An improved Schwarz Lemma at the boundary

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Abstract: We obtain a new boundary Schwarz inequality, for analytic functions mapping the unit disk to itself. The result contains and improves a number of known estimates.

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MSC: 30C80

1 Introduction

Denote by $\Delta \subset \mathbb{C}$ the open unit disk, and let $f: \Delta \to \Delta$ be analytic. We assume that there is $x \in \partial \Delta$ and $\beta \in \mathbb{R}$ such that

$$\liminf_{z \to x} \frac{1-|f(z)|}{1-|z|} = \beta. \quad (1)$$

By pre-composing with a rotation we may suppose that $x=1$, and by post-composing with a rotation we may suppose that $f(1)=1$. Then Julia’s Lemma (e.g. [1, 2]) gives

$$\frac{|1-f(z)|^2}{1-|f(z)|^2} \leq \beta \frac{1-|z|^2}{1-|z|^2} \quad \forall z \in \Delta.

This inequality has an appealing geometric interpretation, which we do not use here. But two immediate consequences which we do use, are that $\beta > 0$ and that the radial derivative of $f$ exists at $1 \in \partial \Delta$:

$$\lim_{r \to 1} \frac{f(r)-f(1)}{r-1} = f'(1) \quad \text{with} \quad |f'(1)| = \beta. \quad (2)$$

(There are many other consequences of Julia’s Lemma, the most important being contained in the Julia-Carathéodory Theorems.)

Assuming the normalization $f(0)=0$, we evidently have $\beta \geq 1$. But even better, Osserman [3] showed that in this case

$$\beta \geq 1 + \frac{1-|f'(0)|}{1+|f'(0)|}. \quad (3)$$

(A proof of (3) can also be found in [4], which is motivated by the influential paper [5].) Now Osserman’s inequality was in fact anticipated by Ünkelbach [6], who had already obtained the better estimate

$$\beta \geq \frac{2(1-\text{Re} f'(0))}{1-|f'(0)|^2} = 1 + \frac{|1-f'(0)|^2}{1-|f'(0)|^2}. \quad (4)$$

However, [3] also contains a non-normalized version, which reduces to (3) if $f(0)=0$, viz.

$$\beta \geq \frac{2(1-|f(0)|)^2}{1-|f'(0)|^2 + |f'(0)|}. \quad (5)$$

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Since the appearance of Osserman’s paper, a good number of authors have refined and generalized these estimates – as discussed in the next section. The aim here is to provide a different and very elementary approach, which contains and improves many of these modifications. But first we recall some results which are of use in the sequel.

The well-known Schwarz’s Lemma, which is a consequence of the Maximum Principle, says that if \( f : \Delta \to \Delta \) is analytic with \( f(0) = 0 \), then
\[
|f(z)| \leq |z| \quad \forall z \in \Delta, \quad \text{and consequently} \quad |f'(0)| \leq 1.
\]
To remove the normalization \( f(0) = 0 \), one applies Schwarz’s Lemma to \( \phi_{f(a)} \circ f \circ \phi_a \) where \( \phi_a \) is the automorphism of \( \Delta \) which interchanges \( a \) and 0:
\[
\phi_a(z) = \frac{a - z}{1 - \bar{a} z}.
\]
This gives the Schwarz-Pick Lemma which says that for \( f : \Delta \to \Delta \) analytic,
\[
|f(w) - f(z)| \leq \frac{|w - z|}{1 - |w z|} \quad \forall z, w \in \Delta.
\]
Consequently, the hyperbolic derivative satisfies
\[
|f^*(z)| \leq 1 \quad \forall z \in \Delta, \quad \text{where} \quad f^*(z) = \frac{1 - |z|^2}{1 - |f(z)|^2} f'(z).
\]
It is the Schwarz-Pick Lemma that does most of the work in proving Julia’s Lemma. But another consequence of the Schwarz-Pick Lemma is the following (e.g. [7–9]), which we shall also rely upon.

**Lemma 1.1** (Dieudonné’s Lemma). Let \( f : \Delta \to \Delta \) be analytic, with \( f(z) = w \) and \( f(z_1) = w_1 \). Then
\[
|f'(z) - c| \leq r,
\]
where
\[
c = \frac{\phi_w (w_1) (1 - |\phi_w (z_1)|^2) - |w|^2}{\phi_w (z_1) (1 - |\phi_w (w_1)|^2) - |z|^2}, \quad r = \frac{|\phi_w (z_1)|^2}{|\phi_w (w_1)|^2} \frac{1 - |w|^2}{1 - |z|^2}.
\]

## 2 Main result

We remove the dependence on \( f(0) \), while improving many estimates which do contain \( f(0) \). We shall rely on Dieudonné’s Lemma, the Schwarz-Pick Lemma, and Julia’s Lemma.

**Theorem 2.1.** Let \( f : \Delta \to \Delta \) be analytic with \( f(z) = w \) and \( f(1) = 1 \) as in (I). Then
\[
\beta \geq 2 \frac{1 - |w|^2 |1 - |z|^2|}{1 - |w|^2 |1 - |z|^2|} \frac{1 - \text{Re} \left( f^*(z) \frac{1 - \bar{w} - \bar{z}}{1 - |w|^2} \right)}{1 - |f^*(z)|^2}.
\]

**Proof.** Using the easily verified identity
\[
1 - |\phi_w(\lambda)|^2 = \frac{(1 - |a|^2)(1 - |\lambda|^2)}{|1 - \bar{a}\lambda|^2},
\]
we get, in Dieudonné’s Lemma,
\[
c = \frac{w_1 - w}{1 - \bar{w}w_1} \frac{1 - |z|^2}{1 - |z_1|^2} \frac{|1 - \bar{w}w_1|^2}{1 - |w_1|^2} = \frac{w_1 - w}{1 - \bar{w}w_1} \frac{1 - |z|^2}{1 - |z_1|^2} \frac{1 - |w_1|^2}{1 - |w_1|^2},
\]
and
\[
    r = \frac{(1 - |\phi(w)z|^2) - (1 - |\phi(z_1)|^2)}{|\phi(z_1)|^2} = \frac{1}{\phi(z_1)} \left( 1 - \frac{1 - |z|^2}{1 - |w|^2} \right) \frac{1 - |w|^2}{1 - |z|^2}.
\]
then having \( z_1 \to 1 \) along a sequence for which \( \beta \) in (1) is attained, we get
\[
    c \to c = \left( \frac{1 - w}{1 - z} \right)^2 \quad \text{and} \quad r \to r = \frac{1 - |w|^2}{1 - |z|^2} = \frac{1 - |w|^2}{\beta (1 - |z|^2)}.
\]
That is,
\[
    |f'(z) - c| \leq r. \tag{8}
\]
Now, upon squaring both sides of this inequality, there is some cancellation:
\[
    |f'(z)|^2 - 2 \text{Re} \left( \frac{f(z)}{1 - z} \right)^2 \leq \frac{2 (1 - |w|^2)^2}{\beta} \frac{1 - |w|^2}{|1 - z|^2}.
\]
That is,
\[
    \left( \frac{1 - |w|^2}{1 - |z|^2} \right)^2 \left( |f''(z)|^2 - 1 \right) \leq \frac{2 (1 - |w|^2)(1 - |z|^2)^2}{\beta},
\]
By the Schwarz-Pick Lemma each side of this last inequality is nonpositive, so isolating \( \beta \) we get (6).

\textbf{Remark 2.2.} Having \( z \to 1 \) radially in line (8), and using (2), we obtain
\[
    \lim_{r \to 1} f'(r) = f'(1).
\]
From this, and using \( |r| = 1 \Rightarrow \frac{1 - \text{Re}(\sigma r)}{1 - |\sigma r|^2} \geq \frac{1}{1 - |\sigma|^2} \), follows the rather comforting fact that the right-hand side of (6) tends to \( \beta \) as \( z \to 1 \) radially.

\textbf{Remark 2.3.} In Lemma 6.1 of [8] is the estimate
\[
    \beta \geq \frac{2}{1 + |f'(z)|} \frac{1 - |f(z)|}{1 + |f(z)|} \tag{9}
\]
which contains (5), but is quite mild if \( |z| \) or \( |f(z)| \) is near 1. Anyway, \( |r| = 1 \Rightarrow \frac{1 - \text{Re}(\sigma r)}{1 - |\sigma|^2} \geq \frac{1}{1 - |\sigma|^2} \) shows that (6) improves (9).

\textbf{Remark 2.4.} Now take \( z = 0 \), so that (6) reads
\[
    \beta \geq \frac{2 |1 - f(0)|^2}{1 - |f(0)|^2} \frac{1 - \text{Re} \left( f'(0) \frac{1 - f(0)}{1 - f(0)} \right)}{1 - |f'(0)|^2}. \tag{10}
\]
This may be regarded as an non-normalized version of (4). Indeed, taking also \( f(0) = 0 \) recovers (4). This is the same estimate which results from having \( z = 0 \) in Theorem 5 of [10]. However, that result (which is arrived at by very nonelementary means) contains \( f(0) \) even for \( z \neq 0 \), a deficiency from which Theorem 2.1 does not suffer.

\textbf{Remark 2.5.} Using again \( |r| = 1 \Rightarrow \frac{1 - \text{Re}(\sigma r)}{1 - |\sigma|^2} \geq \frac{1}{1 - |\sigma|^2} \) in (10), we get
\[
    \beta \geq \frac{2 |1 - f(0)|^2}{1 - |f(0)|^2} \frac{1 - \text{Re} \left( f'(0) \frac{1 - f(0)}{1 - f(0)} \right)}{1 - |f'(0)|^2},
\]
which improves (5), analogously to how (4) improves (3).

\textbf{Remark 2.6.} But using just \( |r| = 1 \Rightarrow \frac{1 - \text{Re}(\sigma r)}{1 - |\sigma|^2} \geq \frac{1}{1 - |\sigma|^2} \) in (10), then \( \frac{1 - f(0)}{1 - f'(0)} = \text{Re} \left( \frac{1 + f(0)}{1 - f'(0)} \right) \), we get
\[
    \beta \geq \frac{2}{\text{Re} \left( \frac{1 - f(0)}{1 - f'(0)} \right)}, \tag{11}
\]
which improves (5) more effectively. Estimate (11) was obtained differently in each of [11] and [12].
3 Consequences

Cases for which $z = w = 0$ (i.e., $f(0) = 0$) are obviously contained in the remarks above, but when this holds we can do a little better, as follows.

Corollary 3.1. Let $f : \Delta \to \Delta$ be analytic with $f(0) = 0$ and $f(1) = 1$ as in (1). Then

$$\beta \geq 1 + \frac{2|1 - f'(0)|^2}{1 - |f'(0)|^2 + |f''(0)|/2} \frac{1 + \text{Re} \left( \frac{f''(0)}{2(1 - |f'(0)|^2)} \right)}{1 - |f''(0)|/2}.$$  \hfill (12)

Proof. We introduce $f''(0)$, in standard fashion: Set

$$g(\lambda) = \frac{f(\lambda)}{\lambda} \quad \text{(with } g(0) := f'(0)), \quad \text{and } h(\lambda) = \phi_{g(0)}(g(\lambda)).$$

Then $h$ is analytic on $\Delta$ with $h(0) = 0$, and by Schwarz’s Lemma $h : \Delta \to \Delta$. Here we have

$$h'(0) = \frac{-f''(0)}{2(1 - |f'(0)|^2)}.$$ \hfill (13)

A calculation using the identity (7) and the assumption (1) gives

$$\liminf_{z \to 1} \frac{1 - |h(z)|}{1 - |z|} = (\beta - 1) \frac{1 - |f'(0)|^2}{1 - |f'(0)|^2} = \beta,$$ \hfill (14)

say. Then in (6), i.e. (4), replacing $f$ with $h$ and $\beta$ with $\beta$, we obtain

$$\beta \geq 1 + \frac{|1 - f'(0)|^2}{1 - |f'(0)|^2} \frac{2(1 - \text{Re} h'(0))}{1 - |h'(0)|^2}.$$ 

Inserting (13) and a little tidying yields (12), as desired. \hfill $\Box$

Remark 3.2. Corollary 3.1 improves

$$\beta \geq 1 + \frac{2(1 - |f'(0)|^2)}{1 - |f'(0)|^2 + |f''(0)|/2},$$ \hfill (15)

which was obtained by Dubinin [13] using a proof which relies directly on (3). (Incidentally, Schwarz’s Lemma applied to $h$ gives $|f''(0)|/2 \leq 1 - |f'(0)|^2$, from which it is readily seen that (15) improves (3).)

Remark 3.3. We add finally that using (4) in the form

$$\beta \geq 1 + \frac{|1 - f'(0)|^2}{1 - |f'(0)|^2},$$

then replacing $f$ with $h$ and $\beta$ with $\beta$ here, and using (13) and (14), we get another way of expressing (12):

$$\beta \geq 1 + \frac{|1 - f'(0)|^2}{1 - |f'(0)|^2} \left( 1 + \frac{|1 + \frac{f''(0)}{2(1 - |f'(0)|^2)}|^2}{1 - \frac{|f''(0)|^2}{2(1 - |f'(0)|^2)}} \right)$$

$$= 1 + \frac{|1 - f'(0)|^2}{1 - |f'(0)|^2} \left( 1 + \frac{|1 + \frac{f''(0)}{2(1 - |f'(0)|^2)}|^2}{1 - \frac{|f''(0)|^2}{2(1 - |f'(0)|^2)}} \right) \frac{|1 - f'(0)|^2}{1 - |f'(0)|^2 + |f''(0)|/2}.$$
References