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Uniqueness theorems for L-functions in the extended Selberg class

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Abstract: In this paper, we obtain uniqueness theorems of L-functions from the extended Selberg class, which generalize and complement some recent results due to Li, Wu-Hu, and Yuan-Li-Yi.

Keywords: Meromorphic function, L-function, Selberg class, Value distribution

MSC: 11M36, 30D35, 30D30

1 Introduction

The Riemann hypothesis as one of the millennium problems has been given a lot of attention by many scholars for a long time. Selberg guessed that the Riemann hypothesis also holds for the L-function in the Selberg class. Such an L-function based on the Riemann zeta function as a prototype is defined to be a Dirichlet series

\[ L(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \]  

of a complex variable \( s = \sigma + it \) satisfying the following axioms [1]:

(i) Ramanujan hypothesis: \( a(n) \ll n^\varepsilon \) for every \( \varepsilon > 0 \).

(ii) Analytic continuation: There exists a nonnegative integer \( m \) such that \( (s - 1)^m L(s) \) is an entire function of finite order.

(iii) Functional equation: \( L \) satisfies a functional equation of type

\[ \Lambda L(s) = \omega L(1 - \overline{s}), \]

where

\[ \Lambda L(s) = L(s) Q K \prod_{j=1}^{K} \Gamma(\lambda_j s + \nu_j) \]

with positive real numbers \( Q, \lambda_j, \) and complex numbers \( \nu_j, \omega \) with \( \text{Re} \nu_j \geq 0 \) and \( |\omega| = 1 \).

(iv) Euler product: \( \log L(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s} \), where \( b(n) = 0 \) unless \( n \) is a positive power of a prime and \( b(n) \ll n^\theta \)

for some \( \theta < \frac{1}{2} \).

It is mentioned that there are many Dirichlet series but only those satisfying the axioms (i)-(iii) are regarded as the extended Selberg class [1, 2]. All the L-functions which are studied in this article are from the extended...
Theorem 1.1 (see [1]). If two L-functions with \( a(1) = 1 \) share a complex value \( c \neq \infty \) CM, then they are identically equal.

Remark 1.2. In [5], the authors gave an example that \( L_1 = 1 + \frac{2}{s^2} \) and \( L_2 = 1 + \frac{3}{s^2} \), which showed that Theorem 1.1 is actually false when \( c = 1 \).

In 2011, Li [6] considered values which are shared CM and got

Theorem 1.3 (see [6]). Let \( L_1 \) and \( L_2 \) be two L-functions satisfying the same functional equation with \( a(1) = 1 \) and let \( a_1, a_2 \in \mathbb{C} \) be two distinct values. If \( L_1^{-1}(a_j) = L_2^{-1}(a_j), j = 1, 2 \), then \( L_1 \equiv L_2 \).

In 2001, Lahiri [7] put forward the concept of weighted sharing as follows.

Let \( k \) be a nonnegative integer or \( \infty \), \( c \in \mathbb{C} \cup \{ \infty \} \). We denote by \( E_k(c, f) \) the set of all zeros of \( f - c \), where a zero of multiplicity \( m \) is counted \( m \) times if \( m \leq k \) and \( k + 1 \) times if \( m > k \). If \( E_k(c, f) = E_k(c, g) \), we say that \( f \) and \( g \) share the value \( c \) with weight \( k \) (see [7]).

In 2015, Wu and Hu [8] removed the assumption that both L-functions satisfy the same functional equation in Theorem 1.3. By including weights, they had shown the following result.

Theorem 1.4 (see [8]). Let \( L_1 \) and \( L_2 \) be two L-functions, and let \( a_1, a_2 \in \mathbb{C} \) be two distinct values. Take two positive integers \( k_1, k_2 \) with \( k_1 k_2 > 1 \). If \( E_{k_1}(a_j, L_1) = E_{k_2}(a_j, L_2) \), \( j = 1, 2 \), then \( L_1 \equiv L_2 \).

In 2003, the following question was posed by C.C. Yang [9].

Question 1.5 (see [9]). Let \( f \) be a meromorphic function in the complex plane and \( a, b, c \) are three distinct values, where \( c \neq 0, \infty \). If \( f \) and the Riemann zeta function \( \zeta \) share \( a, b \) CM and \( c \) IM, will then \( f \equiv \zeta \)?

The L-function is based on the Riemann zeta function as the model. It is then valuable that we study the relationship between an L-function and an arbitrary meromorphic function [10–14]. This paper concerns the problem of how meromorphic functions and L-functions are uniquely determined by their c-values. Firstly, we introduced the following theorem.

Theorem 1.6 (see [10]). Let \( a \) and \( b \) be two distinct finite values and \( f \) be a meromorphic function in the complex plane with finitely many poles. If \( f \) and a nonconstant L-function \( L \) share a CM and \( b \) IM, then \( L \equiv f \).

Then, using the idea of weighted sharing, we will prove the following theorem.

Theorem 1.7. Let \( f \) be a meromorphic function in the complex plane with finitely many poles, let \( L \) be a nonconstant L-function, and let \( a_1, a_2 \in \mathbb{C} \) be two distinct values. Take two positive integers \( k_1, k_2 \) with \( k_1 k_2 > 1 \). If \( E_{k_1}(a_j, f) = E_{k_2}(a_j, L) \), \( j = 1, 2 \), then \( L \equiv f \).

Remark 1.8. Note that an L-function itself can be analytically continued as a meromorphic function in the complex plane. Therefore, an L-function will be taken as a special meromorphic function. We can also see that Theorem 1.4 is included in Theorem 1.7.
In 1976, the following question was mentioned by Gross in [15].

**Question 1.9** (see [15]). Must two nonconstant entire functions \( f_1 \) and \( f_2 \) be identically equal if \( f_1 \) and \( f_2 \) share a finite set \( S \)?

Recently, Yuan, Li and Yi [16] considered this question leading to the theorem below.

**Theorem 1.10** (see [16]). Let \( S = \{ \omega_1, \omega_2, \cdots, \omega_l \} \), where \( \omega_1, \omega_2, \cdots, \omega_l \) are all distinct roots of the algebraic equation \( \omega^n + a \omega^m + b = 0 \). Here \( l \) is a positive integer satisfying \( 1 \leq l \leq n, n \) and \( m \) are relatively prime positive integers with \( n \geq 5 \) and \( n > m \), and \( a, b, c \) are nonzero finite constants, where \( c \neq \omega_j \) for \( 1 \leq j \leq l \). Let \( f \) be a nonconstant meromorphic function such that \( f \) has finitely many poles in \( \mathbb{C} \), and let \( L \) be a nonconstant \( L \)-function. If \( f \) and \( L \) share \( S \) CM and \( c \) IM, then \( f \equiv L \).

Concerning shared set, we prove the following theorem.

**Theorem 1.11.** Let \( f \) be an entire function with \( \lim_{\Re(s) \to +\infty} f(s) = k (k \neq \infty) \) and let \( R(a) = 0 \) be a algebraic equation with \( n \geq 2 \) distinct roots, and \( R(k), R(b), R(1) \neq 0 \). Suppose that \( f(s) = \frac{L(s_0)}{b} \) for some \( s_0 \in \mathbb{C} \).

If \( f \) and a nonconstant \( L \)-function \( L \) share \( S \) CM, where \( S = \{ a : R(a) = 0 \} \), then \( R(L) \equiv R(f) \).

Furthermore, we obtain a result which is similar to Theorem 1.10 by different means.

**Theorem 1.12.** Let \( f \) be an entire function with \( \lim_{\Re(s) \to +\infty} f(s) = k (k \neq \infty) \). Let \( S = \{ \omega_1, \omega_2, \cdots, \omega_l \} \in \mathbb{C} \setminus \{ 1, k, b \} \), where \( \omega_1, \omega_2, \cdots, \omega_l \) are all distinct roots of the algebraic equation \( \omega^{n+m} + a \omega^n + \beta = 0 \), \( 1 \leq i \leq n+m \), \( n, m \) are two positive integers with \( n > m + 2 \), \( a, \beta \) are finite nonzero constants. If \( f \) and a nonconstant \( L \)-function \( L \) share \( S \) CM and \( f(s_0) = \frac{L(s_0)}{b} \) for some \( s_0 \in \mathbb{C} \), then \( f \equiv tL \), where \( t \) is a constant such that \( t^d = 1 \), \( d = \gcd(n, m) \).

## 2 Some lemmas

In this section, we present some important lemmas which will be needed in the sequel. Firstly, let \( f \) be a meromorphic function in \( \mathbb{C} \). The order \( \rho(f) \) is defined as follows:

\[
\rho(f) = \lim \sup_{r \to \infty} \frac{\log T(r, f)}{\log r}.
\]

**Lemma 2.1** (see [4], Lemma 1.22). Let \( f \) be a nonconstant meromorphic function and let \( k \geq 1 \) be an integer. Then \( m\left(r, \frac{f^{(k)}}{f}\right) \leq S(r, f) \). Further if \( \rho(f) < +\infty \), then

\[
m\left(r, \frac{f^{(k)}}{f}\right) = O(\log r).
\]

**Lemma 2.2** (see [4], Corollary of Theorem 1.5). Let \( f \) be a nonconstant meromorphic function. Then \( f \) is a rational function if and only if \( \lim_{r \to \infty} \frac{T(r, f)}{\log r} < \infty \).

**Lemma 2.3** (see [4], Theorem 1.19). Let \( T_1(r) \) and \( T_2(r) \) be two nonnegative, nondecreasing real functions defined in \( r > r_0 > 0 \). If \( T_1(r) = O(T_2(r)) \) (\( r \to \infty, r \notin E \)), where \( E \) is a set with finite linear measure, then

\[
\lim \sup_{r \to \infty} \frac{\log^+ T_1(r)}{\log r} \leq \lim \sup_{r \to \infty} \frac{\log^+ T_2(r)}{\log r},
\]

and

\[
\lim \inf_{r \to \infty} \frac{\log^+ T_1(r)}{\log r} \leq \lim \inf_{r \to \infty} \frac{\log^+ T_2(r)}{\log r}.
\]
which imply that the order and the lower order of $T_1(r)$ are not greater than the order and the lower order of $T_2(r)$ respectively.

**Lemma 2.4** (see [4], Theorem 1.14). Let $f$ and $g$ be two nonconstant meromorphic functions. If the order of $f$ and $g$ is $\rho(f)$ and $\rho(g)$ respectively, then

$$\rho(f \cdot g) \leq \max \{\rho(f) \cdot \rho(g)\},$$

$$\rho(f + g) \leq \max \{\rho(f) \cdot \rho(g)\}.$$  

**Lemma 2.5** (see [17], Lemma 2.7). Let $R(\omega) = \omega^n + a\omega^m + b$, where $n, m$ are positive integers satisfying $n > m$, $a, b$ are finite nonzero complex numbers. Then the algebraic equation $R(\omega) = 0$ has at least $n - 1$ distinct roots.

**Lemma 2.6** (see [18], Lemma 8). Let $s > 0$ and $t$ be relatively prime integers, and let $c$ be a finite complex number such that $c^s = 1$. Then there exists one and only one common zero of $\omega^s - 1$ and $\omega^t - c$.

### 3 Proofs of the theorems

#### 3.1 Proof of Theorem 1.7

First of all, we denote by $d$ the degree of $L$. Then $d = 2 \sum_{j=1}^{k} \lambda_j > 0$, where $k$ and $\lambda_j$ are respectively the positive integer and the positive real number in the functional equation of the axiom (iii) of the definition of $L$-functions. According to a result due to Steuding [1], p.150, we have

$$T(r, L) = \frac{d}{\pi} r \log r + O(r).$$  

(2)

Therefore $\rho(L) = 1$ and $S(r, L) = O(\log r)$.

Noting that $f$ has finitely many poles and $L$ at most has one pole at $s = 1$ in the complex plane, it follows that

$$N(r, f) = O(\log r), \quad N(r, L) = O(\log r).$$  

(3)

Because $f$ and $L$ share $a_1$, $a_2$ weighted $k_1$, $k_2$ respectively, by (3), from the first and second fundamental theorems we have

$$T(r, f) \leq \overline{N}\left(r, \frac{1}{f - a_1}\right) + \overline{N}\left(r, \frac{1}{f - a_2}\right) + \overline{N}(r, f) + S(r, f)$$

$$= \overline{N}\left(r, \frac{1}{L - a_1}\right) + \overline{N}\left(r, \frac{1}{L - a_2}\right) + O(\log r) + S(r, f)$$

$$\leq T\left(r, \frac{1}{L - a_1}\right) + T\left(r, \frac{1}{L - a_2}\right) + O(\log r) + S(r, f)$$

$$= 2T(r, L) + O(\log r) + S(r, f).$$  

(4)

Then from (4) and Lemma 2.3 we obtain

$$\rho(f) \leq \rho(L).$$  

(5)

Similarly,

$$\rho(L) \leq \rho(f).$$  

(6)

Combining (5) with (6) yields

$$\rho(f) = \rho(L).$$  

(7)

Thus

$$S(r, f) = O(\log r).$$  

(8)
We introduce two auxiliary functions below.

\[ F_1 = \frac{L'}{L - a_1} - \frac{f'}{f - a_1}, \quad (9) \]

\[ F_2 = \frac{L'}{L - a_2} - \frac{f'}{f - a_2}. \quad (10) \]

Next, we assume that \( F_1 \neq 0 \) and \( F_2 \neq 0 \). By (8) and Lemma 2.1 we get

\[ m(r, F_i) = O(\log r). \quad (11) \]

By the assumption \( L \) and \( f \) share \((a_1, k_1), (a_2, k_2)\), from (3), (9) and (11) we have

\[ k_2 \mathcal{N}(k_{i+1}\left(r, \frac{1}{L - a_2}\right) \leq N\left(r, \frac{1}{F_1}\right) \leq T(r, F_1) + O(1) \leq N(r, F_1) + m(r, F_1) + O(1) \]

\[ \leq k_1 \mathcal{N}(k_{i+1}\left(r, \frac{1}{L - a_1}\right) + \mathcal{N}(r, L) + O(\log r) \]

\[ \leq k_1 \mathcal{N}(k_{i+1}\left(r, \frac{1}{L - a_1}\right) + O(\log r). \quad (12) \]

Similarly, from (3), (10) and (11) we have

\[ k_1 \mathcal{N}(k_{i+1}\left(r, \frac{1}{L - a_1}\right) \leq N\left(r, \frac{1}{F_2}\right) \leq T(r, F_2) + O(1) \leq N(r, F_2) + m(r, F_2) + O(1) \]

\[ \leq k_2 \mathcal{N}(k_{i+1}\left(r, \frac{1}{L - a_2}\right) + \mathcal{N}(r, L) + O(\log r) \]

\[ \leq k_2 \mathcal{N}(k_{i+1}\left(r, \frac{1}{L - a_2}\right) + O(\log r). \quad (13) \]

Combining (12) with (13) yields

\[ \mathcal{N}(k_{i+1}\left(r, \frac{1}{L - a_1}\right) \leq k_1 \mathcal{N}(k_{i+1}\left(r, \frac{1}{L - a_2}\right) + O(\log r) \]

\[ \leq k_1 k_2 \mathcal{N}(k_{i+1}\left(r, \frac{1}{L - a_1}\right) + O(\log r). \quad (14) \]

Since \( k_1 k_2 > 1 \), from (14) we obtain

\[ \mathcal{N}(k_{i+1}\left(r, \frac{1}{L - a_1}\right) = O(\log r). \quad (15) \]

Substituting (15) into (12) implies

\[ \mathcal{N}(k_{i+1}\left(r, \frac{1}{L - a_2}\right) = O(\log r). \quad (16) \]

Set

\[ G = \frac{L - a_1}{f - a_1}. \]

Noting \( L \) and \( f \) share \((a_1, k_1), (a_2, k_2)\), combining (15) with (16) yields

\[ \mathcal{N}(k_{i+1}\left(r, \frac{1}{L - a_1}\right) = \mathcal{N}(k_{i+1}\left(r, \frac{1}{f - a_1}\right) = O(\log r), \]

\[ \mathcal{N}(k_{i+1}\left(r, \frac{1}{L - a_2}\right) = \mathcal{N}(k_{i+1}\left(r, \frac{1}{f - a_2}\right) = O(\log r). \]

Clearly,

\[ \mathcal{N}(r, G) \leq N(r, L) + \mathcal{N}(k_{i+1}\left(r, \frac{1}{L - a_1}\right) = O(\log r), \quad (17) \]

\[ \mathcal{N}\left(r, \frac{1}{G}\right) \leq N(r, f) + \mathcal{N}(k_{i+1}\left(r, \frac{1}{L - a_1}\right) = O(\log r). \quad (18) \]
Set
\[ G_1 = \frac{Q(L - a_1)}{f - a_1}, \]  
(19)
where \( Q \) is a rational function satisfying that \( G_1 \) is a zero-free entire function. From (17) and (18), it is easy to see that such a \( Q \) does exist. By Lemma 2.2 and Lemma 2.4 we get
\[ \rho(G_1) \leq \max\{\rho(Q), \rho(L), \rho(f)\} = 1. \]

By the Hadamard factorization theorem [19], p.384, we know
\[ G_1 = \frac{Q(L - a_1)}{f - a_1} = e^\varphi, \]  
(20)
where \( \varphi \) is a polynomial of degree at most \( \deg(\varphi) \leq 1 \). We may write \( \varphi = a_0s + b_0 \) for some complex numbers \( a_0, b_0 \). In view of (20) and Hayman [3], p.7, we have
\[ T(r, G_1) = T(r, e^{a_0s+b_0}) = O(r). \]  
(21)
By (19), the assumption that \( L \) and \( f \) share \( a_2 \), we get that every \( a_2 \)-point of \( L \) has to be 1-point of \( \frac{G_1}{Q} - 1 \). Now (20), (21) and the first fundamental theorem yield
\[ N\left(r, \frac{1}{L - a_2}\right) \leq N\left(r, \frac{1}{\frac{G_1}{Q} - 1}\right) \leq T\left(r, \frac{1}{\frac{G_1}{Q} - 1}\right) \leq T(r, G_1) + T(r, Q) + O(1) = O(r). \]  
(22)
Similarly, set
\[ G_2 = \frac{L - a_2}{f - a_2}. \]
We also get
\[ N\left(r, \frac{1}{L - a_1}\right) = O(r). \]  
(23)
By (22), (23) and the second fundamental theorem it follows that
\[ T(r, L) \leq N\left(r, \frac{1}{L - a_1}\right) + N\left(r, \frac{1}{L - a_2}\right) + N(r, L) + O(\log r) = O(r). \]  
(24)
This contradicts (2). Thus, \( F_1 \equiv 0 \) or \( F_2 \equiv 0 \). By integration, we have from (9) that
\[ L - a_1 \equiv A(f - a_1), \]
where \( A(\neq 0) \) is a constant. This implies that \( L \) and \( f \) share \( a_1 \) CM. Hence by Theorem 1.6 we deduce Theorem 1.7 holds. If \( F_2 \equiv 0 \), using the same manner, we also have the conclusion.

This completes the proof of Theorem 1.7.

### 3.2 Proof of Theorem 1.11

First we consider the following function
\[ G = \frac{QR(L)}{RF(f)}, \]  
(25)
where
\[ Q(s) = A(s - 1)^m \]  
(26)
is a rational function satisfying that \( G \) has no zeros and no poles in \( \mathbb{C} \); \( A \) is a nonzero finite value; \( m \) is the nonnegative integer in the axiom (ii) of the definition of \( L \)-functions.
We claim that such a $Q$ does exist. By the condition that $f$ and $L$ share $S$ CM, set

$$F = \frac{R(L)}{R(f)}.$$  \hspace{1cm} (27)

We can see that there can be only a pole of $f$ or $L$ such that $F = 0$ or $F = \infty$. Since $f$ has no pole and $L$ has only

one possible pole at $s = 1$, it follows that $F$ has no zero and only one possible pole at $s = 1$. Hence such a $Q$

doestexist.

Next, assume that $a_1, a_2, \cdots, a_n$ are all distinct roots of $R(a)$. Using the first fundamental theorem we get

$$T(r, L - a_i) = T(r, L) + O(1), \hspace{0.5cm} i = 1, 2, \cdots, n.$$  

Noting $n \geq 2$, by the second fundamental theorem we have

$$(n - 1)T(r, f) \leq \sum_{i=1}^{n} N\left(\frac{1}{f - a_i}\right) + \sum_{i=1}^{n} T(r, f) + \sum_{i=1}^{n} N(r, f) + S(r, f)$$

which gives

$$T(r, f) \leq \frac{n}{n - 1} T(r, L) + S(r, f).$$

This together with Lemma 2.3 yields

$$\rho(f) \leq \rho(L). \hspace{1cm} (29)$$

Similarly,

$$\rho(L) \leq \rho(f). \hspace{1cm} (30)$$

By (29), (30) and (2) we obtain

$$\rho(f) = \rho(L) = 1. \hspace{1cm} (31)$$

Also, from the first fundamental theorem we get

$$\rho\left(\frac{1}{f - a_i}\right) = \rho(f) = 1,$$

and then by Lemma 2.2 and Lemma 2.4 we deduce

$$\rho(G) \leq \max(\rho(Q), \rho(L), \rho(f)) = 1.$$  

From the Hadamard factorization theorem [19], p.384 we see

$$G = e^{h(s)}, \hspace{1cm} (32)$$

where $h(s)$ is a polynomial of degree $\deg(h(s)) \leq 1$. One can write

$$\Re h(\sigma + it) = \alpha(t)\sigma + \beta(t), \hspace{1cm} (33)$$

a polynomial in $\sigma$ with $\alpha(t), \beta(t)$ being polynomials in $t$. Now the claim is $\alpha(t) \equiv 0$. From (25), (27) and (32) we get

$$F = \frac{R(L)}{R(f)} = e^{h(s)}Q^{-1}. \hspace{1cm} (34)$$

Since $\lim_{\sigma \to +\infty} L(s) = 1, \lim_{\sigma \to +\infty} f(s) = k(k \neq \infty), R(k) \neq 0$ and $R(1) \neq 0$, it follows that

$$\lim_{\sigma \to +\infty} \frac{R(L)}{R(f)} = C, \hspace{1cm} (35)$$
where $C \neq 0$ is a finite value. If $\alpha(t) \neq 0$, we obtain $\alpha(t_0) \neq 0$ for some value $t_0$. If $\alpha(t_0) > 0$, from (34) we know that
\[
|\frac{R(L)}{R(f)}| = |Q^{-1}|e^{\beta(t)}.
\] (36)

Thus from (26), (33), (35) and (36) we can deduce that, $|C| = \infty$ when $\sigma \to +\infty$ with $t = t_0$, which is a contradiction. Similarly, if $\alpha(t_0) < 0$, we have that, $|C| = 0$ when $\sigma \to +\infty$ with $t = t_0$, which is also a contradiction. Therefore $\alpha(t) \equiv 0$. Now by (33) and (36) we get
\[
|\frac{R(L)}{R(f)}| = |Q^{-1}|e^{\beta(t)}.
\] (37)

Combining (35) with (37) yields
\[
\lim_{\sigma \to +\infty} |Q| = \frac{e^{\beta(t)}}{|C|}
\] (38)
for a fixed $t$. Considering that the limit of $|Q|$ as $\sigma \to +\infty$ is a nonzero finite constant for some value $t$ and $n \geq 2$, in view of (26) we see that $m = 0$, and then $Q(s) \equiv A$. From (38) we have $e^{\beta(t)} = |A||C|$. Thus it follows by (37) that
\[
|\frac{R(L)}{R(f)}| = |C|.
\] (39)

Since $C \neq 0$ is a finite complex number, from (39) we deduce that $\frac{R(L)}{R(f)}$ is a constant. Then by (35) we know that
\[
\frac{R(L)}{R(f)} \equiv C.
\] (40)

From the assumption in the theorem we have $f(s_0) = L(s_0) = b$ for some $s_0 \in \mathbb{C}$. It now follows from (40) that $C = 1$. Thus
\[
\frac{R(L)}{R(f)} \equiv 1.
\] (41)

That is $R(L) \equiv R(f)$.

This completes the proof of Theorem 1.11.

### 3.3 Proof of Theorem 1.12

First, we have that the algebraic equation $\omega^n + m + \alpha \omega + \beta = 0$ has at least $n + m - 1 \geq 3m + 1 \geq 4$ distinct roots in view of Lemma 2.5. By Theorem 1.11, we get
\[
L^{n+m} + \alpha L^n + \beta \equiv f^{n+m} + \alpha f^n.
\] (42)

Set $H = \frac{1}{f}$. Then by (42) we deduce
\[
\frac{1}{\alpha}L^m = \frac{H^n - 1}{H^{n+m} - 1}.
\] (43)

We discuss two cases:

Case 1. $H$ is a constant. If $H^{n+m} \neq 1$, by (43), we get that $L$ is a constant, which contradicts the assumption that $L$ is a nonconstant L-function. Therefore, $H^{n+m} = 1$, and so it follows by (43) that $H^m = H^n = 1$, that is $f^n = L^n$ and $f^m = L^m$. We get $f^d = L^d$.

Case 2. $H$ is a nonconstant meromorphic function. Note that $L$ has at most one pole. Now we discuss the following two subcases again.

Subcase 2.1. $L$ has no poles. Then, from (43) we get that every 1-point of $H^{n+m}$ has to be 1-point of $H^n$. Since $H^{n+m} = H^nH^m$, we have any 1-point of $H^{n+m}$ to be a 1-point of $H^m$. Because $n > m + 2$, it follows that $H$ is a constant, contradicting the assumption.

Subcase 2.2. $L$ has one and only one pole. Then by (43) we know every zero of $H^{n+m} - 1$ has to be zero of $H^n - 1$ with one exception. Put
\[
H^n - 1 = (H - 1)(H - \zeta_1) \cdots (H - \zeta_{n-1}),
\]
Uniqueness theorems for $L$-functions in the extended Selberg class

$H^{n+m} - 1 = (H - 1)(H - \tau_1)\cdots(H - \tau_{n+m-1}),$

where $\zeta_1, \zeta_2, \cdots, \zeta_{n-1}$ are $n - 1$ distinct finite complex numbers satisfying $\zeta_i^n = 1, \zeta_i \neq 1, 1 \leq i \leq n - 1;$ $\tau_1, \tau_2, \cdots, \tau_{n+m-1}$ are $n + m - 1$ distinct finite complex numbers satisfying $\tau_j^{n+m} = 1, \tau_j \neq 1, 1 \leq j \leq n + m - 1.$

Let $m = 1.$ By Lemma 2.6 we see $H^n - 1$ and $H^{n+1} - 1$ have only one common zero, so $H$ cannot be equal to any $n + m - 2$ values of $\{\tau_1, \tau_2, \cdots, \tau_{n+m-1}\}.$ From $n > m + 2$ it follows that $H$ is a constant, contradicting the assumption.

Let $m \geq 2.$ If any 1-point of $H^n$ is a 1-point of $H^{n+m},$ then any 1-point of $H^n$ is a 1-point of $H^m.$ Note that $n > m + 2.$ This contradicts the assumption that $H$ is nonconstant. If there is at least one $\zeta_i \neq \tau_j, 1 \leq i \leq n - 1,$ $1 \leq j \leq n + m - 1,$ then $H$ cannot be equal to any $m + 1$ values of $\{\tau_1, \tau_2, \cdots, \tau_{n+m-1}\}.$ From $m \geq 2,$ we know $H$ is a constant, contradicting the assumption.

This completes the proof of Theorem 1.12.

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