1 Introduction

The term quantale was suggested by C.J. Mulvey at the Oberwolfach Category Meeting ([1]) as a “quantization” of the term locale ([2]). An important moment in the development of the theory of quantales was the realization that quantales give a semantics for propositional linear logic in the same way as Boolean algebras give a semantics for classical propositional logic ([3, 4]). Quantales arise naturally as lattices of ideals, subgroups, or other suitable substructures of algebras ([5, 6]).

Algebraic investigations on qutale-like structures, such as quantales, quantale modules, sup-algebras, S-quantales, etc. have been studied in [5], [7], [8], and [9], respectively. Some categorical considerations are also taken into account ([10], [6]). S-quantales were firstly introduced by Zhang and Laan in [11], which have been shown to play an important role in the theory of injectivity on the category of S-posets. The current paper is devoted to the study of categorical properties of S-quantales.

In this work, S is always a pomonoid, that is, a monoid S equipped with a partial order ≤ such that ss′ ≤ tt′ whenever s ≤ t, s′ ≤ t′ in S. A poset (A, ≤) together with a mapping A × S → A (under which a pair (a, s) maps to an element of A denoted by as) is called an S-poset, denoted by As, if for any a, b ∈ As, s, t ∈ S,

1. a(st) = (as)t,
2. a1 = a,
3. a ≤ b, s ≤ t imply that as ≤ bt.

S-poset morphisms are order-preserving mappings which also preserve the S-action. We denote the category of S-posets with S-poset morphisms by PosS. An S-subposet of an S-poset As is an action-closed subset of As whose partial order is the restriction of the order from As.
Clearly, \( S \)-posets are generalizations of \( S \)-acts, whose relying category is denoted by \( \text{Act}_S \).

Recall that an \( S \)-poset \( A_S \) is an \( S \)-quantale \((11)\) if

1. the poset \( A \) is a complete lattice;
2. \((\bigvee M)s = \bigvee\{ms \mid m \in M\}\) for each subset \( M \) of \( A \) and each \( s \in S \).

An \( S \)-quantale morphism is a mapping between \( S \)-quantales which preserves both \( S \)-actions and arbitrary joins. An \( S \)-subquantale of an \( S \)-quantale \( A_S \) is exactly the relative \( S \)-subposet of \( A_S \) which is closed under arbitrary joins.

We denote the category of \( S \)-quantales with \( S \)-quantale morphisms by \( \text{Quant}_S \). This work is devoted to the presentation of categorical aspects in \( \text{Quant}_S \). We explore limits and colimits, monomorphisms and epimorphisms, respectively, and exhibit adjoint situations accordingly.

**Lemma 1.1.** The bottom of an \( S \)-quantale is a zero element.

**Proof.** The result follows by the fact that for the bottom \( \perp_{A_S} \) of an \( S \)-quantale \( A_S \), and any \( s \in S \),

\[
\perp_{A_S}s = (\bigvee\emptyset)s = \bigvee_{s \in S}(as) = \bigvee\emptyset = \perp_{A_S}.
\]

Since an \( S \)-quantale morphism \( f : A_S \to B_S \) preserves all joins, it follows by the adjoint functor theorem that it has a right adjoint \( f_* : B_S \to A_S \), satisfying

\[
f(a) \leq b \iff a \leq f_*(b),
\]

for all \( a \in A_S \), \( b \in B_S \).

**Lemma 1.2.** Let \( f : A_S \to B_S \) be an \( S \)-quantale morphism. Then \( f \) preserves the bottom.

**Proof.** Denote by \( \perp_{A_S} \) the bottom of \( A_S \). Then \( \perp_{A_S} \leq f_*(b) \), for every \( b \in B_S \). By (1), we have \( f(\perp_{A_S}) \leq b \).

## 2 Limits and colimits in \( \text{Quant}_S \)

### Products and coproducts

**Proposition 2.1.** The product of a family of \( S \)-quantales is their cartesian product with componentwise action, and order.

**Proposition 2.2.** The coproduct of a family of \( S \)-quantales \( \{X_i\}_{i \in I} \) is \( (\coprod_{i \in I} X_i, (\mu_j)_{j \in I}) \), where \( \mu_j : X_j \to \coprod_{i \in I} X_i \), \( j \in I \), is defined by

\[
\mu_j(x) = (\bar{x}_i)_{i \in I}, \quad \text{where} \quad \bar{x}_i = \begin{cases} x & i = j, \\ \perp_{X_i} & i \neq j. \end{cases}
\]

**Proof.** Clearly, \( \mu_j \) is an \( S \)-quantale morphism for every \( j \in I \). Let \( f_j : X_j \to Q_S \), \( j \in I \), be \( S \)-quantale morphisms. Define a mapping \( \psi : \coprod_{i \in I} X_i \to Q_S \) by

\[
\psi((x_i)_{i \in I}) = \bigvee_{i \in I} f_i(x_i),
\]

for any \( (x_i)_{i \in I} \in \prod_{i \in I} X_i \). It is easy to see that \( \psi \) preserves \( S \)-actions. For arbitrary indexed set \( K \), we have

\[
\psi\left(\bigvee_{k \in K} (x_{ik})_{i \in I}\right) = \psi\left(\bigvee_{k \in K} (\bigvee_{i \in I} x_{ik})\right) = \bigvee_{i \in I} \psi\left(\bigvee_{k \in K} x_{ik}\right) = \bigvee_{k \in K} \psi((x_{ik})_{i \in I}).
\]

Moreover, by Lemma 1.2, \( f_i(\perp_{X_i}) = \perp_{Q_S} \), for each \( i \in I \). Hence

\[
\psi(\mu_j(x)) = \psi((\bar{x}_i)_{i \in I}) = \bigvee_{i \in I} f_i(\bar{x}_i) = f_j(x),
\]
for any \( j \in I, x \in X_j \).

Finally, suppose that there exists an \( S \)-quantale morphism \( \phi : \prod_{i \in I} X_i \to Q_S \) such that \( \phi \mu_i = f_i \), for every \( i \in I \). Then, for each \( (x_i)_{i \in I} \in \prod_{i \in I} X_i \), one gets that

\[
\phi((x_i)_{i \in I}) = \phi\left( \bigvee_{i \in I} \mu_i(x_i) \right) = \bigvee_{i \in I} \phi \mu_i(x_i) = \bigvee_{i \in I} f_i(x_i) = \psi((x_i)_{i \in I}),
\]

and hence \( \phi = \psi \) as needed.

\[ \square \]

### Equalizers, coequalizers, pullbacks, and pushouts

**Proposition 2.3.** Let \( f, g : A_S \to B_S \) be morphisms of \( S \)-quantales. The equalizer of \( f \) and \( g \) is given by \( E = \{ a \in A_S \mid f(a) = g(a) \} \), with action and order inherited from \( A_S \).

**Proof.** Clearly, \( E \) is an \( S \)-poset, and a complete lattice. So it is an \( S \)-quantale by the fact that \( f \) and \( g \) preserve arbitrary joins. Let \( i : E \to A \) be the inclusion mapping. For any morphism \( e : E \to A \) with \( fe = ge \), since \( e(E) \subseteq E \), it follows that \( \overline{e} \), which is the codomain restriction of \( e \), is the unique morphism fulfilling \( i \overline{e} = e \). \( \square \)

By [12] Theorem 12.3, we immediately get that \( \text{Quant}_S \) is complete.

**Proposition 2.4.** The category \( \text{Quant}_S \) is complete.

Let \( \rho \) be a congruence on \( S \)-quantale \( A_S \). In a natural way, the quotient \( A/\rho \) constitutes an \( S \)-quantale equipped with the order defined by a \( \rho \)-chain, where the joins in \( A/\rho \) are

\[
\bigvee_{i \in I} \left[ a_i \right]_\rho = \left[ \bigvee_{i \in I} a_i \right]_\rho,
\]

and the canonical mapping \( \pi : A_S \to (A/\rho)_S \) becomes an \( S \)-quantale morphism, provided that \( \rho = \ker \pi \) ([9]).

For \( H \subseteq A_S \times A_S \), the corresponding \( S \)-quantale congruence generated by \( H \), will be denoted by \( \theta(H) \).

**Proposition 2.5.** Let \( f, g : A_S \to B_S \) be morphisms of \( S \)-quantales. The coequalizer of \( f \) and \( g \) is the quotient \( (B/\theta(H))_S \), where \( H = \{ (f(a), g(a)) \mid a \in A_S \} \).

**Proof.** Let \( f, g : A_S \to B_S \) be morphisms of \( S \)-quantales, \( H = \{ (f(a), g(a)) \mid a \in A_S \} \), \( \pi \) be the canonical mapping from \( B_S \) to \( (B/\theta(H))_S \). Clearly, \( \pi f = \pi g \). For any \( S \)-quantale morphism \( h : B_S \to C_S \) satisfying \( hf = hg \), we obtain that \( \ker \pi \subseteq \ker h \), since \( (f(a), g(a)) \in \ker h \), for \( a \in A_S \).

Now define a mapping \( \overline{h} : (B/\theta(H))_S \to C_S \) by

\[
\overline{h}([b]_{\theta(H)}) = h(b),
\]

for \([b]_{\theta(H)} \in (B/\theta(H))_S \). Clearly \( \overline{h} \) is an \( S \)-act morphism and preserves arbitrary joins by (3). It is quite routine to check that \( \overline{h} \) is the unique morphism satisfying \( h \pi = h \). \( \square \)

**Proposition 2.6.** Let \( f : A_S \to C_S, g : B_S \to C_S \) be morphisms of \( S \)-quantales. The pullback of \( f \) and \( g \) is the \( S \)-subposet \( P = \{ (a, b) \in (A \times B)_S \mid f(a) = g(b) \} \) of \( (A \times B)_S \), together with the restricted projections of \( P_S \) into \( A_S \) and \( B_S \).

**Proof.** It is known that \( P_S \) is an \( S \)-quantale. For any \( S \)-quantale \( Q_S \) and an pair of morphisms \( f_1 : Q_S \to A_S, f_2 : Q_S \to B_S \) with \( f_1 \circ q = g \circ f_2 \), one has that \((f_1(q), f_2(q)) \in P_S \), for any \( q \in Q_S \). Now define a mapping \( \varphi : Q_S \to P_S \) by

\[
\varphi(q) = (f_1(q), f_2(q)),
\]

for \( q \in Q_S \). One gets that

\[
\varphi(q)s = (f_1(q), f_2(q))s = (f_1(q)s, f_2(q)s) = (f_1(qs), f_2(qs)) = \varphi(qs),
\]
for each \( q \in Q_S, s \in S \), and
\[
\varphi \left( \bigvee_{i \in I} q_i \right) = \left( f_1 \left( \bigvee_{i \in I} q_i \right), f_2 \left( \bigvee_{i \in I} q_i \right) \right) = \left( \bigvee_{i \in I} f_1(q_i), \bigvee_{i \in I} f_2(q_i) \right) = \bigvee_{i \in I} \varphi(q_i),
\]
for all \( q_i \in Q_S, i \in I \). If \( \pi_A : P_S \to A_S \) and \( \pi_B : P_S \to B_S \) are the restricted projections, then \( f \pi_A = g \pi_B \). Straightforward checking shows that \( \varphi \) is the unique morphism satisfying \( \pi_A \varphi = f \) and \( \pi_B \varphi = f \).

**Proposition 2.7.** Let \( f : A_S \to B_1, g : A_S \to B_2 \) be morphisms of \( S \)-quantales. The pushout of \( f \) and \( g \) is 
\[
\left( (B_1 \times B_2) / \theta(H) \right)_S,
\]
where \( \mu_1 : (B_1 \times B_2) / \theta(H) \to B_1 \times B_2 \), \( \pi_1 : (B_1 \times B_2) / \theta(H) \to B_1 \times B_2 \) are the restricted projections, then \( \mu_1 \) is injective as needed. According to the above result of Proposition 2.2, the coequalizer of \( \mu_1 f \) and \( \mu_2 g \) is the quotient \( \left( (B_1 \times B_2) / \theta(H) \right)_S \), where \( \theta = \{ (\mu_1 f(a), \mu_2 g(a)) \mid a \in A_S \} \), by Proposition 2.5. The result follows immediately by [12] Remark 11.31.

\[\Box\]

### 3 Monomorphisms

This section contributes to the presentation of several kinds of monomorphisms in the category \( \text{Quant}_S \). It is shown that different from the case of \( S \)-posets (see [13]), monomorphisms in \( \text{Quant}_S \) coincide with order-embeddings, which are precisely injective morphisms. It thus leads to the strengthening results that these classes of monomorphisms are also in accordance with those labeled regular and extremal in \( \text{Quant}_S \), which are exactly the category-theoretic embeddings when \( \text{Quant}_S \) is considered as a concrete category over Set, \( \text{Act}_S \), and \( \text{Pos}_S \), respectively.

**Proposition 3.1.** Let \( f : A_S \to B_S \) be a morphism of \( S \)-quantales. Then the following statements are equivalent:

1. \( f \) is a monomorphism;
2. \( f \) is injective;
3. \( f \) is an order-embedding.

**Proof.** It is enough to show the implications (1) \( \Rightarrow \) (2) and (1) \( \Rightarrow \) (3) hold.

Let \( f : A_S \to B_S \) be a monomorphism of \( S \)-quantales. Consider \( S \)-subquantale ker\( f \) of the product \( (A \times A)_S \), and the restricted projection mappings \( h_i : \ker\ f \to A \), \( i = 1, 2 \). For any \( (x, y) \in \ker\ f \), equalities
\[
f h_1(x, y) = f(x) = f(y) = fh_2(x, y)
\]
 imply that \( fh_1 = fh_2 \) and hence \( h_1 = h_2 \) by assumption. Therefore, \( x = h_1(x, y) = h_2(x, y) = y \), and hence \( f \) is injective as needed.

It remains to prove that \( f \) is an order-embedding whenever it is a monomorphism. Suppose that \( f(a_1) \leq f(a_2) \) for \( a_1, a_2 \in A_S \). Then
\[
f(a_2) = f(a_1) \lor f(a_2) = f(a_1 \lor a_2).
\]
According to the above result of \( f \) being injective, we soon obtain that \( a_1 \leq a_2 \), and thus \( f \) is an order-embedding.

**Lemma 3.2.** Each inclusion mapping in \( \text{Quant}_S \) is a regular monomorphism.

**Proof.** Suppose that \( A_S \) is an \( S \)-subquantale of \( B_S \). Let \( (B \times B)_S, (\mu_1, \mu_2) \) be the coproduct of \( (B_S, B_S) \), described as in Proposition 2.2. Write
\[
R = \{ ((a, 1), (1, a)) \mid a \in A_S \},
\]
where \( 1 \) is the bottom element of \( B_S \). Then the relation \( \rho \), which is defined by
\[
\rho = \{ (x \lor a, y \lor b), (x' \lor a', y' \lor b') \mid x, y, x', y' \in B_S, a, b, a', b' \in A_S, x \lor b = x' \lor b', y \lor a = y' \lor a' \}
\]
is the smallest congruence relation on \( B \times B \) containing \( R \). So \(((B \times B)/\rho)_S\) becomes an \( S\)-quantale equipped with a suitable order defined by a \( \rho \)-chain, and the canonical mapping \( \pi: (B \times B)_S \to ((B \times B)/\rho)_S \) given by \( \pi(x, y) = [(x, y)]_\rho \), for each \((x, y) \in (B \times B)_S\), is a morphism.

Next we show that the inclusion mapping \( \iota_A: A \to B \) is the equalizer of \( \pi_{\mu_1} = \pi_{\mu_2} \). Suppose that \( h: E_S \to B_S \) is any monomorphism satisfying \( \pi_{\mu_1} h = \pi_{\mu_2} h \). Then for any \( e \in E_S \), the equalities

\[
[(h(e), \bot)]_\rho = \pi(h(e), \bot) = \pi_{\mu_1} h(e) = \pi_{\mu_2} h(e) = \pi(\bot, h(e)) = [(\bot, h(e))]_\rho
\]

indicate that \(((h(e), \bot), (\bot, h(e))) \in \rho \). According to the definition of \( \rho \), we deduce that \((h(e), \bot) = (x \lor a, y \lor b) \) and \((\bot, h(e)) = (x \lor a', y \lor b') \) for some \( x, y, x', y' \in B_S \), \( a, a', b, b' \in A_S \). Suppose \( y = b = \bot, x = a' = \bot \), and correspondingly,

\[
a = \bot \lor a = y \lor a = y' \lor a = y',
\]

and

\[
x = x \lor \bot = \bot \lor b' = b'.
\]

Therefore, we have \( h(e) = x \lor a \lor b' \lor a \in A_S \), i.e., \( h(E) \subseteq A_S \). As a consequence, \( \overline{h} = h: E \to A \) is the unique morphism satisfying \( \iota_A \overline{h} = h \).

**Theorem 3.3.** Let \( f: A_S \to B_S \) be a morphism of \( S\)-quantales. Then the following assertions are equivalent:

1. \( f \) is a regular monomorphism;
2. \( f \) is an extremal monomorphism;
3. \( f \) is a monomorphism;
4. \( f \) is a \( \text{Quant}_S \)-embedding over \( \text{Set} \);
5. \( f \) is a \( \text{Quant}_S \)-embedding over \( \text{Act}_S \);
6. \( f \) is a \( \text{Quant}_S \)-embedding over \( \text{Pos}_S \).

**Proof.** (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) are general category-theoretic results.

(3) \( \Rightarrow \) (4). Suppose that \( f: A_S \to B_S \) is a monomorphism. Let \( g: C_S \to A_S \) be a mapping with \( fg: C_S \to B_S \) being an \( S\)-quantale morphism. Then \( g \) preserves arbitrary joins by the fact that for \( a_i \in C_S \), \( i \in I \),

\[
fg \left( \bigvee_{i \in I} a_i \right) = \bigvee_{i \in I} fg(a_i) = f \left( \bigvee_{i \in I} g(a_i) \right),
\]

and \( f \) being injective by Proposition 3.1. Similarly, we get that \( g \) preserves \( S\)-actions. Thus \( f \) is initial and then an \( S\)-quantale embedding over \( \text{Set} \).

(4) \( \Rightarrow \) (3), (4) \( \Rightarrow \) (5) \( \Rightarrow \) (6) are clear.

(6) \( \Rightarrow \) (4). Let \( f: A_S \to B_S \) be a \( \text{Quant}_S \)-embedding over \( \text{Pos}_S \), \( g: C_S \to A_S \) a mapping provided that \( fg: C_S \to B_S \) is a morphism in \( \text{Quant}_S \). We are going to show that \( g \) is an \( S\)-poset morphism. This is the case since

\[
fg(\alpha s) = fg(\alpha)s = f(g(\alpha)s),
\]

for any \( \alpha \in A_S \), \( s \in S \), and

\[
fg(\alpha_2) = fg(\alpha_1 \lor \alpha_2) = fg(\alpha_1) \lor fg(\alpha_2) = f(g(\alpha_1) \lor g(\alpha_2)),
\]

for \( \alpha_1 \leq \alpha_2 \) in \( A_S \). Note that the monomorphisms in \( \text{Pos}_S \) are just the \( S\)-poset morphisms with injective underlying mappings, we immediately achieve that \( g(\alpha s) = g(\alpha)s \) and \( g(\alpha_1) \leq g(\alpha_2) \). Therefore, \( g \) is an \( S\)-poset morphism as required.

(3) \( \Rightarrow \) (1). This follows by [12] Proposition 7.53 (2) and Lemma 3.2.

**4 Epimorphisms**

Dual to discussions on monomorphisms studied in Section 3, this section is intended to motivate our investigation on relationships between various type of epimorphisms in \( \text{Quant}_S \). However, the characterization of
epimorphisms in $\text{Quant}_S$ is quite complicated. So we merely cite the result and the reader is suggested to find complete illustrations in [14].

**Proposition 4.1** ([14] Th. 4.2). Epimorphisms in $\text{Quant}_S$ are exactly onto morphisms.

**Theorem 4.2.** For a morphism $f : A_S \to B_S$ of $\text{S}$-quantales, the following statements are equivalent:

1. $f$ is a regular epimorphism;
2. $f$ is an extremal epimorphism;
3. $f$ is an epimorphism;
4. $f$ is a $\text{Quant}_S$-quotient morphism over $\text{Set}$;
5. $f$ is a $\text{Quant}_S$-quotient morphism over $\text{Acts}_S$;
6. $f$ is a $\text{Quant}_S$-quotient morphism over $\text{Pos}_S$.

**Proof.** (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are clear.

(3) $\Rightarrow$ (4). Let $g : B_S \to C_S$ be a mapping between $\text{S}$-quantales such that $gf$ is an $\text{S}$-quantale morphism. Let us verify that $g$ is an $\text{S}$-quantale morphism, as well. It is easy to see that $g$ is an $\text{S}$-poset morphism. Since $f$ is an epimorphism, it is onto by Proposition 4.1. Hence we may assume that for any $M \subseteq B_S$, $\forall M = f(a)$ for some $a \in A_S$. By the reason that $f$ preserves arbitrary joins, we have

$$f(a) = \bigvee M = \bigvee_{x \in f^{-1}(M)} f(x) = f\left(\bigvee_{x \in f^{-1}(M)} x\right).$$

Consequently,

$$g(\bigvee M) = gf(a) = gf\left(\bigvee_{x \in f^{-1}(M)} x\right) = \bigvee_{x \in f^{-1}(M)} gf(x) = \bigvee_{m \in M} g(m).$$

(4) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) $\Rightarrow$ (6) are clear.

(6) $\Rightarrow$ (2). Let $f : A_S \to B_S$ be a $\text{Quant}_S$-quotient morphism over $\text{Pos}_S$. Suppose that $g : A_S \to C_S$ and $h : C_S \to B_S$ are $\text{S}$-quantale morphisms such that $f = hg$ and $h$ is a monomorphism. Then $h$ is injective by Proposition 3.1. Note that $f$ is a $\text{Pos}_S$-epimorphism by hypotheses, and hence is surjective. So $h$ is surjective, as well, and thus bijective. Now, considering the inverse mapping $h^{-1}$ with $g = h^{-1}f$, we remain to show that $h^{-1}$ is an $\text{S}$-poset morphism. In fact, $f$ being onto indicates that $h^{-1}$ is action-preserving. Observe that

$$h\left(h^{-1}(b) \lor h^{-1}(b')\right) = hh^{-1}(b) \lor hh^{-1}(b') = b \lor b' = hh^{-1}(b'),$$

for any $b \leq b'$ in $B_S$. Thus $h^{-1}(b) \lor h^{-1}(b') = h^{-1}(b')$, which expresses that $h^{-1}$ is an $\text{S}$-poset morphism, and hereby an $\text{S}$-quantale morphism by assumption. \hfill $\square$

## 5 Adjoint situations

The final part is devoted to observation on the adjoint situation between $\text{Pos}$ and $\text{Quant}_S$. By a *free $\text{S}$-quantale on a poset $P$* we mean an $\text{S}$-quantale $Q_S$ together with a monotone mapping $\psi : P \to Q_S$ with the universal property that given any $\text{S}$-quantale $A_S$ and a monotone mapping $f : P \to A_S$, there exists a unique $\text{S}$-quantale morphism $\overline{f} : Q_S \to A_S$ such that $f$ can be factored through.

**Lemma 5.1** ([13] Th. 10). For a given poset $P$ and a pomonoid $S$, the free $\text{S}$-poset on $P$ is given by $P \times S$, with componentwise order and the action $(x, s)t = (x, st)$, for every $x \in P$, $s, t \in S$.

Let $(P \times S)_S$ be the free $\text{S}$-poset presented in Lemma 5.1. Write

$$\Omega(P \times S) = \{D \subseteq P \times S \mid D = D\downarrow\},$$
Theorem 5.5. Proposition 5.2. Let $S$ be a pomonoid, $P$ be a poset. Then $(\mathcal{Q}(P \times S), *, \subseteq)$ is an $S$-quantale.

Proof. Observe first that

$$(D \ast t_1) \ast t_2 = \{(p, s) \in P \times S \mid (p, s) \leq (p_1, s_1 t_2) \text{ for some } (p_1, s_1) \in D \ast t_1\}$$

for any $t_1, t_2 \in S$, $D \in \mathcal{Q}(P \times S)$, and $D \ast 1 = (D1) \downarrow = D$. This shows that $(\mathcal{Q}(P \times S), *)$ is an $S$-act. Clearly, $D \ast s \subseteq D \ast t$, whenever $D_1 \subseteq D_2$ in $\mathcal{Q}(P \times S)$, and $s \leq t$ in $S$. So $\mathcal{Q}(P \times S)$ is an $S$-poset. It is straightforward to check that $(\bigcup_{i \in I} D_i) \ast t = \bigcup_{i \in I} (D_i \ast t)$ for every $D_i \in \mathcal{Q}(P \times S), i \in I, t \in S$. □

Lemma 5.3 comes true directly by the definition of $\mathcal{Q}(P \times S)$.

Lemma 5.4. Let $S$ be a pomonoid, $P$ be a poset. Then $p \downarrow \ast t_1 = (p \downarrow 1) \ast t$ holds in $\mathcal{Q}(P \times S)_S$ for every $p \in P, t \in S$.

Proof. It is clear that $(q, s) \in (p \downarrow 1) \ast t$ for every $(q, s) \in p \downarrow \ast t_1$, since $(q, s) \subseteq (p, t)$. On the other hand, for any $(q, s) \in (p \downarrow 1) \ast t$, $(q, s) \subseteq (p_1, s_1 t) = (p_1, s_1 t) t$ for some $(p_1, s_1) \in p \downarrow 1$, it follows that $(q, s) \subseteq (p, 1) t = (p, t)$. Hence $(q, s) \in p \downarrow \ast t_1$. □

Theorem 5.5. Let $S$ be a pomonoid, $P$ be a poset. Then the free $S$-quantale on $P$ is given by the $S$-quantale $\mathcal{Q}(P \times S)_S$.

Proof. Define a mapping $\tau : P \rightarrow \mathcal{Q}(P \times S)_S$ by $\tau(p) = p \downarrow 1$. Obviously, $\tau$ is order-preserving. Let $Q_S$ be an $S$-quantale, $f : P \rightarrow Q_S$ be any monotone mapping. Define a mapping $\tilde{f} : \mathcal{Q}(P \times S)_S \rightarrow Q_S$ by

$$\tilde{f}(D) = \bigvee \{f(p) s \mid (p, s) \in D\},$$

for every $D \in \mathcal{Q}(P \times S)_S$. We claim that $\tilde{f}$ is the unique $S$-quantale morphism with the property that $\tilde{f}_\tau = f$.

It is clear that $\tilde{f}$ preserves $S$-actions. Take $D_i \in \mathcal{Q}(P \times S)_S, i \in I$, then equalities

$$\tilde{f}\left(\bigcup_{i \in I} D_i\right) = \bigvee \{f(p) s \mid (p, s) \in \bigcup_{i \in I} D_i\} = \bigvee_{i \in I} \{\bigvee f(p) s \mid (p, s) \in D_i\} = \bigvee_{i \in I} \tilde{f}(D_i)$$

indicate that $\tilde{f}$ preserves arbitrary joins. Evidently, for any $p \in P$,

$$\tilde{f}_\tau(p) = \tilde{f}(p \downarrow 1) = \bigvee \{f(q) s \mid (q, s) \in p \downarrow 1\} \leq f(p),$$

while the fact that $f(p)$ being one of the terms in the sup that defines $\tilde{f}_\tau(p)$ guarantees the opposite implication. Suppose that $\tilde{f} : \mathcal{Q}(P \times S)_S \rightarrow Q_S$ is an $S$-quantale morphism such that $\tilde{f}_\tau = f$. Then by Lemma 5.3 and Lemma 5.4, we achieve that

$$\tilde{f}(D) = \tilde{f}\left(\bigcup_{(p, s) \in D} (p \downarrow s_1)\right) = \bigvee_{(p, s) \in D} \tilde{f}(p \downarrow s_1) = \bigvee_{(p, s) \in D} \tilde{f}(p \downarrow s_1) \ast s$$
there is a free functor

Proposition 5.7.

to be left adjoint to the forgetful functor.

where

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