Weak group inverse

Hongxing Wang* and Jianlong Chen

Open Mathematics

Research Article

https://doi.org/10.1515/math-2018-0100
Received December 13, 2017; accepted September 12, 2018.

Abstract: In this paper, we introduce the weak group inverse (called as the WG inverse in the present paper) for square complex matrices of an arbitrary index, and give some of its characterizations and properties. Furthermore, we introduce two orders: one is a pre-order and the other is a partial order, and derive several characterizations of the two orders. The paper ends with a characterization of the core EP order using WG inverses.

Keywords: Group inverse, Weak group inverse, WG order, CE partial order, Core-EP decomposition

MSC: 15A09, 15A57, 15A24

1 Introduction

In this paper, we use the following notations. The symbol \( \mathbb{C}_{m,n} \) is the set of \( m \times n \) matrices with complex entries; \( A^*, \mathcal{R}(A) \) and \( \text{rk}(A) \) represent the conjugate transpose, range space (or column space) and rank of \( A \in \mathbb{C}_{m,n} \), respectively. Let \( A \in \mathbb{C}_{n,n} \) be singular, the smallest positive integer \( k \) satisfying \( \text{rk}(A^{k+1}) = \text{rk}(A^k) \) is called the index of \( A \) and is denoted by \( \text{Ind}(A) \). The index of a non-singular matrix \( A \) is 0 and the index of a null matrix is 1. The symbol \( \mathbb{C}^n_n \) stands for a set of \( n \times n \) matrices of index less than or equal to 1. The Moore-Penrose inverse of \( A \in \mathbb{C}_{m,n} \) is defined as the unique matrix \( X \in \mathbb{C}_{n,m} \) satisfying the equations:

\[
(1) \ AXA = A, \quad (2) \ XAX = X, \quad (3) \ (AX)^* = AX, \quad (4) \ (XA)^* = XA,
\]

and is denoted as \( X = A^\dagger; \ P_A \) stands for the orthogonal projection \( P_A = AA^\dagger \). A matrix \( X \) such that \( AXA = A \) is called a generalized inverse of \( A \). The Drazin inverse of \( A \in \mathbb{C}_{n,n} \) is defined as the unique matrix \( X \in \mathbb{C}_{n,n} \) satisfying the equations

\[
(6^\dagger) \ AX^{k+1}A = A^k, \quad (2) \ XAX = X, \quad (5) \ AX = XA,
\]

and is usually denoted as \( X = A^D \), where \( k = \text{Ind}(A) \). In particular, when \( A \in \mathbb{C}^n_n \), the matrix \( X \) is called the group inverse of \( A \), and is denoted as \( X = A^g \) (see [1]). The core inverse of \( A \in \mathbb{C}^n_n \) is defined as the unique matrix \( X \in \mathbb{C}_{n,n} \) satisfying

\[
AX = AA^\dagger, \quad \mathcal{R}(X) \subseteq \mathcal{R}(A)
\]

and is denoted as \( X = A^c \) [2]. When \( A \in \mathbb{C}^n_n \), we call it a core invertible (or group invertible) matrix.

Several generalizations of the core inverse have been introduced, for example, the DMP inverse[3] the BT inverse[4] and the core-EP inverse[5], etc. Let \( A \in \mathbb{C}_{n,n} \) with \( \text{Ind}(A) = k \). The DMP inverse of \( A \) is \( A^{D^*} = A^DAA^\dagger \) [3]. The BT inverse of \( A \) is \( A^* = (A^2A)^\dagger \) [4, Definition 1]. The core-EP inverse of \( A \) is

---

*Corresponding Author: Hongxing Wang: School of Mathematics, Southeast University, Nanjing, 210096, China and School of Science, Guangxi Key Laboratory of Hybrid Computation and IC Design Analysis, Guangxi University for Nationalities, Nanning, 530006, China, E-mail: winghongxing0902@163.com
Jianlong Chen: School of Mathematics, Southeast University, Nanjing, 210096, China, E-mail: jlchen@seu.edu.cn

Lemma 2.3

where $T$ index and was introduced by Wang [13]. We record it as:

Very similar to core-nilpotent decomposition is the core-EP decomposition of a square matrix of arbitrary matrices $A$ and was introduced by Wang [13]. We record it as:

Furthermore, it is known that the index of a group invertible matrix is less than or equal to 1, that is, a matrix is core invertible if and only if it is group invertible. Although the generalizations of the core inverse have attracted huge attention, the generalizations of group inverse have not received the same kind of attention. Therefore, it is of interest to inquire whether one can do something similar to the group inverse and that too by using some matrix decompositions as a tool as it has been used to study generalizations of core inverse.

In this paper, our main tool is the core-EP decomposition. By using this decomposition, we introduce a generalization of the group inverse for square matrices of an arbitrary index. We also give some of its characterizations, properties and applications.

2 Preliminaries

In this section, we present some preliminary results.

Lemma 2.1 ([1]). Let $A \in \mathbb{C}_{n,n}$ with $\text{Ind}(A) = k$. Then

$$A^\circ = A^k \left( (A^+) \left( A^{k+1} \right)^\circ \right)^\circ.$$  \hspace{1cm} (1)

The following decomposition is attributed to Hartwig and Spindelböck [11] and is called Hartwig-Spindelböck decomposition

Lemma 2.2 ([11, Hartwig-Spindelböck Decomposition]). Let $A \in \mathbb{C}_{n,n}$ with $\text{rk}(A) = r$. Then there exists a unitary matrix $U$ such that

$$A = U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^*,$$  \hspace{1cm} (2)

where $\Sigma = \text{diag} (\sigma_1 I_{r_1}, \sigma_2 I_{r_2}, \ldots, \sigma_t I_{r_t})$ is the diagonal matrix of singular values of $A$, $\sigma_1 > \sigma_2 > \ldots > \sigma_t > 0$, $r_1 + r_2 + \cdots + r_t = r$, and $K \in \mathbb{C}_{r_1,n}, L \in \mathbb{C}_{r_t,n-r}$ satisfy $KK^* + LL^* = I_r$.

Furthermore, $A$ is core invertible if and only if $K$ is non-singular, [2]. When $A \in \mathbb{C}_{n,n}$, it is easy to check that

$$A^\circ = U \begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*,$$  \hspace{1cm} (3)

$$A^\circ = U \begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix} U^*,$$  \hspace{1cm} (4)

where $T = \Sigma K$ and $S = \Sigma L$.

The core-nilpotent decomposition of a square matrix is widely used in matrix theory [1, 12] and just to remind ourselves it is given as:

Lemma 2.3 ([12, Core-nilpotent Decomposition]). Let $A \in \mathbb{C}_{n,n}$ with $\text{Ind}(A) = k$, then $A$ can be written as the sum of matrices $\tilde{A}_1$ and $\tilde{A}_2$, i.e. $A = \tilde{A}_1 + \tilde{A}_2$, where

$$\tilde{A}_1 \in \mathbb{C}_{n,n}, \tilde{A}_2^k = 0 \text{ and } \tilde{A}_1 \tilde{A}_2 = \tilde{A}_2 \tilde{A}_1 = 0.$$

Very similar to core-nilpotent decomposition is the core-EP decomposition of a square matrix of arbitrary index and was introduced by Wang [13]. We record it as:

Lemma 2.4 ([13, Core-EP Decomposition]). Let $A \in \mathbb{C}_{n,n}$ with $\text{Ind}(A) = k$, then $A$ can be written as the sum of matrices $A_1$ and $A_2$, i.e. $A = A_1 + A_2$, where
(i) $A_1 \in \mathbb{C}^{n 	imes n}$;
(ii) $A_2^k = 0$;
(iii) $A_1^* A_2 = A_2 A_1 = 0$.
Here one or both of $A_1$ and $A_2$ can be null.

Lemma 2.5 ([13]). Let the core-EP decomposition of $A \in \mathbb{C}^{n \times n}$ be as in Lemma 2.4. Then there exists a unitary matrix $U$ such that

\[
A_1 = U \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} U^*, \quad A_2 = U \begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix} U^*,
\]

where $T$ is non-singular, and $N$ is nilpotent. Furthermore, the core-EP inverse of $A$ is

\[
A^\circ = U \begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*.
\]

3 WG inverse

In this section, we apply the core-EP decomposition to introduce a generalized group inverse (i.e. the WG inverse) and consider some characterizations of the generalized inverse.

3.1 Definition and properties of the WG inverse

Let $A \in \mathbb{C}^{n \times n}$ with $\text{Ind}(A) = k$, and consider the system of equations

\[
(2') \quad AX^2 = X, \quad (3^c) \quad AX = A^\circ A.
\]

Theorem 3.1. The system of equations (7) is consistent and has a unique solution

\[
X = U \begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix} U^*.
\]

Proof. Let $A \in \mathbb{C}^{n \times n}$ with $\text{Ind}(A) = k$. Since $A^\circ = A^k \left( (A^*)^k A^{k+1} \right)^{-1} (A^*)^k$, $\mathcal{R} (A^\circ A) \subseteq \mathcal{R} (A)$. Therefore, (3$^c$) is consistent. Let $A$ be as in (5). From (6), we obtain

\[
(A^\circ)^2 A = U \begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix} U^*
\]

and

\[
A \left( (A^\circ)^2 A \right) = A^\circ A,
\]

that is, $(A^\circ)^2 A$ is a solution to (3$^c$).

It is obvious that (2') is consistent. Applying (9), we have

\[
A \left( (A^\circ)^2 A \right)^2 = (A^\circ)^2 A,
\]

that is, $(A^\circ)^2 A$ is a solution to (2').

Therefore, from (9), (10) and (11), we derive that (7) is consistent and (8) is a solution of (7).

---

1 Since $A^\circ A$ is core invertible, we use the symbol 3$c$ in (7).
Furthermore, suppose that both \(X\) and \(Y\) satisfy (7), then
\[
X = AXY^2 = A^o AX = A^o A^o A = A^o AY = AY^2 = Y,
\]
that is, the solution to the system of equations (7) is unique. \(\square\)

**Definition 3.2.** Let \(A \in \mathbb{C}_{n,n}\) be a matrix of index \(k\). The WG inverse of \(A\), denoted as \(A^w\), is defined to be the solution to the system (7).

**Remark 3.3.** When \(A \in \mathbb{C}^*_{n,n}\), we have \(A^w = A^d\).

**Remark 3.4.** In [14, Definition 1], the notion of weak Drazin inverse was given: let \(A \in \mathbb{C}_{n,n}\) and \(\text{Ind}(A) = k\), then \(X\) is a weak Drazin inverse of \(A\) if \(X\) satisfies (6\(^k\)). Applying (8), it is easy to check that the WG inverse \(A^w\) is a weak Drazin inverse of \(A\).

**Remark 3.5.** Let \(A \in \mathbb{C}_{n,n}\). Applying Theorem 3.1, it is easy to check \(A^w A A^w = A^w\) and \(R(A^w) = R(A^k)\).

More details about the weak Drazin inverse can be seen in [14–16].

In the following example, we explain that the WG inverse is different from the Drazin, DMP, core-EP and BT inverses.

**Example 3.6.** Let \(A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\). It is easy to check that \(\text{Ind}(A) = 2\), the Moore-Penrose inverse \(A'\) and the Drazin inverse \(A^D\) are
\[
A' = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad A^D = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
\]
the DMP inverse \(A^{d,t}\) and the BT inverse \(A^\circ\) are
\[
A^{d,t} = A^D AA' = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A^\circ = (A^2 A')' = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
\]
and the core-EP inverse \(A^\circ\) and the WG inverse \(A^w\) are
\[
A^\circ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A^w = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\]

**3.2 Characterizations of the WG inverse**

**Theorem 3.7.** Let \(A \in \mathbb{C}_{n,n}\) be as in (5). Then
\[
A^w = A^D = U \begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix} U^*.
\]

**Proof.** Let \(A = \overline{A}_1 + \overline{A}_2\) be the core-nilpotent decomposition of \(A \in \mathbb{C}_{n,n}\). Then \(A^D = \overline{A}_1^D\). Applying Lemma 2A, (5) and (8), we derive (12). \(\square\)
Theorem 3.8. Let \( A \in \mathbb{C}_{n,n} \) with \( \text{Ind}(A) = k. \) Then

\[
A^\# = (AA^\# A)^\# = (A^\#)^2 A = (A^2)^\# A.
\] (13)

Proof. Let \( A \) be as in (5). Then

\[
AA^\# A = U \begin{bmatrix} TS & T^{-1} 0 \\ 0 N & 0 0 \end{bmatrix} \begin{bmatrix} TS \\ 0 N \end{bmatrix} U^* = U \begin{bmatrix} TS & T^{-1} \end{bmatrix} U^*,
\]

\[
(A^\#)^2 = \left( U \begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* \right)^2 = U \begin{bmatrix} T^{-2} & 0 \\ 0 & 0 \end{bmatrix} U^*,
\]

\[
(A^2)^\# = \left( U \begin{bmatrix} T^2 TS + SN & 0 \\ 0 & N^2 \end{bmatrix} U^* \right)^\# = U \begin{bmatrix} T^{-2} & 0 \\ 0 & 0 \end{bmatrix} U^*.
\]

It follows from Theorem 3.7 that

\[
(AA^\# A)^\# = \left( U \begin{bmatrix} T S & 0 \\ 0 & 0 \end{bmatrix} U^* \right)^\# = U \begin{bmatrix} T^{-1} & T^{-2} S \\ 0 & 0 \end{bmatrix} U^* = A^\#,
\]

\[
(A^\#)^2 A = (A^2)^\# A = U \begin{bmatrix} T^{-2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} TS \\ 0 N \end{bmatrix} U^* = U \begin{bmatrix} T^{-1} & T^{-2} S \\ 0 & 0 \end{bmatrix} U^* = A^\#.
\]

Therefore, we obtain (13). \( \square \)

Theorem 3.9. Let \( A \in \mathbb{C}_{n,n} \) with \( \text{Ind}(A) = k. \) Then

\[
A^\# = A^k (A^{k+2})^\# A = (A^2 P_{A^k})^\dagger A.
\] (14)

Proof. Let \( A \) be as in (5). Then

\[
A^k = U \begin{bmatrix} T^k & \Phi \\ 0 & 0 \end{bmatrix} U^*,
\] (15)

where \( \Phi = \sum_{i=1}^{k} T^i S N^{k-i}. \) It follows that

\[
A^k (A^{k+2})^\# A = U \begin{bmatrix} T^k & \Phi \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T^{-(k+2)} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} TS \\ 0 N \end{bmatrix} U^* = U \begin{bmatrix} T^{-1} & T^{-2} S \\ 0 & 0 \end{bmatrix} U^* = A^\#,
\]

\[
P_{A^k} = A^k (A^k)^\dagger = U \begin{bmatrix} I_{k(k)} & 0 \\ 0 & 0 \end{bmatrix} U^*,
\]

\[
(A^2 P_{A^k})^\dagger A = U \begin{bmatrix} T^2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} TS \\ 0 N \end{bmatrix} U^* = A^\#.
\]

Therefore, we have (14). \( \square \)

It is known that the Drazin inverse is one generalization of the group inverse. We will see the similarities and differences between the Drazin inverse and the WG inverse from the following corollaries.

Corollary 3.10. Let \( A \in \mathbb{C}_{n,n} \) with \( \text{Ind}(A) = k. \) Then

\[
\text{rk} \left( A^\# \right) = \text{rk} \left( A^\# \right) = \text{rk} \left( A^k \right).
\]
It is well known that \((A^2)^D = (A^D)^2\), but the same is not true for the WG inverse. Applying the core-EP decomposition (5) of \(A\), we have
\[
A^2 = U \begin{bmatrix} T^2 & T S + SN \\ 0 & N^2 \end{bmatrix} U^* \tag{18}
\]
and
\[
(A^2)^\# = U \begin{bmatrix} T^{-2} & T^{-4} (T S + SN) \\ 0 & 0 \end{bmatrix} U^*, \quad (A^\#)^2 = U \begin{bmatrix} T^{-2} & T^{-3} S \\ 0 & 0 \end{bmatrix} U^*. \tag{19}
\]

Therefore, \((A^2)^\# = (A^\#)^2\) if and only if \(T^{-4} (T S + SN) = T^{-3} S\). Since \(T\) is invertible, we derive the following Corollary 3.11.

**Corollary 3.11.** Let \(A \in \mathbb{C}_{n,n}\) be as in (5). Then \((A^2)^\# = (A^\#)^2\) if and only if \(SN = 0\).

The commutativity is one of the main characteristics of the group inverse. The Drazin inverse too has the characteristic. It is of interest to inquire whether the same is true or not for the WG inverse. Applying the core-EP decomposition (5) of \(A\), we have
\[
AA^\# = U \begin{bmatrix} T S & T^{-1} T^{-2} S \\ 0 & N \end{bmatrix} U^* = U \begin{bmatrix} I & T^{-1} S \\ 0 & 0 \end{bmatrix} U^*; \tag{20a}
\]
\[
A^\# A = U \begin{bmatrix} T^{-1} T^{-2} S & T S \\ 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} I & T^{-1} S + T^{-2} SN \\ 0 & 0 \end{bmatrix} U^*. \tag{20b}
\]

Therefore, we have the following Corollary 3.12.

**Corollary 3.12.** Let the core-EP decomposition of \(A \in \mathbb{C}_{n,n}\) be as in (5). Then \(AA^\# = A^\# A\) if and only if \(SN = 0\).

**Corollary 3.13.** Let \(A \in \mathbb{C}_{n,n}\) with \(\text{Ind}(A) = k\), the core-EP decomposition of \(A\) be as in (5) and \(SN = 0\). Then
\[
A^\# = A^D = (A^{k+1})^\# A^k = (A^{t+1})^\# A^t,
\]
where \(t\) is an arbitrary positive integer.

**Proof.** Let the core-EP decomposition of \(A \in \mathbb{C}_{n,n}\) be as in (5).

By applying \(SN = 0\) and \(\text{Ind}(A) = k\), we have
\[
A^{-1} = U \begin{bmatrix} T^{-1} & T^{-2} S \\ 0 & N^{-1} \end{bmatrix} U^*, \quad A^k = U \begin{bmatrix} T^k & T^{k-1} S \\ 0 & 0 \end{bmatrix} U^*, \quad A^{k+1} = U \begin{bmatrix} T^{k+1} & T^k S \\ 0 & 0 \end{bmatrix} U^*.
\]

It follows from applying (1), (4) and (6) that
\[
(A^{k+1})^\# = (A^{k+1})^D = U \begin{bmatrix} T^{-(k+1)} & T^{-(k+2)} S \\ 0 & 0 \end{bmatrix} U^*,
\]
\[
A^D = (A^{k+1})^\# A^k = U \begin{bmatrix} T^{-(k+1)} & T^{-(k+2)} S \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T^k & T^{k-1} S \\ 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} T^{-1} & T^{-2} S \\ 0 & 0 \end{bmatrix} U^* = A^\#.
\]

Therefore, \(A^\# = A^D = (A^{k+1})^\# A^k\).

Let \(t\) be an arbitrary positive integer. By applying \(SN = 0\), we have
\[
A^t = U \begin{bmatrix} T^t & T^{t-1} S \\ 0 & N^t \end{bmatrix} U^*, \quad A^{t+1} = U \begin{bmatrix} T^{t+1} & T^t S \\ 0 & N^{t+1} \end{bmatrix} U^*.
\]
It follows from Lemma 2.5 that
\[
\begin{align*}
(A^{t+1})^\circ &= U\begin{bmatrix} T^{-(t+1)} & 0 \\ 0 & 0 \end{bmatrix} U^* , \\
(A^{t+1})^\circ A^t &= U\begin{bmatrix} T^{-(t+1)} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T^t & T^{t-1}S \\ 0 & N^t \end{bmatrix} U^* = A^\circ ,
\end{align*}
\] (21)
Therefore, we derive \( A^\circ = (A^{t+1})^\circ A^t \), in which \( t \) is an arbitrary positive integer. \( \square \)

### 4 Two orders

Recall the definitions of the minus partial order, sharp partial order, Drazin order and core-nilpotent partial order [12]:

\[
A \preceq B : A, B \in \mathbb{C}_{m,n}, \text{ rk}(B - A) = \text{rk}(B) - \text{rk}(A) , \quad (22)
\]

\[
A \preceq B : A, B \in \mathbb{C}_{m,n}, A^2 = BA , \quad (23)
\]

\[
A \preceq B : A, B \in \mathbb{C}_{m,n}, \vec{A}_1 \preceq \vec{B}_1 , \quad (24)
\]

\[
A \preceq B : A, B \in \mathbb{C}_{m,n}, \vec{A}_1 \preceq \vec{B}_1 \text{ and } \vec{A}_2 \preceq \vec{B}_2 , \quad (25)
\]
in which \( A = \vec{A}_1 + \vec{A}_2 \) and \( B = \vec{B}_1 + \vec{B}_2 \) are the core-nilpotent decompositions of \( A \) and \( B \), respectively. Similarly, in this section, we apply the core-EP decomposition to introduce two orders: one is the WG order and the other is the CE partial order.

#### 4.1 WG order

Consider the binary relation:

\[
A \preceq_{\text{WG}} B : A, B \in \mathbb{C}_{m,n}, \text{ if } A_1 \preceq B_1 , \quad (26)
\]
in which \( A = A_1 + A_2 \) and \( B = B_1 + B_2 \) are the core-EP decompositions of \( A \) and \( B \), respectively.

Reflexivity of the relation is obvious. Suppose \( A \preceq_{\text{WG}} B \) and \( B \preceq_{\text{WG}} C \), in which \( A = A_1 + A_2 , B = B_1 + B_2 \) and \( C = C_1 + C_2 \) are the core-EP decompositions of \( A \), \( B \) and \( C \), respectively. Then \( A_1 \preceq B_1 \) and \( B_1 \preceq C_1 \). Therefore \( A_1 \preceq C_1 \). It follows from (26) that \( A \preceq_{\text{WG}} C \).

**Example 4.1.** Let

\[
A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} , \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}
\]

Although \( A \preceq_{\text{WG}} B \) and \( B \preceq_{\text{WG}} A \), \( A \neq B \). Therefore, the anti-symmetry of the binary operation (26) does not hold in general.

Therefore, we have the following Theorem 4.2.

**Theorem 4.2.** The binary relation (26) is a pre-order. We call this pre-order the weak-group (WG for short) order.

**Remark 4.3.** The WG order coincides with the sharp partial order on \( \mathbb{C}_{m,n}^\circ \).
We give below two examples to show that WG order is different from Drazin order and that either of two orders does not imply the other order.

**Example 4.4.** Let $A$ and $B$ be as in Example 4.1. We have

$$
A^D = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
$$

It is easy to check that $A^\text{WG} \leq B$.

Since $A^D A \neq A^D B$, we derive $A \not\leq D B$. Therefore, the WG order does not imply the Drazin order.

**Example 4.5.** Let

$$
\begin{align*}
\tilde{A} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & \tilde{B} &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, & P &= \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
A &= P\tilde{A}P^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & B &= P\tilde{B}P^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
A_1 &= \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & A_2 &= 0, & B_1 &= \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & B_2 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\end{align*}
$$

in which $A = A_1 + A_2$ and $B = B_1 + B_2$ are the core-EP decompositions of $A$ and $B$, respectively. Then $A \not\leq B$ and $A_1 \not\leq B_1$. Therefore, the Drazin order does not imply the WG order.

It is well known that $A \leq B \Rightarrow A^D \leq B^D$, but the same is not true for the WG order as the following example shows:

**Example 4.6.** Let $A$ and $B$ be as in Example 4.1, then

$$
A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B^2 = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
$$

We derive $A^2 \not\leq B^2$. Therefore, $A \not\leq B \Rightarrow A^2 \not\leq B^2$.

**Theorem 4.7.** Let $A, B \in \mathbb{C}_{n,n}$. Then $A \leq B$ if and only if there exists a unitary matrix $\tilde{U}$ such that

$$
\begin{align*}
A &= \tilde{U} \begin{bmatrix} T & S_1 & \tilde{S}_2 \\ N_{11} & N_{12} & 0 \\ 0 & N_{21} & N_{22} \end{bmatrix} \tilde{U}^*, \\
B &= \tilde{U} \begin{bmatrix} T & \tilde{S}_1 & -T^{-1}S_1T_1 & \tilde{S}_2 - T^{-1}S_1S_1 \\ 0 & T_1 & S_1 & 0 \\ 0 & 0 & N_2 & 0 \end{bmatrix} \tilde{U}^*,
\end{align*}
$$

where $T$ and $T_1$ are invertible, $\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}$ and $N_2$ are nilpotent.
Proof. Assume $A \leq B$. Let $A = A_1 + A_2$ and $B = B_1 + B_2$ be the core-EP decompositions of $A$ and $B$, $A_1$ and $A_2$ be as given in (5), and partition

$$U^*B_1U = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}.$$  \hfill (28)

Applying (12) gives

$$A_1A_1^\# = U \begin{bmatrix} I & T^{-1}S \\ 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} I & T^{-1}S \\ 0 & 0 \end{bmatrix} U^*;$$

$$B_1A_1^\# = U \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} I & T^{-1}S \\ 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} I & T^{-1}S \\ 0 & 0 \end{bmatrix} U^*.$$  \hfill (29)

Since $A \leq B$, $A_1 \leq B_1$. It follows from $A_1A_1^\# = B_1A_1^\#$ that

$$B_{11} = T \text{ and } B_{21} = 0.$$  \hfill (30)

By applying (12) and (29), we have

$$A_1^\#A_1 = U \begin{bmatrix} I & T^{-1}S \\ 0 & 0 \end{bmatrix} U^*,$$

$$A_1^\#B_1 = U \begin{bmatrix} I & T^{-1}B_{12} + T^{-2}SB_{22} \\ 0 & 0 \end{bmatrix} U^*.$$  \hfill (31)

It follows from $A_1^\#A_1 = A_1^\#B_1$ that

$$T^{-1}\left(S - T^{-1}SB_{22} - B_{12}\right) = 0.$$  \hfill (32)

Therefore,

$$B_{12} = S - T^{-1}SB_{22},$$  \hfill (33)

in which $B_{22}$ is an arbitrary matrix of an appropriate size. From (29) and (30), we obtain

$$B_1 = U \begin{bmatrix} T & S \\ 0 & B_{22} \end{bmatrix} U^*.$$  \hfill (34)

Since $B_1$ is core invertible and $T$ is non-singular, $B_{22}$ is core invertible. Let the core-EP decomposition of $B_{22}$ be as

$$B_{22} = U_1 \begin{bmatrix} T_1 & S_1 \\ 0 & 0 \end{bmatrix} U_1^*,$$  \hfill (35)

where $T_1$ is invertible. Denote

$$\bar{U} = U \begin{bmatrix} I & 0 \\ 0 & U_1 \end{bmatrix}.$$  \hfill (36)

It is easy to see that $\bar{U}$ is a unitary matrix. Let $SU_1$ be partitioned as follows:

$$SU_1 = \begin{bmatrix} \bar{S}_1 & \bar{S}_2 \end{bmatrix},$$

where the number of columns of $\bar{S}_1$ coincides with the size of the square matrix $T_1$. Then

$$A_1 = \bar{U} \begin{bmatrix} T & \bar{S}_1 \\ 0 & \bar{S}_2 \end{bmatrix} U^*.$$  \hfill (37)
and

\[
B_1 = U \begin{bmatrix}
T & S - T^{-1}SB_{22} \\
0 & U_1 \begin{bmatrix} T_1 & S_1 \\ 0 & 0 \end{bmatrix} U_1^* 
\end{bmatrix} U^*
= U \begin{bmatrix}
I & 0 \\
0 & U_1 \begin{bmatrix} T_1 & S_1 \\ 0 & 0 \end{bmatrix} U_1^* 
\end{bmatrix} \begin{bmatrix} I & 0 \\
0 & U_1^* 
\end{bmatrix} U^*
= \bar{U} \begin{bmatrix}
T [\bar{S}_1 \bar{S}_2] - T^{-1} [\bar{S}_1 \bar{S}_2] \begin{bmatrix} T_1 & S_1 \\ 0 & 0 \end{bmatrix} \\
0 \begin{bmatrix} T_1 & S_1 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \bar{U}^*
= \bar{U} \begin{bmatrix}
T \bar{S}_1 - T^{-1} \bar{S}_1 T_1 \bar{S}_2 - T^{-1} \bar{S}_1 S_1 \\
0 & T_1 & S_1 \\
0 & 0 & 0
\end{bmatrix} \bar{U}^*.
\]

(34)

From (26), (33) and (34), we derive (27a) and (27b).

\[\square\]

### 4.2 CE partial order

Consider the binary relation:

\[A \subseteq_{CE} B : A, B \in \mathbb{C}_{n,n}, A_\#_1 \preceq B_1 \text{ and } A_\#_2 \preceq B_2,\]

(35)

in which \(A = A_1 + A_2\) and \(B = B_1 + B_2\) are the core-EP decompositions of \(A\) and \(B\), respectively.

**Definition 4.8.** Let \(A, B \in \mathbb{C}_{n,n}\). We say that \(A\) is below \(B\) under the core-EP-minus (CE for short) order if \(A\) and \(B\) satisfy the binary relation (35).

When \(A\) is below \(B\) under the CE order, we write \(A \subseteq_{CE} B\).

**Remark 4.9.** According to (26) and (35) we derive that the CE order implies the WG order, that is,

\[A \subseteq_{CE} B \Rightarrow A \subseteq_{WG} B.\]

(36)

Furthermore,

\[A \subseteq_{CE} B \iff A \subseteq_{WG} B \text{ and } A_\#_2 \preceq B_2.\]

(37)

**Theorem 4.10.** The CE order is a partial order.

**Proof.** Reflexivity is trivial.

Let \(A \subseteq_{CE} B, B \subseteq_{CE} C\) and \(A = A_1 + A_2, B = B_1 + B_2\) and \(C = C_1 + C_2\) are the core-EP decompositions of \(A, B\) and \(C\), respectively. Then \(A_\#_1 \preceq B_\#_1 \preceq C_1\) and \(A_\#_2 \preceq B_\#_2 \preceq C_2\). Therefore, \(A_\#_1 \preceq C_1\) and \(A_\#_2 \preceq C_2\). It follows from Definition 4.8 that \(A \subseteq_{CE} C\).

If \(A \subseteq_{CE} B\) and \(B \subseteq_{CE} A\), then \(A_1 = B_1\) and \(A_2 = B_2\), that is, \(A = B\).

\[\square\]

**Theorem 4.11.** Let \(A, B \in \mathbb{C}_{n,n}\). Then \(A \subseteq_{CE} B\) if and only if there exists a unitary matrix \(\bar{U}\) satisfying

\[A = \bar{U} \begin{bmatrix} T \bar{S}_1 & \bar{S}_2 \\
0 & 0 & 0 \\
0 & 0 & N_{22} 
\end{bmatrix} \bar{U}^*,\]

(38a)
Applying (39) to (40) we obtain

\[
B = \bar{U} \begin{bmatrix}
T \bar{S}_1 - T^{-1} \bar{S}_1 T_1 & \bar{S}_2 - T^{-1} \bar{S}_1 S_1 \\
0 & T_1 & S_1 \\
0 & 0 & N_2
\end{bmatrix} \bar{U}^*,
\]  

(38b)

where \(T\) and \(T_1\) are invertible, \(N_{22}\) and \(N_2\) are nilpotent, and \(N_{22} \preceq N_2\).

**Proof.** Let \(A \preceq B\), and \(A = A_1 + A_2\) and \(B = B_1 + B_2\) are the core-EP decompositions of \(A\) and \(B\), respectively. Then \(A_1 \preceq B_1\) and \(A_2 \preceq B_2\). By applying Lemma 2.5, Theorem 4.7 and \(A_1 \preceq B_1\), we have

\[
A_1 = \bar{U} \begin{bmatrix}
T \bar{S}_1 & \bar{S}_2 \\
0 & 0 \\
0 & 0
\end{bmatrix} \bar{U}^*, \quad A_2 = \bar{U} \begin{bmatrix}
0 & 0 & 0 \\
0 & N_1 & N_2 \\
0 & N_21 & N_22
\end{bmatrix} \bar{U}^*,
\]

\[
B_1 = \bar{U} \begin{bmatrix}
T \bar{S}_1 - T^{-1} \bar{S}_1 T_1 & \bar{S}_2 - T^{-1} \bar{S}_1 S_1 \\
0 & T_1 & S_1 \\
0 & 0 & 0
\end{bmatrix} \bar{U}^*, \quad B_2 = \bar{U} \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & N_2
\end{bmatrix} \bar{U}^*,
\]

where \(\bar{U}, T, T_1, \begin{bmatrix}N_{11} & N_{12} \\ N_{21} & N_{22}\end{bmatrix}\) and \(N_2\) are as in Theorem 4.7.

Since \(A_2 \preceq B_2\), we have \(\text{rk } (B_2 - A_2) = \text{rk } (B_2) - \text{rk } (A_2)\), that is,

\[
\text{rk } \begin{bmatrix}0 & 0 \\ 0 & N_2\end{bmatrix} = \text{rk } (N_2) - \text{rk } \begin{bmatrix}N_{11} & N_{12} \\ N_{21} & N_{22}\end{bmatrix}.
\]  

(39)

In addition, it is easy to check that

\[
\text{rk } (N_2) - \text{rk } \begin{bmatrix}N_{11} & N_{12} \\ N_{21} & N_{22}\end{bmatrix} \leq \text{rk } (N_2) - \text{rk } (N_{22})
\]

\[
\leq \text{rk } (N_2 - N_{22}) \leq \text{rk } \begin{bmatrix}0 & 0 \\ 0 & N_2\end{bmatrix} - \begin{bmatrix}N_{11} & N_{12} \\ N_{21} & N_{22}\end{bmatrix}.
\]  

(40)

Applying (39) to (40) we obtain

\[
\text{rk } (N_{22}) = \text{rk } \begin{bmatrix}N_{11} & N_{12} \\ N_{21} & N_{22}\end{bmatrix},
\]  

(41)

\[
\text{rk } (N_2) - \text{rk } (N_{22}) = \text{rk } (N_2 - N_{22}).
\]  

(42)

Therefore, we obtain

\[
N_{22} \preceq N_2.
\]  

(43)

Since \(N_{22} \preceq N_2\), there exist nonsingular matrices \(P\) and \(Q\) such that

\[
N_{22} = P \begin{bmatrix}D_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix} Q, \quad N_2 = P \begin{bmatrix}D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & 0\end{bmatrix} Q,
\]

where \(D_1\) and \(D_2\) are nonsingular, (see [12, Theorem 3.7.3]). It follows that

\[
\text{rk } (N_{22}) = \text{rk } (D_1) \quad \text{and} \quad \text{rk } (N_2) - \text{rk } (N_{22}) = \text{rk } (D_2).
\]  

(44)

Denote

\[
N_{12} = \begin{bmatrix}M_{12} & M_{13} & M_{14}\end{bmatrix} Q \quad \text{and} \quad N_{21} = P \begin{bmatrix}M_{21} \\ M_{31} \\ M_{41}\end{bmatrix}.
\]  

(45)
Then

\[
\begin{bmatrix}
N_{11} & N_{12} \\
N_{21} & N_{22}
\end{bmatrix} = \begin{bmatrix}
I_{rk(N_{11})} & 0 \\
0 & P
\end{bmatrix} \begin{bmatrix}
N_{11} & M_{12} & M_{13} & M_{14} \\
M_{21} & D_1 & 0 & 0 \\
M_{31} & 0 & 0 & 0 \\
M_{41} & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
I_{rk(N_{11})} & 0 \\
0 & Q
\end{bmatrix}
\]

and

\[
\text{rk}\left(\begin{bmatrix}
N_{11} & N_{12} \\
N_{21} & N_{22}
\end{bmatrix}\right) = \text{rk}(D_1) + \text{rk}\left(\begin{bmatrix}
M_{13} & M_{14}
\end{bmatrix}\right) + \text{rk}\left(\begin{bmatrix}
M_{31} \\
M_{41}
\end{bmatrix}\right)
+ \text{rk}\left(N_{11} - M_{12}D_1^{-1}M_{21}\right)
\]

It follows from (44) and (41) that

\[
M_{13} = 0, \quad M_{14} = 0, \quad M_{31} = 0 \quad \text{and} \quad M_{41} = 0.
\]

Therefore,

\[
\begin{bmatrix}
-N_{11} & -N_{12} \\
-N_{21} & N_2 - N_{22}
\end{bmatrix} = \begin{bmatrix}
I_{rk(N_{11})} & 0 \\
0 & P
\end{bmatrix} \begin{bmatrix}
-N_{11} & -M_{12} & 0 & 0 \\
-M_{21} & 0 & 0 & 0 \\
0 & 0 & D_2 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
I_{rk(N_{11})} & 0 \\
0 & Q
\end{bmatrix}.
\]

By applying (41), (44) and \(\begin{bmatrix}
N_{11} & N_{12} \\
N_{21} & N_{22}
\end{bmatrix} \leq \begin{bmatrix}
0 & 0 \\
0 & N_2
\end{bmatrix}\), we derive that

\[
\text{rk}\left(\begin{bmatrix}
0 & 0 \\
0 & N_2
\end{bmatrix} - \begin{bmatrix}
N_{11} & N_{12} \\
N_{21} & N_{22}
\end{bmatrix}\right) = \text{rk}\left(\begin{bmatrix}
N_{11} & M_{12} \\
M_{21} & 0
\end{bmatrix}\right) + \text{rk}(D_2)
= \text{rk}(N_2) - \text{rk}(N_{22})
= \text{rk}(D_2).
\]

Therefore, \(\begin{bmatrix}
N_{11} & M_{12} \\
M_{21} & 0
\end{bmatrix}\) = 0, that is, \(N_{11} = 0, \quad M_{12} = 0 \quad \text{and} \quad M_{21} = 0\). By applying (45) and (46), we obtain \(N_{11} = 0, \quad N_{12} = 0 \quad \text{and} \quad N_{21} = 0\). So, we obtain (38a) and (38b).

Let \(A \quad \text{and} \quad B\) be of the forms as given in (38a) and (38b). It is easy to check that \(A = A_1 + A_2\) and \(B = B_1 + B_2\) are the core-EP decompositions of \(A \quad \text{and} \quad B\), respectively, and

\[
A_1 = \bar{U} \begin{bmatrix}
\bar{T} & \bar{S}_1 & \bar{S}_2 \\
0 & 0 & 0
\end{bmatrix} \bar{U}^*; \quad A_2 = \bar{U} \\
0 & 0 & 0
\]

\[
B_1 = \bar{U} \begin{bmatrix}
\bar{T} & \bar{S}_1 - T^{-1}\bar{S}_2 T_1 \bar{S}_2 - T^{-1}\bar{S}_1 S_1 \\
0 & T_1 & S_1
\end{bmatrix} \bar{U}^*; \quad B_2 = \bar{U} \\
0 & 0 & 0
\]

It follows from (23) and \(N_{22} \leq N_2\) that \(A_1 \leq B_1\) and \(A_2 \leq B_2\). Therefore, \(A \leq B\).

**Remark 4.12.** In Ex. 4.5, it is easy to check that \(A^\perp \leq B\). Since \(A_1 \leq B_1\), we have \(A \leq B\). Therefore, the core-nilpotent partial order does not imply the CE partial order.

**Corollary 4.13.** Let \(A, B \in \mathbb{C}_{n,n}\). If \(A \leq B\), then \(A \leq B\).

**Proof.** Let \(A, B \in \mathbb{C}_{n,n}\). Then \(A \quad \text{and} \quad B\) have the forms as given in Theorem 4.11. According to \(A \leq B\), we have \(N_{22} \leq N_2\), that is,

\[
\text{rk}(N_2 - N_{22}) = \text{rk}(N_2) - \text{rk}(N_{22})
\]
Since \( T \) and \( T_1 \) are invertible, it follows that
\[
\begin{align*}
\text{rk}(B) &= \text{rk}(T) + \text{rk}(T_1) + \text{rk}(N_2); \\
\text{rk}(A) &= \text{rk}(T) + \text{rk}(N_{22}); \\
\text{rk}(B - A) &= \text{rk} \left( \begin{pmatrix}
0 & -T^{-1}S_1T_1 & -T^{-1}S_1S_1 \\
0 & T_1 & S_1 \\
0 & 0 & N_2 - N_{22}
\end{pmatrix} \right) \\
&= \text{rk} \left( \begin{pmatrix}
I_{\text{rk}(T)} & T^{-1}S_1 & 0 \\
0 & I_{\text{rk}(T_1)} & 0 \\
0 & 0 & I_{n - \text{rk}(T) - \text{rk}(T_1)}
\end{pmatrix} \right) \\
&= \text{rk} \left( \begin{pmatrix}
T_1 & S_1 \\
0 & N_2 - N_{22}
\end{pmatrix} \right) = \text{rk} \left( \begin{pmatrix}
T_1 & 0 \\
0 & N_2 - N_{22}
\end{pmatrix} \right)
\end{align*}
\] (48)

Therefore, by applying (22), (47) and (48) we derive \( \text{rk}(B - A) = \text{rk}(B) - \text{rk}(A) \), that is, \( A \preceq B \).

\[ \square \]

5 **Characterizations of the core-EP order**

As is noted in [13], the core-EP order is given:

\[ A \preceq B : A, B \in \mathbb{C}_{n,n}, A^\circ A = A^\circ B = AA^\circ = BA^\circ. \] (49)

Some characterizations of the core-EP order are given in [13].

**Lemma 5.1** ([13]). *Let \( A, B \in \mathbb{C}_{n,n} \) and \( A \preceq B \). Then there exists a unitary matrix \( U \) such that
\[
A = U \begin{bmatrix}
T_1 & T_2 & S_1 \\
0 & N_{11} & N_{12} \\
0 & N_{21} & N_{22}
\end{bmatrix} U^*, \quad B = U \begin{bmatrix}
T_1 & T_2 & S_1 \\
0 & T_3 & S_2 \\
0 & 0 & N_2
\end{bmatrix} U^*,
\]
(50)

where \( \begin{bmatrix} N_{11} & N_{12} \\
N_{21} & N_{22} \end{bmatrix} \) and \( N_2 \) are nilpotent, \( T_1 \) and \( T_3 \) are non-singular.*

**Theorem 5.2.** *Let \( A, B \in \mathbb{C}_{n,n} \). Then \( A \preceq B \) if and only if
\[
AA^\circ = BA^\circ \text{ and } A^\circ A^\circ = B^\circ A^\circ.
\]
(51)

**Proof.** Let \( A \) be as given in (5), and denote
\[
U^* BU = \begin{bmatrix} B_1 & B_2 \\
B_3 & B_4 \end{bmatrix}.
\]
(52)

By applying (20a) and
\[
BA^\circ = U \begin{bmatrix} B_1 & B_2 \\
B_3 & B_4 \end{bmatrix} \begin{bmatrix} T^{-1} & T^{-2}S \\
0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} B_1T^{-1} & B_1T^{-2}S \\
B_3T^{-1} & B_3T^{-2}S \end{bmatrix} U^*,
\]

we have \( AA^\circ = BA^\circ \) if and only if
\[ B_1 = T \text{ and } B_3 = 0. \]
It follows that
\[
A^* A^\omega = U \begin{bmatrix} T^* & 0 \\ S^* & N^* \end{bmatrix} \begin{bmatrix} T^{-1} & T^{-2} S \\ 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} T^* T^{-1} & T^* T^{-2} S \\ S^* T^{-1} & S^* T^{-2} S \end{bmatrix} U^*,
\]
\[
B^* A^\omega = U \begin{bmatrix} T^* & 0 \\ B_2^* & B_4^* \end{bmatrix} \begin{bmatrix} T^{-1} & T^{-2} S \\ 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} T^* T^{-1} & T^* T^{-2} S \\ B_2^* T^{-1} & B_2^* T^{-2} S \end{bmatrix} U^*.
\]

Therefore, \( AA^\omega = BA^\omega \) and \( A^* A^\omega = B^* A^\omega \) if and only if
\[
B_1 = T, B_3 = 0, B_2 = S, \text{ and } B_4 \text{ is arbitrary},
\]
that is, \( A \) and \( B \) satisfy \( AA^\omega = BA^\omega \) and \( A^* A^\omega = B^* A^\omega \) if and only if there exists a unitary matrix \( U \) such that
\[
A = U \begin{bmatrix} T & S \\ 0 & N \end{bmatrix} U^*, \quad B = U \begin{bmatrix} T & S \\ 0 & B_4 \end{bmatrix} U^*,
\]
where \( N \) is nilpotent, \( T \) is non-singular and \( B_4 \) is arbitrary. Therefore, by applying Lemma 5.1, we derive the characterization (51) of the core-EP order.

\[\square\]

**Disclosure statement**

No potential conflict of interest was reported by the authors.

**Funding**

This work was supported partially by Guangxi Natural Science Foundation [grant number 2018GXNSFAA138181], China Postdoctoral Science Foundation [grant number 2015M581690], High level innovation teams and distinguished scholars in Guangxi Universities, Special Fund for Bagui Scholars of Guangxi [grant number 2016A17] and National Natural Science Foundation of China [grant number 11771076].

**Acknowledgement:** The authors wish to extend their sincere gratitude to the referees for their precious comments and suggestions.

**References**


