii) we have $a^{m+n} \triangledown x \in X$ and so $x \in D_\alpha(D_\alpha(X))$. Hence $D_\alpha(X) = D_\alpha(D_\alpha(X))$.

(vi). Consider $F$ a filter of $L$. Then $F$ is a nonempty upset and by (ii) we have $1 \in D_\alpha(F)$. Let $x, x \rightarrow y \in D_\alpha(F)$, then there are $m, n \in \mathbb{N}$ such that $(a^{*n})^m \rightarrow x^{*n} \in F$ and $(a^{*m})^n \rightarrow (x \rightarrow y)^{**} \in F$. By (r5) and (r17) we obtain successively $(a^{*n})^m \rightarrow x^{*n} \rightarrow ((a^{*n})^m \rightarrow y^{**}) = ((a^{*n})^m \rightarrow x^{*n} \rightarrow ((a^{*n})^m \rightarrow (a^{*n})^m \rightarrow y^{**}) \rightarrow ((a^{*n})^m \rightarrow x^{*n} \rightarrow ((a^{*n})^m \rightarrow y^{**}) \rightarrow ((a^{*n})^m \rightarrow x^{*n} \rightarrow ((a^{*n})^m \rightarrow y^{**}) \rightarrow ((a^{*n})^m \rightarrow x^{*n} \rightarrow y^{**}) \rightarrow y^{**} \in F$.

Since $F$ is a filter and $(a^{*n})^m \rightarrow x^{*n} \in F$, then $(a^{*n})^m \rightarrow y^{**} \in F$, hence $y \in D_\alpha(F)$. We deduce that $D_\alpha(F)$ is a filter of $L$.

(vii). Consider $a \leq s$ and $x \in D_\alpha(X)$, then there exists $n \in \mathbb{N}$ such that $(s^{*n})^n \rightarrow x^{*n} \in X$. By (r6) we obtain successively $a \leq s^* \leq s^{**}, (a^{*n})^m \leq (s^{*n})^n, (s^{*n})^n \rightarrow x^{*n} \leq (a^{*n})^m \rightarrow x^{*n}$. Since $(s^{*n})^n \rightarrow x^{*n} \in X$, then $(a^{*n})^m \rightarrow x^{*n} \in X$, hence $x \in D_\alpha(X)$. We deduce that $D_\alpha(X) \subseteq D_\alpha(x)$.

(viii). Consider $x \in L$. Then we have the equivalences:

$\forall x \in D_\alpha(\cup \{X_\alpha | \alpha \in I\})$ iff $a^n \triangledown x \in \cup \{X_\alpha | \alpha \in I\}$ iff

$a^n \triangledown x \in X_\alpha$, for some $n \in \mathbb{N}$, and $\alpha \in I$ iff

$x \in D_\alpha(X_\alpha)$, for some $n \in \mathbb{N}$ and $\alpha \in I$ iff

$x \in \cup \{D_\alpha(\{X_\alpha | \alpha \in I\})\}$. Hence $D_\alpha(\cup \{X_\alpha | \alpha \in I\}) = \cup \{D_\alpha(\{X_\alpha | \alpha \in I\})\}$.

(ix). Following (iv), since $\cap \{X_i | i = 1, 2, ..., n\} \subseteq X_i$, for any $i \in \{1, 2, ..., n\}$, then $D_\alpha(\cap \{X_i | i = 1, 2, ..., n\}) \subseteq D_\alpha(X_i)$, for any $i \in \{1, 2, ..., n\}$, hence $D_\alpha(\cap \{X_i | i = 1, 2, ..., n\}) \subseteq \cap \{D_\alpha(X_i) | i = 1, 2, ..., n\}$. Consider $x \in \cap \{D_\alpha(X_i) | i = 1, 2, ..., n\}$. Then there exist $m_1, m_2, ..., m_n \in \mathbb{N}$ such that $(a^{*n})^m \rightarrow x^{*n} \in X_i$, for any $i \in \{1, 2, ..., n\}$.

Consider now $p = \max\{m_1, m_2, ..., m_n\}$. By (r6), since $m_1 \leq p$, for any $i \in \{1, 2, ..., n\}$, then $(a^{*n})^m \rightarrow x^{*n} \leq (a^{*n})^p \rightarrow x^{*n}$. Since $(a^{*n})^m \rightarrow x^{*n} \in X_i$, then $(a^{*n})^p \rightarrow x^{*n} \in X_i$, for any $i \in \{1, 2, ..., n\}$. It follows that $(a^{*n})^p \rightarrow x^{*n} \in \cap \{X_i | i = 1, 2, ..., n\}$, hence $x \in D_\alpha(\cap \{X_i | i = 1, 2, ..., n\})$. We deduce that $D_\alpha(\cap \{X_i | i = 1, 2, ..., n\}) \subseteq \cap \{D_\alpha(X_i) | i = 1, 2, ..., n\}$.

(x). We have successively:

$D_\alpha(D_\alpha(X)) = \{t \in L | a^n \triangledown t \in D_\alpha(X),$ for some $n \in \mathbb{N}\} = \{t \in L | a^n \triangledown t \in X,,$ for some $m, n \in \mathbb{N}\}$

$\cap \{t \in L | (a^{*n})^m \rightarrow (a^m \triangledown t)^{**} \in X,,$ for some $m, n \in \mathbb{N}\}$

$\cap \{t \in L | (a^{*n})^m \rightarrow (a^m \triangledown t)^{**} \in X,,$ for some $m, n \in \mathbb{N}\}$

$\cap \{t \in L | (a^{*n})^m \rightarrow (a^m \triangledown t)^{**} \in X,,$ for some $m, n \in \mathbb{N}\}$

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$\cap \{t \in L | (a^{*n})^m \rightarrow (a^m \triangledown t)^{**} \in X,,$ for some $m, n \in \mathbb{N}\}$

$= \{t \in L | a^n \triangledown t \in D_\alpha(X),$ for some $m, n \in \mathbb{N}\} = D_\alpha(D_\alpha(X))$.

(xi). It follows from (viii) and (ix).

In universal algebra, for a nontrivial lattice $A$, a unary mapping $f : P(A) \rightarrow P(A)$ is called lattice modal operator on $A$ if it satisfies the conditions: For all $A_1, A_2 \subseteq A$,

(1) $A_1 \subseteq f(A_1)$;

(2) $f(f(A_1)) = f(A_1)$;

(3) $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$.

A lattice modal operator is called monotone if it satisfies:

(m) If $A_1 \subseteq A_2$, then $f(A_1) \subseteq f(A_2)$.

Following Theorem 3.12 (ii), (iv), (v), and (viii) we deduce:

Corollary 3.13. Let $a \in L$. Then the map $D_\alpha : U(L) \rightarrow U(L)$ is a lattice monotone modal operator.
Proof. Following Theorem 3.12 (ii), (iv), (v), and (viii) we deduce that the map $D_a : U(L) \to U(L)$ is a latticeal monotone modal operator.

We recall that, for a residuated lattice $L$, by $U(L)$ we denote the set of all upsets of $L$.

**Corollary 3.14.** For any $a \in L$, the map $D_a : U(L) \to U(L)$ is a closure operator and $D_a = D_{a^{**}}$.

**Proof.** Following Theorem 3.12 (i), (ii), (iv) and (v), we deduce that $D_a$ is a closure operator.

Consider $x$ an upset of $L$. Following Theorem 3.12 (vii), since by (r16) we have $a \leq a^{**}$, then $D_{a^{**}}(X) \subseteq D_a(X)$. Consider now $x \in D_a(X)$. Then there exists $n \in \mathbb{N}$ such that $(a^{**})^n \to x^{**} \in X$. By (r15) we have $a^{****} = a^{**}$, then $(a^{****})^n \to x^{**} \in X$, hence $x \in D_{a^{**}}(X)$. We deduce that $D_a(\cdot) = D_{a^{**}}(\cdot)$.

**Lemma 3.15.** Let $a \in L$ and $F$ be a filter of $L$. Then $D_a(F) = D_1(F)$ iff $a^{**} \in F$.

**Proof.** $\Rightarrow$. Consider $a \in L$ and $F$ a filter of $L$ such that $D_a(F) = D_1(F)$. Following Theorem 3.12 (ii), we have $a \in D_a(F) = D_1(F)$, then $a \in D_1(F)$, and we have successively $a \in D_1(F)$, $(1^{**})^n \to a^{**} \in F$, $1^n \to a^{**} \in F$, $1 \to a^{**} \in F$, $a^{**} \in F$.

$\Leftarrow$. Suppose $a^{**} \in F$. By Theorem 3.12 (vii), since $a \leq 1$, then $D_1(F) \subseteq D_a(F)$. Consider now $x \in D_a(F)$. Then there exists $n \in \mathbb{N}$ such that $(a^{**})^n \to x^{**} \in X$. Since $a^{**} \in F$ and $F$ is a filter of $L$, then $(a^{**})^n \in F$, for any $n \in \mathbb{N}$. By $(a^{**})^n \in F$, $(a^{**})^n \to x^{**} \in F$, we deduce $x^{**} \in F$. Since $x^{**} = (1^{**})^n \to x^{**} \in F$, then $x \in D_1(F)$, hence $D_a(F) \subseteq D_1(F)$. We deduce that $D_a(F) = D_1(F)$.

**Corollary 3.16.** Let $L$ be an involutive residuated lattice, $a \in L$ and $F$ be a filter of $L$. Then $D_a(F) = F$ iff $a \in F$.

**Proof.** $\Rightarrow$. Consider $a \in L$ and $F$ a filter of $L$ such that $D_a(F) = F$. Following Theorem 3.12 (ii), we have $a \in D_a(F) = F$, then $a \in F$.

$\Leftarrow$. Suppose $a \in F$. By Theorem 3.12 (ii), $a \in F \subseteq D_a(F)$. Consider now $x \in D_a(F)$. Then there exists $n \in \mathbb{N}$ such that $(a^{**})^n \to x^{**} \in F$. Since $a \leq a^{**} \in F$ and $F$ is a filter of $L$, then $(a^{**})^n \in F$, for any $n \in \mathbb{N}$. By $(a^{**})^n \in F$, $(a^{**})^n \to x^{**} \in F$, we deduce $x^{**} \in F$, hence $x \in F$. Therefore, $D_a(F) \subseteq F$. We deduce that $D_a(F) = F$.

**Remark 3.17.** (i). By Theorem 3.12 (vi), if $F$ is a proper filter, then $D_a(F)$ is a proper filter or $D_a(F) = L$. We consider the lattice $L = \{0, a, c, d, m, 1\}$ with $0 < a < m < 1$, $0 < c < d < m < 1$, but a incomparable with $c$ and $d$.

Then ([12], page 233) $L$ becomes a residuated lattice relative to the following operations:

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Then (1113) $L$ becomes a residuated lattice relative to the following operations:
We have \( \{a\} = \{a, m, 1\} \) is a proper filter of \( L \), \( D_m(\{a\}) = \{x \in L \mid (m^\alpha)^n \to x^\alpha \in \{a\}, \text{ for some } n \in \mathbb{N}\} = \{x \in L \mid 1 \to x^\alpha \in \{a\}\} = \{a\} \) and \( D_c(\{a\}) = \{x \in L \mid (c^\alpha)^n \to x^\alpha \in \{a\}, \text{ for some } n \in \mathbb{N}\} = \{x \in L \mid c \to x^\alpha \} = L \).

Moreover, \( \{a\} \) is a maximal filter of \( L \) and \( D_x(\{a\}) = L \neq \{a\} \). So, if \( M \) is a maximal filter, then \( D_x(M) = M \) or \( D_x(M) = L \), for any \( x \in L \).

(ii). There are residuated lattices \( L \) such that for a proper filter \( F \) of \( L \), there is \( a \in L \) and \( D_a(F) = L \). Indeed, we consider \( L = \{0, a, b, c, d, e, f, m, 1\} \) with \( 0 < a < c < m < 1, 0 < a < e < m < 1, 0 < b < c < m < 1, 0 < b < f < m < 1, 0 < d < e < m < 1 \) and elements \( \{a, b\}, \{a, f\}, \{a, d\}, \{b, d\}, \{b, e\}, \{d, c\}, \{e, c\}, \{c, f\} \) and \( \{e, f\} \) are pairwise incomparable.

![Diagram of the residuated lattice](image)

Then \( (\{1\}) \) \( L \) is a residuated lattice with the following operations:

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Let \( F = \{1\} \). Since \( D_m(\{1\}) = \{x \in L \mid (m^\alpha)^n \to x^\alpha \in \{1\}, \text{ for some } n \in \mathbb{N}\} = \{x \in L \mid 1 \in \{1\}\} = L \).

(iii). There are residuated lattices \( L \) such that for a prime filter \( F \), there is \( a \in L \) such that \( D_a(F) \) is not prime. Indeed, we consider \( L = \{0, a, b, n, c, d, 1\} \) with \( 0 < a < n < d < 1, 0 < b < n < c < 1 \), but \( \{a, b\} \) and \( \{c, d\} \) are pairwise incomparable.

![Diagram of the residuated lattice](image)
Then ([12], page 191) \( L \) becomes a residuated lattice relative to the operations:

<table>
<thead>
<tr>
<th>→</th>
<th>0 a b n c d 1</th>
<th>⊗</th>
<th>0 a b n c d 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1 1 1 1 1 1</td>
<td>0</td>
<td>0 0 0 0 0 0</td>
</tr>
<tr>
<td>a b</td>
<td>1 1 1 1</td>
<td>a</td>
<td>0 a 0 a a a</td>
</tr>
<tr>
<td>b a</td>
<td>1 1 1 1 1</td>
<td>b</td>
<td>0 0 b b b b</td>
</tr>
<tr>
<td>n 0 a b</td>
<td>1 1 1 1</td>
<td>n</td>
<td>0 a b n n n n</td>
</tr>
<tr>
<td>c 0 a b d 1 d 1</td>
<td>c</td>
<td>0 a b n c n c</td>
<td></td>
</tr>
<tr>
<td>d 0 a b c c</td>
<td>1 1 d</td>
<td>d</td>
<td>0 a b n n n n</td>
</tr>
<tr>
<td>1 0 a b n c d 1</td>
<td>1</td>
<td>0 a b n c d 1</td>
<td></td>
</tr>
</tbody>
</table>

Clearly, \( \langle c \rangle = \{ c, 1 \} \) is a prime filter of \( L \). Since \( D_n(\langle c \rangle) = \{ x \in L | (n^{**})^m \to x^{**} \in \langle c \rangle \}, \) for some \( m \in \mathbb{N} \} = \{ x \in L | 1 \to x^{*} \in \langle c \rangle \} = \{ n, c, d, 1 \}, \) hence \( D_n(\langle c \rangle) = \{ n, c, d, 1 \}. \) Since \( n = a \lor b, \) but \( a \notin D_n(\langle c \rangle) \) and \( b \notin D_n(\langle c \rangle) \), we deduce \( D_n(\langle c \rangle) \) is not prime.

(iv). There are residuated lattices \( L \) such that for a filter \( F \), there is \( a \in L \) such that \( D_n(F) \) is prime, but \( F \) is not prime. Indeed, we consider \( L = \{ 0, a, b, c, d, e, f, 1 \} \) with \( 0 < a < c < e < 1, 0 < b < e < f < 1 \) \( 0 < b < c < e < 1, 0 < b < d < e < 1, \) and elements \( \{ a, b \}, \{ a, d \}, \{ a, f \}, \{ c, d \}, \{ c, f \}, \) and \( \{ e, f \} \) are pairwise incomparable.

Then [12] \( L \) becomes a MTL-algebra with the following operations:

<table>
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<tr>
<th>→</th>
<th>0 a b c d e f 1</th>
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<tr>
<td>0</td>
<td>1 1 1 1 1 1</td>
<td>0</td>
<td>0 0 0 0 0 0</td>
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<tr>
<td>a e f 1 f 1 f 1</td>
<td>a</td>
<td>0 a 0 a a a</td>
<td></td>
</tr>
<tr>
<td>b e e 1 1 1 1</td>
<td>b</td>
<td>0 0 0 0 0 b b</td>
<td></td>
</tr>
<tr>
<td>c d e f 1 f 1 f 1</td>
<td>c</td>
<td>0 a 0 a a a b c</td>
<td></td>
</tr>
<tr>
<td>d c c c c 1 1 1</td>
<td>d</td>
<td>0 0 0 0 0 d d d</td>
<td></td>
</tr>
<tr>
<td>e b c b c f 1 f 1</td>
<td>e</td>
<td>0 a 0 a a a d e</td>
<td></td>
</tr>
<tr>
<td>f a a c c e e 1 1</td>
<td>f</td>
<td>0 0 b b d d f f</td>
<td></td>
</tr>
<tr>
<td>1 0 a b c d e f 1</td>
<td>1</td>
<td>0 a b c d e f 1</td>
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</table>

Since \( \langle e \rangle = \{ e, 1 \}, \) with \( e = c \lor d, \) but \( c \notin \langle e \rangle \) and \( d \notin \langle e \rangle, \) then \( \langle e \rangle \) is not a prime filter. Since \( D_d(\langle e \rangle) = \{ x \in L | (d^{**})^n \to x^{**} \in e \}, \) for some \( n \in \mathbb{N} \} = \{ x \in L | d \to x^{*} \in \langle e \rangle \} = \{ d, e, f, 1 \}. \) Hence \( D_d(\langle e \rangle) = \{ d, e, f, 1 \}, \) which is a prime filter of \( L, \) but \( \langle e \rangle \) is not prime.

(v). If \( M \in \text{Max}(L) \) is a maximal filter and \( D_n(M) \) is a proper filter, then \( D_n(M) = M. \) Indeed, by Theorem 3.12 (ii), \( M \subseteq D_n(M) \) and \( D_n(M) \) is a proper filter. We get that if \( M \) is a maximal filter, then \( M = D_n(M) \).
There are residuated lattices \( L \) such that \( D_\alpha(M) \) is a maximal filter, but \( M \notin \text{Max}_i(L) \). We consider \( L = \{0, n, a, b, c, d, 1\} \) with \( 0 < n < a < b < c, d < 1 \), but \( c \) and \( d \) are incomparable.

Then (12) \( L \) becomes a distributive residuated lattice relative to the following operations:

\[
\begin{array}{ccccccccc}
\rightarrow & 0 & n & a & b & c & d & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
n & 1 & n & 1 & 1 & 1 & 1 & 1 & 1 \\
a & n & n & a & 1 & 1 & 1 & 1 & 1 \\
b & n & n & a & b & b & b & b & b \\
c & 0 & n & a & d & 1 & d & c & d \\
d & n & n & a & c & c & 1 & 1 & d \\
1 & 0 & n & n & a & b & c & d & 1
\end{array}
\]

Clearly, \( \langle a \rangle = \{a, b, c, d, 1\} \) is the unique maximal filter of \( L \) and \( \langle b \rangle = \{b, c, d, 1\} \) is a filter of \( L \). Since \( D_\alpha((b)) = \{x \in L | c^{\ast \ast} \rightarrow x^{\ast \ast} \in \langle b \rangle, \text{ for some } n \in \mathbb{N}\} = \{x \in L | x^{\ast \ast} \in \langle b \rangle\} = \{a, b, c, d, 1\} = \langle a \rangle \). Therefore, \( D_\alpha((b)) = \langle a \rangle \) is the unique maximal filter of \( L \), but \( \langle b \rangle \) is not maximal.

**Proposition 3.18.** Let \( a \in L \) and \( F \) be a filter of \( L \). Then:

(i) if \( M \) is a maximal filter of \( L \) and \( a^{\ast \ast} \in M \), then \( D_\alpha(M) = M \);

(ii) \( D_1(F) \) is a prime filter if \( D_1(F) \) is prime;

(iii) \( (a^n)^* \in F \), for some \( n \in \mathbb{N} \), if \( D_1(F) = L \);

(iv) if \( M \) is a maximal filter of \( L \), then \( D_\alpha(M) = L \) if \( a \in L \setminus M \).

Moreover, if \( M \) is a maximal filter of \( L \), then \( D_\alpha(M) = M \) if \( a \in M \).

**Proof.** (i). Let \( M \) be a maximal filter of \( L \) and \( a^{\ast \ast} \in M \). Following Theorem 3.12 (ii) and (vi) we have \( D_\alpha(M) \) is a filter and \( M \subseteq D_\alpha(M) \). We must prove that \( D_\alpha(M) \) is a proper filter of \( L \). If \( 0 \in D_\alpha(M) \), then by (r5) and (r15) we obtain successively \( 0 \in D_\alpha(M) \), \( (a^{\ast \ast})^n \rightarrow 0^{\ast \ast} \in M \), \( (a^{\ast \ast})^n \rightarrow 0 \in M \), \( (a^{\ast \ast})^{n-1} \rightarrow a^{\ast \ast} \rightarrow 0 \in M \), \( n = 1 \rightarrow (a^{\ast \ast}) \rightarrow M \), \( (a^{\ast \ast})^{n-1} \rightarrow a^{\ast \ast} \rightarrow a \in M \). Since \( a^{\ast \ast} \in M \) and \( M \) is a filter of \( L \), then \( (a^{\ast \ast})^{n-1} \) is a filter of \( M \). Since \( a^n = a^{\ast \ast} \in M \) and \( M \) is a filter of \( L \), then \( 0 = a^n \ast a^{\ast \ast} \in M \), a contradiction. Hence \( D_\alpha(M) \neq L \).

By hypothesis, \( M \) is a maximal filter and \( M \subseteq D_\alpha(M) \), then \( D_\alpha(M) = M \).

(ii). Let \( F \) be a filter of \( L \). Following Theorem 3.12 (vi), \( D_1(F) \) and \( D_2(F) \) are filters of \( L \). Suppose \( D_1(F) \) is a prime filter of \( L \) and \( x \lor y \in D_1(F) \). If \( x \lor y \in D_1(F) \), then \( (1^{\ast \ast})^n \rightarrow (x \lor y)^{\ast \ast} = (x \lor y)^{\ast \ast} \in F \), for any \( n \in \mathbb{N} \). Since \( F \) is a filter and \( (x \lor y)^{\ast \ast} \subseteq (a^{\ast \ast})^n \rightarrow (x \lor y)^{\ast \ast} \subseteq (a^{\ast \ast})^n \), we deduce \( x \lor y \in D_1(F) \).

Since \( x \lor y \in D_1(F) \) and \( D_1(F) \) is prime, then \( x \lor y \in D_1(F) \). It follows that \( x^{\ast \ast} \in E \) or \( y^{\ast \ast} \in F \). We have successively \( x^{\ast \ast} \in E \), \( x^{\ast \ast} \subseteq (a^{\ast \ast})^n \rightarrow x^{\ast \ast} \in E \) or \( y^{\ast \ast} \in F \), \( y^{\ast \ast} \subseteq (a^{\ast \ast})^n \rightarrow y^{\ast \ast} \in E \), then \( x \in D_\alpha(F) \) or \( y \in D_\alpha(F) \), for some \( n \in \mathbb{N} \). Hence \( D_\alpha(F) \) is prime.

Now, suppose \( D_\alpha(F) \) is prime and for \( a = 1 \) we deduce \( D_1(F) \) is prime, too.

(iii). Suppose \( (a^n)^* \in F \), for some \( n \in \mathbb{N} \). Following Theorem 3.12 (ii), \( F \subseteq D_\alpha(F) \), then \( (a^n)^* \in D_\alpha(F) \).

By Theorem 3.12 (ii) and (vi), we obtain \( a \in D_\alpha(F) \) and \( D_\alpha(F) \) is a filter, then \( a^n \in D_\alpha(F) \), for any \( n \in \mathbb{N} \). Since \( a^n \in D_\alpha(F) \), \( (a^n)^* \in D_\alpha(F) \) and \( D_\alpha(F) \) is a filter, then \( 0 = a^n \ast (a^n)^* \in D_\alpha(F) \). Hence \( D_\alpha(F) = L \).
Now, suppose $D_a(F) = L$. Then $0 \in D_a(F)$, $(a^{**})^n \to 0^* \in F$, $(a^{**})^n \to 0 \in F$, for some $n \in \mathbb{N}$.

We prove by induction that $(a^{**})^{n+1} \to 0 = a^{n+1} \to 0$, for $n \in \mathbb{N}$. Clear, for $n = 0$. Suppose $(a^{**})^n \to 0 = a^n \to 0$, for $n \in \mathbb{N}$. By $(r_s)$, $(a^{**})^{n+1} \to 0 = ((a^{**})^n \circ a^{**}) \to 0 = a^{n+1} \to 0 = (a^{**})^n \to 0 = a^{**} \to 0 = (a^0 \to 0) = a^n \to 0 = (a \to 0) = a^{n+1} \to 0$.

Since $(a^{**})^n \to 0 \in F$, then $a^n \to 0 \in F$, that is $(a^n)^* \in F$.

(iv). Consider $M$ a maximal filter of $L$.

" \Rightarrow ". Consider $D_a(M) = L$. If $a \in M$, then we get $a^{**} \in M$, and by (i) we obtain $L = D_a(M) = M$, a contradiction. We deduce $a \notin M$, that is, $a \in L \setminus M$.

" \Leftarrow ". Consider $a \in L \setminus M$. Since $M$ is a maximal filter, then following Proposition 2.8 there is $n \in \mathbb{N}$ such that $(a^n)^* \in M$, and by (iii) we obtain $D_a(M) = L$.

Now, the fact that $D_a(M) = M$ iff $a \in M$ is routine.

Georgescu et al. (2015)[15] called Gelfand resideduated lattices those residuated lattices in which any prime filter is included in a unique maximal filter. They are also called normal resideduated lattices. Examples of Gelfand resideduated lattices are Boolean algebras, BL-algebras and Stonean resideduated lattices (see [8]).

**Proposition 3.19.** Let $a \in L$ and $P \in \text{Spec}(L)$ be a prime filter. If $P = D_a(P)$, then $L$ is Gelfand (normal) resideduated lattice. The converse does not hold.

**Proof.** Let $P$ be a prime filter of $L$ such that $P = D_a(P)$. Using Zorn’s Lemma we deduce that $P$ is contained in a maximal filter. Suppose that there are two distinct maximal filters $M_1$ and $M_2$ such that $P \subseteq M_1$ and $P \subseteq M_2$. Since $M_1 \neq M_2$, there is $a \in M_1$ such that $a \notin M_2$. By Theorem 3.12 (ii) and (vi) we have that $a \in D_a(P)$ and $D_a(P)$ is a filter, then $(a^{**})^n \in D_a(P) = P$, for any $n \in \mathbb{N}$. Hence $(a^n)^* \in P$, for any $n \in \mathbb{N}$.

Following Proposition 2.8, there is $n \geq 1$ such that $(a^n)^* \in M_2$. Then $(a^n)^* \notin M_2$, hence $(a^n)^* \notin P$, a contradiction.

For the converse we consider the resideduated lattice $L$ from Remark 3.17 (v). The prime filter of $L$ are $(a) = \{a, b, c, d, 1\}$, $(b) = \{b, c, d, 1\}$, $(c) = \{c, 1\}$ and $(d) \{d, 1\}$. The maximal filter of $L$ is $(a)$, which include the all other prime filters. Hence $L$ is Gelfand, but $D_c(b) = (a) \neq (b)$.

**Proposition 3.20.** Let $F \in \mathcal{F}(L)$ and $a \in F$. For $x \in D_a(F)$ the following assertions are equivalent:

(i) $D_a(F) = F$;

(ii) $x^{**} \in F$, then $x \in F$.

**Proof.** (i) \Rightarrow (ii). Consider $D_a(F) = F$. By hypothesis $F$ is a filter and $a \in F$, then $a^{**} \in F$, $(a^{**})^n \in F$, for every $n \in \mathbb{N}$. Since $F$ is a filter and $(a^{**})^n \in F$, $(a^{**})^n \to x^{**} \in F$, then $x^{**} \in F$. We obtain successively $F = D_a(F) = \{x \in L \mid (a^{**})^n \to x^{**} \in F\}$, for some $n \in \mathbb{N}$, that is, for all $x \in L$.

(ii) \Rightarrow (i). By hypothesis $F$ is a filter and $a \in F$, then $a^{**} \in F$, $(a^{**})^n \in F$, for every $n \in \mathbb{N}$. Since $F$ is a filter and $(a^{**})^n \in F$, $(a^{**})^n \to x^{**} \in F$, then $x^{**} \in F$. We obtain successively $D_a(F) = \{x \in L \mid (a^{**})^n \to x^{**} \in F\}$, for some $n \in \mathbb{N}$, that is, $x \in D_a(F)$.

**Theorem 3.21.** Let $a \in L$ and $F$ be a filter of $L$. For $x \in D_a(F)$ the following assertions are equivalent:

(i) $D_a(F) = F$;

(ii) $x^{**} \in F$ iff $x \in F$.

**Proof.** (i) \Rightarrow (ii). Consider $D_a(F) = F$. By hypothesis $F$ is a filter and $x \in D_a(F) = F$, by $(r_{16})$ we obtain $x \leq x^{**} \in F$, so, if $x \in F$, then $x^{**} \in F$. Now, we prove that if $x^{**} \in F$, then $x \in F$. Since $F$ is a filter and $x^{**} \in F$, $x^{**} \leq (a^{**})^n \to x^{**} \in F$, for any $n \in \mathbb{N}$, then $x \in D_a(F) = F$.

(ii) \Rightarrow (i). By Theorem 3.12, (ii) we have $F \subseteq D_a(F)$. Now, we consider $x \in D_a(F)$ such that $x^{**} \in F$ iff $x \in F$. Since $x^{**} \in F$ and $x^{**} \leq (a^{**})^n \to x^{**} \in F$, for any $n \in \mathbb{N}$, then we obtain successively $D_a(F) = \{x \in L \mid (a^{**})^n \to x^{**} \in F\}$, for some $n \in \mathbb{N}$, that is, $x \in D_a(F)$, that is, $x \in D_a(F) = F$. Therefore, $D_a(F) = F$. 


Lemma 3.27. Examples of residuated lattices which satisfy the conditions from Proposition 3.20 and Theorem 3.21 are Boolean algebras, MV-algebras and involutive residuated lattices.

Corollary 3.23. The set \( \tau_a = \{ D_a(X) | X \in U(L) \} \) is a topology on \( L \) and \( (L, \tau_a) \) is a topological space.

Proof. Let \( a \in L \). Clearly, \( D_a(\varnothing) = \varnothing \) and \( D_a(L) = L \). Following Theorem 3.12 (viii) and (ix) the set \( \tau_a = \{ D_a(X) | X \in U(L) \} \) is a topology on \( L \) and \( (L, \tau_a) \) is a topological space. \( \square \)

Proposition 3.24. The set \( \beta_a = \{ D_a(\uparrow x) | x \in L \} \) is a base for the topology \( \tau_a \) on \( L \).

Proof. Let \( Z \) be an open subset of \( (L, \tau_a) \). Then there is \( X \in U(L) \) such that \( Z = D_a(X) \). Since \( X \) is an upset, then \( X = \cup \{ \uparrow x | x \in X \} \) and by Theorem 3.12 (viii), we deduce that \( D_a(X) = \cup \{ D_a(\uparrow x) | x \in X \} \). Hence \( \beta_a \) is a base for the topology \( \tau_a \) on \( L \). \( \square \)

Corollary 3.25. Every open set \( X \) relative to the topology \( \tau_a \) is an upset. But the converse does not hold.

Proof. Following Corollary 3.23 and Proposition 3.24 we have \( (L, \tau_a) \) is a topological space with \( \beta_a \) a base for the topology \( \tau_a \). Since every union of elements of \( \beta_a \) is an upset and every open set \( X \) relative to \( \tau_a \) can be written as a union of elements of \( \beta_a \), then \( X \) is an upset.

For the converse we consider the residuated lattice \( L \) from Remark 3.17 (v), where \( \uparrow n = \{ n, a, b, c, d, 1 \} \) is an upset of \( L \). The upsets of \( L \) are \( \uparrow 0 = \varnothing, \uparrow n = \{ n, a, b, c, d, 1 \}, \uparrow a = \{ a, b, c, d, 1 \}, \uparrow b = \{ b, c, d, 1 \}, \uparrow c = \{ c, 1 \}, \uparrow d = \{ d, 1 \} \) and \( \uparrow 1 = \{ 1 \} \). Since \( U(L) = \{ \uparrow 0, \uparrow n, \uparrow a, \uparrow b, \uparrow c, \uparrow d, \uparrow 1 \} \). Since the base of topology \( \tau_a \) on \( L \) is the set \( \beta_a = \{ D_a(\uparrow x) | x \in L \} \), it is easy to verify that \( \uparrow n \) can not be written as a union of elements of \( \beta_a \). Therefore, \( \uparrow n \) is not an open set relative to the topology \( \tau_a \). \( \square \)

Proposition 3.26. If \( u, v \in L \) such that \( u \preceq v \), then the topology \( \tau_v \) is finer than topology \( \tau_u \).

Proof. We denote by \( \beta_u, \beta_v \) basis of the topology \( \tau_u, \tau_v \), respectively.

Consider \( t \in L \) and \( D_u(\uparrow t) \subseteq \beta_v \) an element of the basis \( \beta_v \). Then there exists \( n \in \mathbb{N} \) such that \( (u^{**})^n \rightarrow t^{**} \in \uparrow t \). Since \( \uparrow n = \{ n, a, b, c, d, 1 \} \) is an upset of \( L \), \( \uparrow 0 = \varnothing, \uparrow n = \{ n, a, b, c, d, 1 \}, \uparrow a = \{ a, b, c, d, 1 \}, \uparrow b = \{ b, c, d, 1 \}, \uparrow c = \{ c, 1 \}, \uparrow d = \{ d, 1 \} \) and \( \uparrow 1 = \{ 1 \} \). Since \( U(L) = \{ \uparrow 0, \uparrow n, \uparrow a, \uparrow b, \uparrow c, \uparrow d, \uparrow 1 \} \). Since the base of topology \( \tau_a \) on \( L \) is the set \( \beta_a = \{ D_a(\uparrow x) | x \in L \} \), it is easy to verify that \( \uparrow n \) can not be written as a union of elements of \( \beta_a \). Therefore, \( \uparrow n \) is not an open set relative to the topology \( \tau_a \). \( \square \)

Lemma 3.27. If \( X \) is a non-empty subset of \( L \) and \( a \in L \), then \( X \) is a compact subset of \( (L, \tau_a) \) iff \( X \subseteq \{ \downarrow \{ x_i | i \in I \}, \downarrow \{ x_i | i \in I \} \} \), for some \( x_1, x_2, \ldots, x_n \in X \) and \( i \in I \).

Proof. \( \Rightarrow \). Suppose \( X \subseteq \{ \downarrow \{ x_i | i \in I \}, \downarrow \{ x_i | i \in I \} \} \), for some \( x_1, x_2, \ldots, x_n \in X \) and \( \{ D_a(X_i) | i \in I \} \) be a family of open subsets of \( L \) whose union contains \( X \). For any \( j \in \{ 1, 2, \ldots, n \} \), there is \( i_j \in I \) such that \( x_{i_j} \in D_a(X_j) \). Following Theorem 3.12 (i) and (iv), we have \( D_a(x_{i_j}) \subseteq D_a(X_j) \), for any \( j \in \{ 1, 2, \ldots, n \} \). By Theorem 3.12 (viii) we obtain \( X \subseteq \{ \downarrow \{ x_i | i \in I \}, \downarrow \{ x_i | i \in I \} \} = D_a(x_{i_1}) \cup D_a(x_{i_2}) \cup \ldots \cup D_a(x_{i_n}) \subseteq D_a(X_{i_1}) \cup D_a(X_{i_2}) \cup \ldots \cup D_a(X_{i_n}) \cup \ldots \cup D_a(X_{i_n}) \). Hence \( X \) is compact.

\( \Rightarrow \). Now, suppose \( X \) be a compact subset of \( L \). Since \( X \subseteq \{ \downarrow \{ x_i | i \in I \}, \downarrow \{ x_i | i \in I \} \} \), then by Theorem 3.12(ii), \( X \subseteq \{ \downarrow \{ x_i | i \in I \} \in X \} \). Hence \( \{ D_a(\uparrow x) | x \in X \} \) is a family of open subsets of \( L \) whose union contains \( X \).
By hypothesis, there are \(x_1, x_2, \ldots, x_n \in X\) such that \(X \subseteq D_a(\uparrow x_1) \cup D_a(\uparrow x_2) \cup \ldots \cup D_a(\uparrow x_n)\). By Theorem 3.12(viii) we obtain \(X \subseteq D_a(\uparrow \{x_1, x_2, \ldots, x_n\})\).

**Theorem 3.28.** The topological space \((L, \tau_a)\) is connected.

**Proof.** Consider \(X\) a non-empty subset of \(L\) such that is both closed and open relative to the topology \(\tau_a\). If \(X\) is an open set, by Corollary 3.25, then \(X\) is an upset. If \(0 \in X\) and \(X\) is an upset, then \(X = L\). If \(0 \in L - X\), since \(X\) is closed, we get \(L - X\) is open, by Corollary 3.25, \(L - X\) is an upset of \(L\), since \(0 \in L - X\), then \(L - X = L\). Hence \(X = \emptyset\), a contradiction. We conclude that \((\emptyset, L)\) is the set of all subsets of \(L\) which are both closed and open. That is, \((L, \tau_a)\) is connected.

### 4 Some Properties of Quotient Topology on Residuated Lattices

**Proposition 4.1.** Let \(P, Q \in \mathcal{F}_i(L)\) and \(a \in L\) such that \(Q \subseteq P\). Then \(D_{[a]}(P/Q) = D_a(P)/Q\), where \([a] = \{aQ \mid a \in L\}\).

**Proof.** Consider \(x \in L\). Then
\[
D_{[a]}(P/Q) = \{[x] \in L/Q \mid [a]^m \otimes [x] \in P/Q, \text{ for some } m \in \mathbb{N}\} = \{[x] \in L/Q \mid D_{[a]}([a]^m \otimes x) \in P/Q\} = \{[x] \in L/Q \mid a^m \otimes x \in P\} = D_a(P)/Q.
\]

Following Proposition 3.24 we obtain the following result.

**Remark 4.2.** (i) Let \(F \in \mathcal{F}_i(L)\) and \(a \in L\). Then \((L/F, \tau_{a/F})\) is a topological space, where the set \(\tau_{a/F} = \{D_{a/F}(X/F) \mid X \in U(L)\}\) is a topology on \(L/F\) and the set \(\{D_{a/F}(\uparrow x/F) \mid x/F \in L/F\}\) is a base for the topology \(\tau_{a/F}\) on \(L/F\).

If \(\pi : L \to L/F\) is the canonical morphism defined by \(\pi(x) = [x]\), then \(\pi\) is a continuous map. Indeed, if \(O\) is an open set with respect the topology \(\tau_{a/F}\), then \(O = \bigcup \{D_{a/F}(\uparrow x/F) \mid x/F \in L/F\}\), and \(\pi^{-1}(O) = \bigcup \pi^{-1}(D_{a/F}(\uparrow x/F)) = \bigcup D_a(\pi^{-1}(\uparrow x/F))\), which is an open set on \((L, \tau_a)\).

(ii) Let \(F \in \mathcal{F}_i(L)\) and \(a \in L\). Then \((L/F, \tau_{a})\) is a topological space, where the set \(\tau_a = \{D_a(X/F) \mid X \in U(L)\}\) is a topology on \(L/F\) and the set \(\{D_a(\uparrow x/F) \mid x/F \in L/F\}\) is a base for the topology \(\tau_a\).

If \(\pi : L \to L/F\) is the canonical morphism defined by \(\pi(x) = [x]\), then \(\pi\) is a continuous map. Indeed, if \(O\) is an open set with respect the topology \(\tau_a\), then \(O = \bigcup \{D_a(\uparrow x) \mid x \in L\}\), and \(\pi^{-1}(O) = \bigcup \pi^{-1}(D_a(\uparrow x)) = \bigcup \pi^{-1}(D_a(\uparrow x/F))\), which is an open set on \((L, \tau_a)\).

**Theorem 4.3.** Let \(F\) be a filter of \(L\) and \(a, x \in L\). Then:

(i) \(D_a(\uparrow x/F) \subseteq D_{a[F]}(\uparrow x/F)\).

(ii) The topology \(\tau_{a/F}\) is finer than \(\tau_a\), where \(\tau_a\) is the quotient topology of \(L/F\).

**Proof.** (i) \(D_{a[F]}(\uparrow x/F) = \{u/F \in L/F \mid x/F \subseteq (a/F)^* \subseteq (u/F)^*\}\) = \{u/F \in L/F \mid x \to [(a^*)^* \to u^*] \in F\} and \(D_a(\uparrow x/F) = \{v/F \mid v \in D_a(\uparrow x)\}\). Let \(u/F \in D_{a[F]}(\uparrow x/F)\). Then there exists \(v \in D_a(\uparrow x)\) such that \(u/F = v/F\) and \(x \to [(a^*)^* \to v^*] = 1\), for some \(n \in \mathbb{N}\). Since \(u \equiv v\), then \(x \to [(a^*)^* \to u^*] = x \to [(a^*)^* \to v^*] = 1\), that is \(x \to [(a^*) \to u^*] \in F\), hence \(u/F \in D_a(\uparrow x/F)\). Therefore, \(D_a(\uparrow x/F) \subseteq D_{a[F]}(\uparrow x/F)\).

(ii) By Proposition 3.24 the set \(\{D_{a[F]}(\uparrow x/F) : x/F \in L/F\}\) is a base for the topology \(\tau_{a/F}\) on \(L/F\). Moreover, the set \(\{D_a(\uparrow x/F) : x \in L\}\) is a base for the topology \(\tau_a\). Now, the proof of (ii) is straightforward by (i).

Let \(L, L'\) be residuated lattices. On \(L \times L'\) we consider the relation of order \((x, y) \leq (x', y')\) iff \(x \leq x'\) and \(y \leq y'\) and the operations

\[
\begin{align*}
(x, y) \wedge (x', y') &= (x \wedge x', y \wedge y'), \\
(x, y) \vee (x', y') &= (x \vee x', y \vee y'), \\
(x, y) \odot (x', y') &= (x \odot x', y \odot y'), \\
(x, y) \rightarrow (x', y') &= (x \rightarrow x', y \rightarrow y') \quad \text{for all } x, y \in L, x', y' \in L'.
\end{align*}
\]
Then $L \times L'$ with the above operations is a residuated lattice called direct product of $L$ and $L'$.

**Lemma 4.4** ([17]). Let $L, L'$ be residuated lattices. Then $K$ is a filter of $L \times L'$ iff there exist $P \in \mathcal{F}_L(L)$ and $Q \in \mathcal{F}_L(L')$ such that $K = P \times Q$.

**Proof.** "$\Rightarrow$". If $K \in \mathcal{F}_L(L \times L')$, we have $P = \{ x \in L : (x, x') \in K \text{ for some } x' \in L' \}$ and $Q = \{ x' \in L' : (x, x') \in K \text{ for some } x \in L \}$.

Clearly, $K = P \times Q$. Since $(1, 1) \in K$ we get $1 \in P$. Let $x, y \in P$. Then there exist $x', y' \in L'$ such that $(x, x'), (y, y') \in K$. Thus, $(x, x') \circ (y, y') = (x \circ y, x' \circ y') \in K$, so $x \circ y \in P$.

Consider $x \leq y$ and $x \in P$. Then there exists $x' \in L'$ such that $(x, x') \in K$. Since $(x, x') \leq (y, y)$, then $y \in P$, hence $P \in \mathcal{F}_L(L)$. Similarly, $Q \in \mathcal{F}_L(L')$.

"$\Leftarrow$". Let $K = P \times Q$ for some $P \in \mathcal{F}_L(L)$ and $Q \in \mathcal{F}_L(L')$. Clearly, $K = L \times L'$. We consider $(x, y) \in K, (p, q) \in K$. Then $x, p \in P$ and $y, q \in Q$, that is $x \circ p \in P, y \circ q \in Q$. Therefore, $(x, y) \circ (p, q) = (x \circ p, y \circ q) \in P \times Q = K$.

Now, we consider $(x, y) \in K$ such that $(x, y) \leq (p, q)$. Then $x \leq p$ with $x \in P$ and $y \leq q$ with $y \in Q$. Since $P \in \mathcal{F}_L(L)$ and $Q \in \mathcal{F}_L(L')$, then $p \in P$ and $q \in Q$, that is $(p, q) \in K$. Hence $K \in \mathcal{F}_L(L \times L')$.

**Lemma 4.5.** Let $P, Q \in \mathcal{F}_L(L)$ and $a \in L$. Then $D_a(P \times D_a(Q)) = D_a(P \times Q)$.

**Proof.** By Lemma 4.4, $D_a(P \times D_a(Q)) = 
\{ x \in L : a^n \otimes x \in P, \text{ for some } n \in \mathbb{N} \} \times \{ y \in L : a^m \otimes y \in Q, \text{ for some } m \in \mathbb{N} \} = \{ x \in L \times L \times L | (a^t \otimes x, a^t \otimes y) \in P \times Q, \text{ for some } t \geq \max \{ m, n \} \}$.

Consider the topological spaces $(L, \tau_a)$ and $(L, \tau_b)$.

**Theorem 4.6.** Let $P, Q \in \mathcal{F}_L(L)$ and $a, b \in L$. Then there exists a homeomorphism from $\frac{L \times L}{D_a(P \times D_a(Q))}$ to $\frac{L \times L}{D_b(P \times D_b(Q))}$.

**Proof.** We define $\phi : L \times L \to \frac{L \times L}{D_a(P \times D_a(Q))} \times \frac{L \times L}{D_a(P \times D_a(Q))}$, by $\phi(x, y) = (\frac{x}{D_a(P \times D_a(Q))}, \frac{y}{D_a(P \times D_a(Q))})$. Clearly, $\phi$ is onto. Let $(x, y) \in L \times L$. Then $(x, y) \in \ker(\phi)$ iff $x/1_D_a(P) = 1/1_D_a(Q)$ and $y/1_D_a(Q) = 1/1_D_a(P)$ iff $x \in D_a(P)$ and $y \in D_a(Q)$. Hence $\ker(\phi) = D_a(P \times D_a(Q))$. It follows easily by Lemma 4.5 that $D_a(P \times D_a(Q)) = D_b(P \times Q)$. Consider the map $h : \frac{L \times L}{D_a(P \times D_a(Q))} \to \frac{L \times L}{D_b(P \times D_b(Q))}$, defined by $h(x/y) = (x/1_D_a(P \times D_a(Q)), y/1_D_a(P \times D_a(Q)))$, then by the first isomorphism theorem $h$ becomes an isomorphism. If we suppose that $X$ is an open subset of $\frac{L \times L}{D_a(P \times D_a(Q))}$, then there exist $U, V \in \tau_b$ open subsets such that $X = \frac{U \times V}{D_b(P \times D_b(Q))}$. Clearly, $h^{-1} = \frac{U \times V}{D_b(P \times D_b(Q))}$ is an open subset of $\frac{L \times L}{D_a(P \times D_a(Q))}$. Hence $h$ is a continuous map. In the same manner we can prove that $h^{-1}$ is a continuous map. Therefore, $h$ is a homeomorphism.

### 4.1 Uniform topology on quotient residuated lattice $L/J$

In this section based on the work of Ghorbani and Hasankhani (2010)[18] we define and study a uniform topology on the quotient residuated lattice $L/J$, where $J$ is a filter of $L$.

If $X$ is a non-empty set, $U$ and $V$ are subsets of $X \times X$. Then:

$U \circ V = \{(x, y) \in X \times X : (z, y) \in U \text{ and } (x, z) \in V \text{ for some } z \in X \}$,

$U^{-1} = \{(x, y) \in X \times X : (y, x) \in U \}$,

$\Delta = \{(x, x) \in X \times X : x \in X \}$.

In universal algebra if $X$ is a non-empty set, then a non-empty collection $K$ of subsets of $X \times X$ is called a uniformity on $X$ if it satisfies the following conditions:

$(U_1)$ $\Delta \subseteq U$ for any $U \in K$,

$(U_2)$ if $U \in K$, then $U^{-1} \in K$,

$(U_3)$ if $U \in K$, then there exists a set $V \in K$ such that $V \circ V \subseteq U$,

$(U_4)$ if $U, V \in K$, then $U \cap V \subseteq K$,

$(U_5)$ if $U \in K$ and $U \subseteq \nu \subseteq X \times X$, then $V \in K$. 


The pair \((X, K)\) is called a \textit{uniform structure}.

In Ghorbani and Hasankhani (2010)[18], Theorem 2.10, it was proved that if \(A\) is a family of filters of a residuated lattice \(L\) which is closed under intersection. If we consider the sets \(U_F = \{(x, y) \in L : x \cong_F y\}\) for every \(F \in A\) and \(K^* = \{U_F : F \in A\}\), then \(K = \{U \subseteq L : U_F \subseteq U\) for some \(U_F \in K\)\) is a uniformity on \(L\). Let \(U \in K\), define \(U_{[x]} = \{y \in L : (x, y) \in U\}\).

Then \(\tau_A = \{O \subseteq L : \forall x \in O, \exists U \in K\) such that \(U_{[x]} \subseteq O\}\) is a topology on \(L\) and is called the \textit{uniform topology} on \(L\) induced by \(A\).

In [18], Theorem 3.1, it was proved that if \(J\) is a filter of \(L\). For each \(F \in A\), let \(\mathcal{F} = (F \cup J)/J\). Then \(A^* = \{F : F \in A\}\) gives a uniform topology on \(L/J\), where \(U_{\mathcal{F}} = \{(x, [y]) \in L/J \times L/J : [x] \cong_{\mathcal{F}} [y]\}\).

The set \(\tau = \{O \subseteq L/J : \pi^{-1}(O) \subseteq \tau_A\}\), where \(\pi : L \rightarrow L/J\) is the canonical epimorphism, \(\tau\) becomes the quotient topology on \(L/J\), and \(\tau_{A^*}\) becomes the uniform topology induced by \(A^*\) on \(L/J\).

\textbf{Lemma 4.7.} Let \(F, Q \in \mathcal{F}_a(L)\) and \(a \in L\). Then:

(i) \(\langle a \rangle \cup F \subseteq D_a(F)\), but the converse does not hold;

(ii) \(\langle a \rangle \cup Q \cup F \subseteq D_a((Q \cup F))\), but the converse does not hold.

\textbf{Proof.} (i). Let \(F \in \mathcal{F}_a(L)\) and \(a \in L\). If \(x \in \langle a \rangle \cup F\) then \(x \geq a \circ f\), for some \(f \in F\). We obtain successively:

\[
(a^{**})^n \rightarrow x^{**} \quad \overset{(t_r)}{\geq} \quad (a \circ f)^{**} \quad \overset{(t_r)}{\geq} \quad a \circ (a \circ f)^{**} \quad \overset{(t_r)}{=} \quad (a \circ f)^{**} \quad \overset{(t_r)}{=} \quad a \circ [(a \circ f)^* \rightarrow 0] \overset{(t_r)}{=} \quad \frac{a \circ (a \circ f)^*}{0} \overset{(t_r)}{=} \quad \frac{a \circ (a \circ f)}{0} \overset{(t_r)}{=} \quad a \circ [a \circ f^*] \overset{(t_r)}{=} \quad a \circ f \overset{(t_r)}{=} \quad a \circ f^* \overset{(t_r)}{=} \quad 0 \otimes f^* \geq f \in F, \text{ that is } x \in D_a(F). \text{ Thus } \langle a \rangle \cup F \subseteq D_a(F).
\]

For the converse we consider \(L = \{0, n, a, b, c, d, e, f, m, 1\}\) with \(0 < n < a < c < e < m < 1, 0 < n < b < d < f < m < 1\) and the elements \(\{a, b\}, \{c, d\}, \{e, f\}\) are pairwise incomparable.

\[
\begin{array}{c|cccccccc}
 & 0 & n & a & b & c & d & e & f & m \\
\hline
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
n & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
f & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
m & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Let \(F = \{1, m, f\}\), then \(\langle d \rangle \cup F = \{x \in L : x \geq d \circ t \text{ for some } t \in F\} = \{d, e, f, m, 1\}\). But \(D_a(F) = \{x \in L : (d^{**})^n \rightarrow x^{**} \in F = \{L\}\) hence \(\langle d \rangle \cup F \subseteq D_a(F)\).
(ii). Let \( F, Q \in \mathcal{F}_J(L) \) and \( a \in L \). If \( x \in (\{a\} \cup Q \cup F) \) then \( x \geq a \odot Q \odot f \), for some \( q \in Q, f \in F \). We obtain successively \((a^*)^n \to x^{**} \geq (a^*)^n \to (a \odot Q \odot f)^* \) \((a^*)^n \to (a \odot Q \odot f)^* \geq a \to (a \odot Q \odot f)^* = a \to ((a \odot Q \odot f)^* \to 0)^{(\varnothing)} \), \((a \odot Q \odot f)^* \to 0 \) \((a \odot (a \odot (Q \odot f))^* \to 0 \) \((Q \odot f)^* \to 0 = (Q \odot f)^* \geq (Q \odot f) \in (Q \cup F) \), that is \( x \in D_a((Q \cup F)) \). Thus \((\{a\} \cup Q \cup F) \subseteq D_a((Q \cup F)) \).

For the converse we consider the residuated lattice \( L \) from (i), let \( Q = \{1, m, f\} \) and \( F = \{1, m, f\} \), then \([\{b\} \cup Q \cup F] = \{x \in L : \exists q \in Q, t \in F \} = \{b, c, d, e, f, m, 1\} \). But \( D_a((Q \cup F)) = \{x \in L : \forall i \in I \) \( x \geq a \odot Q \odot f \} \), hence \((\{b\} \cup F) \subseteq D_a((Q \cup F)) \).

Clearly, if \( a \in L \) and \( J, F \in \mathcal{F}_J(L) \), then \((J \cup F) \in \mathcal{F}_J(L) \) and \( D_a((J \cup F)) \in \mathcal{F}_J(L) \), too.

**Theorem 4.8.** Let \( a \in L \). The lattice \( (D_a(\mathcal{F}_J(L)), \subseteq) \) is a complete Brouwerian lattice (hence distributive).

**Proof.** Clearly, if \( \{X_i : i \in I\} \) is a family of filters from \( L \), then the infimum of this family is \( \bigwedge_i D_a(X_i) = \bigwedge_i D_a \big|_{\tau} \) and the supremum is \( \bigvee_i D_a(X_i) = \bigvee_i D_a \big|_{\tau} \), which are filters, too.

By Lemma 3.27 we have that an upset \( X \subseteq L \) is compact iff \( X \subseteq D_a((\uparrow \{x_1, x_2, \ldots, x_n\}) \), for some \( x_1, x_2, \ldots, x_n \in X \) and \( i \in I \). Since any filter is an upset we deduce that the compact elements of \( D_a(\mathcal{F}_J(L)) \) are exactly the upsets generated by the principal filters of \( L \), which are filters, too.

Following Theorem 3.12, \((\pi)\) we deduce that the lattice \( (D_a(\mathcal{F}_J(L)), \subseteq) \) is distributive.

Following Lemma 4.7 and Theorem 3.1 from [18] we obtain:

**Lemma 4.9.** Consider \( J \) a filter of \( L \) and \( a \in L \). For each \( F \in \Lambda \), let \( D_a(F) = D_a((J \cup F)) \). Then \( \Lambda = \{D_a((J \cup F)) : F \in \Lambda \} \) gives a uniform topology on \( L/J \), where \( U_{D_a(F)} = \{(x, [y]) \in L/J \times L/J : [x] = D_a(F)^{\pi \tau}(y)\} \).

That is, \( L/J \) is a topological space with the uniform topology \( \tau_\Lambda \) induced by \( \Lambda \).

For a residuated lattice \( L \) and a topology \( \tau \) defined on the set \( L \), the pair \((L, \tau)\) is called a *topological residuated lattice* if the operations \( \wedge, \vee, \odot \) and \( \to \) are continuous with respect to \( \tau \).

In the same manner as Theorem 3.2 from [18] the following result can be proved:

**Theorem 4.10.** For each \( x \in L \) and \( F \in \Lambda \), \( \pi(U_{D_a(F)}[x]) = U_{D_a(F)^\pi}(x) \). Hence each \( U_{D_a(F)^\pi}[x] \) is open in the quotient topology and \( \tau_\Lambda \subseteq \tau \). Moreover, \( \tau_\Lambda \subseteq \tau_\Lambda \subseteq \tau \).

Following Lemma 4.7, Lemma 4.9, Theorem 4.10 and Theorem 3.3 from [18] we obtain:

**Theorem 4.11.** \( \tau = \tau_\Lambda \subseteq \tau_\Lambda \) in \( L/J \). Moreover, if \( J \) is a filter, then \((L/J, \tau_\Lambda)\) is a topological residuated lattice.

We recall that for any non-empty set \( X \subseteq L \), we denote by \( X^c = L \setminus X \). In the same manner as Theorem 3.9 and Theorem 3.10, from [18] the following result can be proved:

**Theorem 4.12.** \((L/J, \tau_\Lambda)\) is Hausdorff iff \( J \) is closed and \((L/J, \tau_\Lambda)\) is discrete iff \( J \) is open.

**Theorem 4.13.** Let \( J \) be a filter of \( L \). If \( J \) is closed relative to the uniform topology \( \tau_\Lambda \), then \((L/J, \tau_\Lambda)\) is a regular space.

**Proof.** Consider \( J \) a filter of \( L \) such that \( J \) is closed relative to the uniform topology \( \tau_\Lambda \), then \((L/J, \tau_\Lambda)\) is Hausdorff. Since every Hausdorff and locally compact topological space is regular, we have to prove that \((L/J, \tau_\Lambda)\) is locally compact.

Let \( x \in L/J \). Since \( L \) is a locally compact space by uniform topology induced by \( \Lambda \), there exists an open neighborhood \( O \) of \( x \) and a compact set \( K \) such that \( x \in O \subseteq K \). Since \( x \) is open and continuous map, we have that \( \pi(O) \) is open and \( \pi(K) \) is compact such that \( [x] \in \pi(O) \subseteq \pi(K) \). Thus \( L/J \) is locally compact at \( [x] \) and hence is locally compact.
5 Some Classes of Semitopological Residuated Lattices

By definition the class of BL-algebras is a subclass of divisible residuated lattices. In [11], O. Zahiri and R. A. Borzooei studied the (semi)topological algebras and for a BL-algebra L they proved that (see Theorem 3.17,[11]) (L, {∨, ∧, ⊙}) is a semitopological BL-algebra and (L, {→}, τa) is a right semitopological BL-algebra, for an element a ∈ L. We prove that their result works in the case of divisible residuated lattices, which is a larger class than BL-algebras.

We notice the following rules of calculus:

Lemma 5.1. If L is a residuated lattice, then for a, p, q ∈ L and m ≥ 1 we have:

\[(e_1)\] (a∗∗)m → (q ∨ p)∗∗ = [(a∗∗)m → (q ∨ p)∗∗]∗∗;
\[(e_2)\] q → [(a∗∗)m → (q ∨ p)∗∗] = 1;
\[(e_3)\] ((a∗∗)m → (p∗∗)) → [(a∗∗)m → (q ∨ p)∗∗] = 1;
\[(e_4)\] q ∧ ((a∗∗)m → (p∗∗)) → ((a∗∗)m → (q ∨ p)) ∧ ((a∗∗)m → (p∗∗)) = 1;
\[(e_5)\] (q ∨ ((a∗∗)m ∧ (p∗∗)) → [(q ∨ p)∗∗] = 1.

Proof. (e1). Following (r35) we have a∗∗ ⊗ (q ∨ p) = (a∗∗)m → (q ∨ p)∗∗ by (r36) we obtain a∗∗ ⊗ (q ∨ p) = (a∗∗ ⊗ (q ∨ p))∗∗ = [(a∗∗)m → (q ∨ p)]∗∗. Hence (a∗∗)m → (q ∨ p)∗∗ = [(a∗∗)m → (q ∨ p)]∗∗.

The rule of calculus (e1) holds if instead of (q ∨ p) we have (q ∧ p), (q ⊗ p), (q → p) or p.

(e2). Since q ≤ q ∨ p ≤ (q ∨ p)∗∗ ≤ (a∗∗)m → (q ∨ p)∗∗, we obtain q → ((q ∨ p)∗∗) = 1.

(e3). Since p∗∗ ≤ (q ∨ p)∗∗ and by (r3), we obtain (a∗∗)m → (p∗∗) ≤ (a∗∗)m → (q ∨ p)∗∗, that is (a∗∗)m → (p∗∗) → ((a∗∗)m → (q ∨ p))∗∗ = 1.

(e4). Since q ≤ (q ∨ p)∗∗ ≤ (a∗∗)m → (q ∨ p)∗∗, then [q ∧ ((a∗∗)m → (p∗∗))] ≤ (((a∗∗)m → (q ∨ p)) ∧ ((a∗∗)m → (p∗∗))) = 1.

(e5). Since q ≤ q ∨ p ≤ (q ∨ p)∗∗ ≤ (a∗∗)m ∧ (p∗∗), then q ⊗ ((a∗∗)m ∧ (p∗∗)) ≤ (q ∨ p)∗∗. By (r18), (q ∨ p)∗∗ ≤ (q ⊗ p)∗∗, then [q ∨ (p∗∗)] ≤ (q ∨ p)∗∗ = 1. Hence [q ∨ ((a∗∗)m ∧ (p∗∗))] ≤ [q ∨ (p∗∗)] = 1.

Definition 5.2. Let τ be a topology on the divisible residuated lattice L. If (L, {∗}, τ), where {∗} ⊆ {∨, ∧, ⊙, →} is a (semi)topological algebra, then (L, {∗}, τ) is a (semi)topological divisible residuated lattice.

For simplicity, if {∗} ⊆ {∨, ∧, ⊙, →}, we consider (L, τ) instead of (L, {∨, ∧, ⊙, →}, τ).

Theorem 5.3. Let L be a divisible residuated lattice and a ∈ L. Then:
(i) (L, {∨, ∧, ⊙, τa}) is a semitopological divisible residuated lattice;
(ii) (L, {→}, τa) is a right semitopological divisible residuated lattice.

Proof. (i). Consider q an arbitrary element of L.

(1). We prove that (L, {∨, τa}) is a semitopological divisible residuated lattice.

Consider the map ϕq : L → L, defined by ϕq(x) = q ∨ x, for any x ∈ L. Following Proposition 3.24 the set ϕa = {Da(x | x ∈ L)} is a base for the topology τa on L. Then it suffices to prove that ϕq−1(Da(x | x ∈ L)) = τa, for any x ∈ L.

Let x ∈ L, then we obtain successively ϕq−1(Da(x | x ∈ L)) = {p ∈ L | ϕq(p) ∈ Da(x | x ∈ L)} = {p ∈ L | (a∗∗)m → (q ∨ p)∗∗ ∈ τa, for some n ∈ N} = {p ∈ L | x ≤ (a∗∗)m → (q ∨ p)∗∗, for some n ∈ N}.

Consider the set A = {p ∈ L | x ≤ (a∗∗)m → (q ∨ p)∗∗}, for some n ∈ N and we show that the set A is an upset of L. Let p ∈ A and p ≤ s, for some s ∈ L. Then there is n ∈ N such that x ≤ (a∗∗)m → (q ∨ p)∗∗. By (r3), (a∗∗)m → (q ∨ p)∗∗ ≤ (a∗∗)m → (q ∨ s)∗∗, and so s ∈ A. Hence A is an upset. Now, we prove that Da(A) = A, then we deduce that A ∈ τa, that is ϕq is a continuous map.

Let p ∈ Da(A). Then there is m ∈ N such that (a∗∗)m → (p∗∗) ∈ A, that is x ≤ (a∗∗)m → (q ∨ ((a∗∗)m → (p∗∗)))∗∗, for some n ∈ N.

We obtain successively

[(a∗∗)m → (q ∨ ((a∗∗)m → (p∗∗)))∗∗] →
Let $p$ be some $a$. Following Theorem 3.12 (ii), we deduce that $D = D_a(A)$, that is $\varphi_q$ is a continuous map.

(2) We prove that $(L, \wedge, \tau_a)$ is a semitopological divisible residuated lattice.

Consider the map $\psi_q : L \rightarrow L$, defined by $\psi_q(x) = q \otimes x$, for any $x \in L$. Following Proposition 3.24 the set $\beta = \{D_a(\uparrow x) \mid x \in L\}$ is a base for the topology $\tau_a$ on $L$. Then it suffices to prove that $\psi_q^{-1}(D_a(\uparrow x)) \in \tau_a$, for any $x \in L$.

Let $x \in L$, then we obtain successively $\psi_q^{-1}(D_a(\uparrow x)) = \{p \in L \mid \psi_q(p) \in D_a(\uparrow x)\} = \{p \in L \mid (a^{**})^{n} \rightarrow (q \wedge p)^{**} \in \uparrow x, \text{ for some } n \in \mathbb{N}\} = \{p \in L \mid (a^{**})^{n} \rightarrow (q \wedge p)^{**} \in \uparrow x, \text{ for some } n \in \mathbb{N}\}$.

Consider the set $B = \{p \in L \mid (a^{**})^{n} \rightarrow (q \wedge p)^{**} \in \uparrow x, \text{ for some } n \in \mathbb{N}\}$ and we show that the set $B$ is an upset of $L$. Let $p \in B$ and $p \leq s$, for some $s \in L$. Then there is $n \in \mathbb{N}$ such that $x \leq (a^{**})^{n} \rightarrow (q \wedge p)^{**}$. By (r4), $(a^{**})^{n} \rightarrow (q \wedge p)^{**} \leq (a^{**})^{n} \rightarrow (q \wedge s)^{**}$, and so $s \in B$. Hence $B$ is an upset. Now, we prove that $D_a(B) = B$, then we deduce that $B \in \tau_a$, that is $\psi_q^{-1}(D_a(\uparrow x)) \in \tau_a$, hence $\psi_q$ is a continuous map.

Let $p \in D_a(B)$. Then there is $m \in \mathbb{N}$ such that $(a^{**})^{m} \rightarrow (p^{**}) \in B$, that is $x \leq (a^{**})^{n} \rightarrow (q \wedge ((a^{**})^{m} \rightarrow (p^{**})))^{**}$, for some $n \in \mathbb{N}$. We obtain successively

$$[(a^{**})^{n} \rightarrow (q \wedge ((a^{**})^{m} \rightarrow (p^{**})))^{**}] \rightarrow [(a^{**})^{m+n} \rightarrow (q \wedge p)^{**}]$$

$$[(a^{**})^{n} \rightarrow (q \wedge ((a^{**})^{m} \rightarrow (p^{**})))^{**}] \rightarrow [(a^{**})^{n} \rightarrow (a^{**})^{m} \rightarrow (q \wedge p)^{**}]$$

$$[q \wedge ((a^{**})^{m} \rightarrow (p^{**}))^{**}] \rightarrow [(a^{**})^{m} \rightarrow (q \wedge p)^{**}]$$

$$[(a^{**})^{m} \rightarrow (p^{**})] \rightarrow ((a^{**})^{m} \rightarrow (q \wedge p)^{**})^{**}$$

$$[(a^{**})^{m} \rightarrow (p^{**})] \rightarrow (a^{**})^{m} \rightarrow (q \wedge p)^{**}$$

$$[(q \wedge ((a^{**})^{m} \rightarrow (p^{**}))) \rightarrow ((a^{**})^{m} \rightarrow (q \wedge p)^{**})^{**} \leq (a^{**})^{m+n} \rightarrow (q \wedge p)^{**}.$$
(a∗∗)n → (q ⊗ p)∗∗ ≤ (a∗∗)n → (q ⊗ s)∗∗, and so s ∈ C. Hence C is an upset. Now, we prove that Da(C) = C, then we deduce that C ∈ τa, that is ψq−1(Da(↑ x) ∈ τa, hence ψq is a continuous map.

Let p ∈ Da(C). Then there is m ∈ N such that (a∗∗)m → (p∗∗) ∈ C, that is x ≤ (a∗∗)n → (q ⊗ ((a∗∗)m → (p∗∗)))∗∗, for some n ∈ N.

We obtain successively
\[(a∗∗)n → (q ⊗ ((a∗∗)m → (p∗∗)))∗∗ \rightarrow [(a∗∗)m+n → (q ⊗ p)∗∗] (τ2)\]
\[\|(a∗∗)n → ((a∗∗)m → (q ⊗ p∗∗))\| \rightarrow [(a∗∗)m → (q ⊗ p)∗∗] (τ2)\]
\[q ⊗ ((a∗∗)m → (p∗∗))∗∗ \rightarrow [(a∗∗)m → (q ⊗ p)∗∗] (τ2)\]
\[q ⊗ ((a∗∗)m → (p∗∗))∗∗ \rightarrow [(a∗∗)m → (q ⊗ p)∗∗] (τ2)\]
\[([q ⊗ ((a∗∗)m → (p∗∗))] → [(a∗∗)m → (q ⊗ p)∗∗])∗ (τ2)\]
\[([q ⊗ ((a∗∗)m → (p∗∗))] → [(a∗∗)m → (q ⊗ p)∗∗])∗ (τ2)\]
\[([q ⊗ ((a∗∗)m → (p∗∗))] → [(q ⊗ p)∗∗])∗ (τ2)\]
\[([q ⊗ ((a∗∗)m → (p∗∗))] → [(q ⊗ p)∗∗])∗ (τ2)\]
\[([q ⊗ ((a∗∗)m → (p∗∗))] → [(q ⊗ p)∗∗])∗ (τ2)\]
\[([q ⊗ ((a∗∗)m → (p∗∗))] → [(q ⊗ p)∗∗])∗ (τ2)\]
\[([q ⊗ ((a∗∗)m → (p∗∗))] → [(q ⊗ p)∗∗])∗ (τ2)\]

By (τ2), (a∗∗)m → (q ⊗ ((a∗∗)m → (p∗∗)))∗∗ ≤ (a∗∗)m+n → (q ⊗ p)∗∗.

We deduce that x ≤ (a∗∗)n → (q ⊗ ((a∗∗)m → (p∗∗)))∗∗ ≤ (a∗∗)m+n → (q ⊗ p)∗∗, hence x ≤ (a∗∗)m+n → (q ⊗ p)∗∗, that is p ∈ C. Following Theorem 3.12 (ii), we deduce that C = Da(C), that is ψq is a continuous map.

Since ∨, ∧, ⊙ are commutative and from (1), (2) and (3) we deduce that (L, {∨, ∧, ⊙}, τa) is a semitopological divisible residuated lattice.

(ii). We prove that (L, {→}, τa) is a right semitopological divisible residuated lattice.

Consider the map ωq : L → L, defined by ωq(x) = q → x, for any x ∈ L. Following Proposition 3.24 the set βa = {Da(↑ x) | x ∈ L} is a base for the topology τa on L. Then it suffices to prove that ωq−1(Da(↑ x) ∈ τa, for any x ∈ L.

Let x ∈ L, then we obtain successively ωq−1(Da(↑ x) = {p ∈ L|ωq(p) ∈ Da(↑ x) = {p ∈ L|(a∗∗)n → (q → p)∗∗ ∈ ↑ x, for some n ∈ N} = {p ∈ L|x ≤ (a∗∗)n → (q → p)∗∗, for some n ∈ N}.

Consider the set D = {p ∈ L|x ≤ (a∗∗)n → (q → p)∗∗, for some n ∈ N} and we show that the set D is an upset of L. Let p ∈ D and p ≤ s, for some s ∈ L. Then there is n ∈ N such that x ≤ (a∗∗)n → (q → p)∗∗. By (τ4), (a∗∗)n → (q → p)∗∗ ≤ (a∗∗)n → (q → s)∗∗, and so s ∈ D. Hence D is an upset. Now, we prove that Da(D) = D, then we deduce that D ∈ τa, that is ωq−1(D(↑ x) ∈ τa, hence ωq is a continuous map.

Let p ∈ Da(D). Then there is m ∈ N such that (a∗∗)m → (p∗∗) ∈ D, that is x ≤ (a∗∗)n → (q → ((a∗∗)m → (p∗∗)))∗∗, for some n ∈ N.

We obtain successively
\[(a∗∗)n → (q → ((a∗∗)m → (p∗∗)))∗∗ (τ2) → (a∗∗)n → (q → ((a∗∗)m → (p∗∗)))∗∗ (τ2)\]
\[(a∗∗)n → (q → ((a∗∗)m → (p∗∗)))∗∗ (τ2) → (a∗∗)n → (q → ((a∗∗)m → (p∗∗)))∗∗ (τ2)\]
\[(a∗∗)n → (q → ((a∗∗)m → (p∗∗)))∗∗ (τ2) → (a∗∗)n → (q → ((a∗∗)m → (p∗∗)))∗∗ (τ2)\]
\[(a∗∗)n → (q → ((a∗∗)m → (p∗∗)))∗∗ (τ2) → (a∗∗)n → (q → ((a∗∗)m → (p∗∗)))∗∗ (τ2)\]
\[(a∗∗)n → (q → ((a∗∗)m → (p∗∗)))∗∗ (τ2) → (a∗∗)n → (q → ((a∗∗)m → (p∗∗)))∗∗ (τ2)\]
\[(a∗∗)n → (q → ((a∗∗)m → (p∗∗)))∗∗ (τ2) → (a∗∗)n → (q → ((a∗∗)m → (p∗∗)))∗∗ (τ2)\]
\[(a∗∗)n → (q → ((a∗∗)m → (p∗∗)))∗∗ (τ2) → (a∗∗)n → (q → ((a∗∗)m → (p∗∗)))∗∗ (τ2)\]
\[(a∗∗)n → (q → ((a∗∗)m → (p∗∗)))∗∗ (τ2) → (a∗∗)n → (q → ((a∗∗)m → (p∗∗)))∗∗ (τ2)\]
\[(a∗∗)n → (q → ((a∗∗)m → (p∗∗)))∗∗ (τ2) → (a∗∗)n → (q → ((a∗∗)m → (p∗∗)))∗∗ (τ2)\]

We deduce that x ≤ (a∗∗)n → (q → ((a∗∗)m → (p∗∗)))∗∗ ≤ (a∗∗)m+n → (q → p)∗∗, hence x ≤ (a∗∗)m+n → (q → p)∗∗, that is p ∈ D. Following Theorem 3.12 (ii), we deduce that D = Da(D), that is ωq is a continuous map.

Since → is not commutative we deduce that (L, {→}, τa) is a right semitopological divisible residuated lattice.
In [3] it was proved that \( L \) is an MV-algebra iff \( L \) is an involutive BL-algebra. Therefore, following Theorem 5.3 and Corollary 5.4 we deduce that:

**Corollary 5.5.** Let \( L \) be an MV-algebra and \( a \in L \). Then:
(i) \( (L, \{\lor, \land, \odot\}, \tau_a) \) is a semitopological MV-algebra;
(ii) \( (L, \{\rightarrow\}, \tau_a) \) is a right semitopological MV-algebra.

**Corollary 5.6.** Let \( L \) be a divisible residuated lattice. For any filter \( F \) of \( L \) and \( a, x \in L \), then:
(i) \( D_a(\uparrow x)/F = D_{a/F}(\uparrow x/F) \);
(ii) The topologies \( \tau_{a/F} \) and \( \tau_a \) on \( L/F \) are the same, where \( \tau_a \) is the quotient topology of \( L/F \).

**Proof.** (i). By Theorem 4.3 (i) we get \( D_a(\uparrow x)/F \subseteq D_{a/F}(\uparrow x/F) \), then it remains to prove \( D_{a/F}(\uparrow x/F) \subseteq D_a(\uparrow x)/F \). Let \( u/F \in D_{a/F}(\uparrow x/F) \). Then there exists \( n \in \mathbb{N} \) such that \( x \rightarrow [(a^{**})^n \rightarrow u^{**}] \in F \). Let \( f = x \rightarrow [(a^{**})^n \rightarrow u^{**}] \). Then by \((r_2)\) we obtain \( x \rightarrow ((a^{*})^n \rightarrow [f \rightarrow u^{**}]) = f \rightarrow [x \rightarrow ((a^{*})^n \rightarrow u^{**})] \). Since \( f \in F \), then \( f/F = 1/F \) and so \( f \rightarrow u) \in F = f/F \rightarrow u/F = 1/F \rightarrow u/F = (1 \rightarrow u)/F = u/F \). That is \( f \rightarrow u) \equiv_F u \). Therefore, \( u/F \in D_a(\uparrow x)/F \).

(ii). By Proposition 3.24 the set \( \{D_{a/F}(\uparrow x/F) \times X \in L/F \} \) is a base for the topology \( \tau_{a/F} \) on \( L/F \). Moreover, the set \( \{D_a(\uparrow x)/F : x \in L \} \) is a base for the topology \( \tau_a \). Now, the proof of (ii) is straightforward by (i).

**Corollary 5.7.** Let \( L \) be an involutive residuated lattice. For any filter \( F \) of \( L \) and \( a, x \in L \), then:
(i) \( D_a(\uparrow x)/F = D_{a/F}(\uparrow x/F) \);
(ii) The topologies \( \tau_{a/F} \) and \( \tau_a \) on \( L/F \) are the same, where \( \tau_a \) is the quotient topology of \( L/F \).

**Proof.** (i). By Theorem 4.3 (i) we get \( D_a(\uparrow x)/F \subseteq D_{a/F}(\uparrow x/F) \), then it remains to prove \( D_{a/F}(\uparrow x/F) \subseteq D_a(\uparrow x)/F \). Let \( u/F \in D_{a/F}(\uparrow x/F) \). Then there exists \( n \in \mathbb{N} \) such that \( x \rightarrow [(a^{**})^n \rightarrow u^{**}] \in F \). Let \( f = x \rightarrow [(a^{*})^n \rightarrow u^{**}] \). Then by \((r_2)\) we obtain \( x \rightarrow ((a^{**})^n \rightarrow [f \rightarrow u^{**}]) = f \rightarrow [x \rightarrow ((a^{*})^n \rightarrow u^{**})] \). Since \( f \in F \), then \( f/F = 1/F \) and so \( f \rightarrow u) \in D_a(\uparrow x) \). Since \( f \in F \), then \( f/F = 1/F \) and so \( f \rightarrow u) \in F = f/F \rightarrow u/F = 1/F \rightarrow u/F = (1 \rightarrow u)/F = u/F \). That is \( f \rightarrow u) \equiv_F u \). Therefore, \( u/F \in D_a(\uparrow x)/F \).

(ii). By Proposition 3.24 the set \( \{D_{a/F}(\uparrow x/F) : x \in L/F \} \) is a base for the topology \( \tau_{a/F} \) on \( L/F \). Moreover, the set \( \{D_a(\uparrow x)/F : x \in L \} \) is a base for the topology \( \tau_a \). Now, the proof of (ii) is straightforward by (i).

6 Conclusions

In Borzooei and Zahiri (2014)[11] the definition of double complemented elements for any filter \( F \) in BL-algebras, initially, introduced by A. Borumand Saeid and S. Motamed (2009)[10], was generalized to the concept of \( D_a(F) \), for any upset \( F \) of the BL-algebra \( L \). Their aim was to show that any BL-algebra \( L \) with that topology is a semitopological BL-algebra, so in that way they could construct many semitopological BL-algebras. Our goal is to extend the study in the case of residuated lattices. Therefore, we work on a special type of topology induced by a modal and closure operator denoted by \( D_a(X) \), for an upset \( X \) of a residuated lattice \( L \), where \( a \) is an element of \( L \).

We discuss briefly the properties and applications of the operator \( D_a(\cdot) \) in residuated lattices. We obtain some of the important topological aspects of these structures such as connectivity and compactness. We study some properties of quotient topologies on residuated lattices by considering two types of quotient topologies denoted by \( \tau_{a/F} \) and \( \tau_{\tilde{a}} \). We study the uniform topology \( \tau_{\tilde{a}} \) and we obtain important characterizations as \( (L/F, \tau_{\tilde{a}}) \) becomes a Hausdorff space iff \( F \) is closed relative to the uniform topology. Also, we study some properties of the direct product of residuated lattices.
Finally, we apply our results on classes of residuated lattices such as *divisible residuated lattices, MV-algebras* and *involutive residuated lattices* and we find that any of this subclasses of residuated lattices with respect to these topologies form semitopological algebras.

In this way we can construct many semitopological algebras on residuated lattices.

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**References**


