1 Introduction

The term quantale was suggested by C.J. Mulvey at the Oberwolfach Category Meeting ([1]) as a “quantization” of the term locale ([2]). An important moment in the development of the theory of quantales was the realization that quantales give a semantics for propositional linear logic in the same way as Boolean algebras give a semantics for classical propositional logic ([3, 4]). Quantales arise naturally as lattices of ideals, subgroups, or other suitable substructures of algebras ([5, 6]).

Algebraic investigations on quandle-like structures, such as quantales, quantale modules, sup-algebras, S-quantales, etc. have been studied in [5], [7], [8], and [9], respectively. Some categorical considerations are also taken into account ([10], [6]). S-quantales were firstly introduced by Zhang and Laan in [11], which have been shown to play an important role in the theory of injectivity on the category of S-posets. The current paper is devoted to the study of categorical properties of S-quantales.

In this work, S is always a pomonoid, that is, a monoid S equipped with a partial order ≤ such that ss' ≤ tt' whenever s ≤ t, s' ≤ t' in S. A poset (A, ≤) together with a mapping A × S → A (under which a pair (a, s) maps to an element of A denoted by as) is called an S-poset, denoted by As, if for any a, b ∈ As, s, t ∈ S,

1. a(st) = (as)t,
2. a1 = a,
3. a ≤ b, s ≤ t imply that as ≤ bt.

S-poset morphisms are order-preserving mappings which also preserve the S-action. We denote the category of S-posets with S-poset morphisms by PosS. An S-subposet of an S-poset As is an action-closed subset of As whose partial order is the restriction of the order from As.
Clearly, S-posets are generalizations of S-acts, whose relying category is denoted by ActS.

Recall that an S-poset A_5 is an S-quantale ([11]) if
(1) the poset A is a complete lattice;
(2) \( (\lor M)s = \lor \{ ms \mid m \in M \} \) for each subset M of A and each s \in S.

An S-quantale morphism is a mapping between S-quantales which preserves both S-actions and arbitrary joins. An S-subquantale of an S-quantale A_5 is exactly the relative S-subposet of A_5 which is closed under arbitrary joins.

We denote the category of S-quantales with S-quantale morphisms by QuantS. This work is devoted to the presentation of categorical aspects in QuantS. We explore limits and colimits, monomorphisms and epimorphisms, respectively, and exhibit adjoint situations accordingly.

**Lemma 1.1.** The bottom of an S-quantale is a zero element.

**Proof.** The result follows by the fact that for the bottom \( \bot_{A_5} \) of an S-quantale A_5, and any s \in S,
\[
\bot_{A_5}s = (\lor \emptyset)s = \lor_{s \in S}(as) = \lor = \bot_{A_5}.
\]

Since an S-quantale morphism \( f : A_5 \to B_5 \) preserves all joins, it follows by the adjoint functor theorem that it has a right adjoint \( f_* : B_5 \to A_5 \), satisfying
\[
f(a) \leq b \iff a \leq f_*(b),
\]
for all \( a \in A_5, b \in B_5 \).

**Lemma 1.2.** Let \( f : A_5 \to B_5 \) be an S-quantale morphism. Then f preserves the bottom.

**Proof.** Denote by \( \bot_{A_5} \) the bottom of A_5. Then \( \bot_{A_5} \leq f_*(b) \), for every \( b \in B_5 \). By (1), we have \( f(\bot_{A_5}) \leq b \).

### 2 Limits and colimits in QuantS

**Proposition 2.1.** The product of a family of S-quantales is their cartesian product with componentwise action, and order.

**Proposition 2.2.** The coproduct of a family of S-quantales \( \{X_i\}_{i \in I} \) is \( \prod_{i \in I} X_i, (\mu_j)_{j \in I} \), where \( \mu_j : X_j \to \prod_{i \in I} X_i, j \in I, \) is defined by
\[
\mu_j(x) = (\mu_{ij}x)_{i \in I}, \text{ where } \mu_{ij}x = \begin{cases} x & i = j, \\ \bot_{X_i} & i \neq j. \end{cases}
\]

**Proof.** Clearly, \( \mu_j \) is an S-quantale morphism for every \( j \in I \). Let \( f_j : X_j \to Q_5, j \in I, \) be S-quantale morphisms. Define a mapping \( \psi : \prod_{i \in I} X_i \to Q_5 \) by
\[
\psi((x_i)_{i \in I}) = \bigvee_{i \in I} f_i(x_i),
\]
for any \( (x_i)_{i \in I} \in \prod_{i \in I} X_i \). It is easy to see that \( \psi \) preserves S-actions. For arbitrary indexed set \( K \), we have
\[
\psi \left( \bigvee_{k \in K} (x_k)_{i \in I} \right) = \psi \left( \bigvee_{i \in I} f_i \left( \bigvee_{k \in K} x_k \right) \right) = \bigvee_{i \in I} f_i \left( \bigvee_{k \in K} x_k \right) = \bigvee_{k \in K} \psi((x_k)_{i \in I}).
\]

Moreover, by Lemma 1.2, \( f_i(\bot_{X_i}) = \bot_{Q_5} \), for each \( i \in I \). Hence
\[
\psi(\mu_j(x)) = \psi((\mu_i)_{i \in I}) = \bigvee_{i \in I} f_i(\mu_i(x)) = f_j(x),
\]
Clearly, Proposition 2.5.

Let \( \phi : \prod_{i \in I} X_i \to Q_S \) such that \( \phi \mu_i = f_i \), for every \( i \in I \). Then, for each \( (x_i)_{i \in I} \in \prod_{i \in I} X_i \), one gets that
\[
\phi((x_i)_{i \in I}) = \phi \left( \bigvee_{i \in I} \mu_i(x_i) \right) = \bigvee_{i \in I} \phi \mu_i(x_i) = \bigvee_{i \in I} f_i(x_i) = \psi((x_i)_{i \in I}),
\]
and hence \( \phi = \psi \) as needed.

**Equalizers, coequalizers, pullbacks, and pushouts**

**Proposition 2.3.** Let \( f, g : A_S \to B_S \) be morphisms of \( S \)-quantales. The equalizer of \( f \) and \( g \) is given by \( E = \{ a \in A_S \mid f(a) = g(a) \} \), with action and order inherited from \( A_S \).

**Proof.** Clearly, \( E \) is an \( S \)-poset, and a complete lattice. So it is an \( S \)-quantale by the fact that \( f \) and \( g \) preserve arbitrary joins. Let \( i : E \to A \) be the inclusion mapping. For any morphism \( e : E \to A \) with \( fe = ge \), since \( e(E) \subseteq E \), it follows that \( \overline{e} \), which is the codomain restriction of \( e \), is the unique morphism fulfilling \( i \overline{e} = e \).

By [12] Theorem 12.3, we immediately get that \( \text{Quant}_S \) is complete.

**Proposition 2.4.** The category \( \text{Quant}_S \) is complete.

Let \( \rho \) be a congruence on \( S \)-quantale \( A_S \). In a natural way, the quotient \( A_S/\rho \) constitutes an \( S \)-quantale equipped with the order defined by a \( \rho \)-chain, where the joins in \( A_S/\rho \) are
\[
\bigvee_{i \in I} a_i = \left[ \bigvee_{i \in I} a_i \right]_\rho,
\]
and the canonical mapping \( \pi : A_S \to (A_S/\rho) \) becomes an \( S \)-quantale morphism, provided that \( \rho = \ker \pi \) ([9]). For \( H \subseteq A_S \times A_S \), the corresponding \( S \)-quantale congruence generated by \( H \), will be denoted by \( \theta(H) \).

**Proposition 2.5.** Let \( f, g : A_S \to B_S \) be morphisms of \( S \)-quantales. The coequalizer of \( f \) and \( g \) is the quotient \( (B/\theta(H))_S \), where \( H = \{ (f(a), g(a)) \mid a \in A_S \} \).

**Proof.** Let \( f, g : A_S \to B_S \) be morphisms of \( S \)-quantales, \( H = \{ (f(a), g(a)) \mid a \in A_S \} \), \( \pi \) be the canonical mapping from \( B_S \) to \( (B/\theta(H))_S \). Clearly, \( \pi f = \pi g \). For any \( S \)-quantale morphism \( h : B_S \to C_S \) satisfying \( h f = h g \), we obtain that \( \ker \pi \subseteq \ker h \), since \( (f(a), g(a)) \in \ker h \), for \( a \in A_S \).

Now define a mapping \( \bar{h} : (B/\theta(H))_S \to C_S \) by
\[
\bar{h}([b]_{\theta(H)}) = h(b),
\]
for \( [b]_{\theta(H)} \in (B/\theta(H))_S \). Clearly \( \bar{h} \) is an \( S \)-act morphism and preserves arbitrary joins by (3). It is quite routine to check that \( \bar{h} \) is the unique morphism satisfying \( \bar{h} \pi = h \).

**Proposition 2.6.** Let \( f : A_S \to C_S \), \( g : B_S \to C_S \) be morphisms of \( S \)-quantales. The pullback of \( f \) and \( g \) is the \( S \)-subposet \( P = \{ (a, b) \in (A \times B)_S \mid f(a) = g(b) \} \) of \( (A \times B)_S \), together with the restricted projections of \( P_S \) into \( A_S \) and \( B_S \).

**Proof.** It is known that \( P_S \) is an \( S \)-quantale. For any \( S \)-quantale \( Q_S \) and an pair of morphisms \( f_1 : Q_S \to A_S \), \( f_2 : Q_S \to B_S \) with \( f_1 = g f_2 \), one has that \( (f_1(q), f_2(q)) \in P_S \), for any \( q \in Q_S \). Now define a mapping \( \varphi : Q_S \to P_S \) by
\[
\varphi(q) = (f_1(q), f_2(q)),
\]
for \( q \in Q_S \). One gets that
\[
\varphi(q)s = (f_1(q), f_2(q))s = (f_1(q)s, f_2(q)s) = (f_1(qs), f_2(qs)) = \varphi(qs),
\]
for each \( q \in Q_S \), \( s \in S \), and
\[
\varphi\left(\bigvee_{i\in I} q_i\right) = \left(f_1\left(\bigvee_{i\in I} q_i\right), f_2\left(\bigvee_{i\in I} q_i\right)\right) = \left(\bigvee_{i\in I} f_1(q_i), \bigvee_{i\in I} f_2(q_i)\right) = \bigvee_{i\in I} \varphi(q_i),
\]
for all \( q_i \in Q_S \), \( i \in I \). If \( \pi_A : P_S \rightarrow A_S \) and \( \pi_B : P_S \rightarrow B_S \) are the restricted projections, then \( f\pi_A = g\pi_B \). Straightforward checking shows that \( \varphi \) is the unique morphism satisfying \( \pi_A \varphi = f_1 \) and \( \pi_B \varphi = f_2 \).

**Proposition 2.7.** Let \( f : A_S \rightarrow B_1 \), \( g : A_S \rightarrow B_2 \) be morphisms of \( S \)-quantales. The pushout of \( f \) and \( g \) is \(((B_1 \times B_2)/\theta(H))_S\), together with \( \mu_1 \) and \( \mu_2 \), where \( \mu_i : B_i \rightarrow (B_1 \times B_2)_S \), \( i = 1, 2 \), are defined as in Proposition 2.2, \( \pi \) is the canonical mapping, \( H = \{(\mu_1 f(a), \mu_2 g(a)) | a \in A_S\} \).

**Proof.** Since \(((B_1 \times B_2)_S, (\mu_1, \mu_2))\) is the coproduct of \((B_1, B_2)\) by Proposition 2.2, the coequalizer of \( \mu_1 f \) and \( \mu_2 g \) is the quotient \(((B_1 \times B_2)/\theta(H))_S\), where \( H = \{(\mu_1 f(a), \mu_2 g(a)) | a \in A_S\} \), by Proposition 2.5. The result follows immediately by [12] Remark 11.31.

### 3 Monomorphisms

This section contributes to the presentation of several kinds of monomorphisms in the category \( \text{Quant}_S \). It is shown that different from the case of \( S \)-posets (see [13]), monomorphisms in \( \text{Quant}_S \) coincide with order-embeddings, which are precisely injective morphisms. It thus leads to the strengthening results that these classes of monomorphisms are also in accordance with those labeled regular and extremal in \( \text{Quant}_S \), which are exactly the category-theoretic embeddings when \( \text{Quant}_S \) is considered as a concrete category over Set, \( \text{Act}_S \), and \( \text{Pos}_S \), respectively.

**Proposition 3.1.** Let \( f : A_S \rightarrow B_S \) be a morphism of \( S \)-quantales. Then the following statements are equivalent:

1. \( f \) is a monomorphism;
2. \( f \) is injective;
3. \( f \) is an order-embedding.

**Proof.** It is enough to show the implications (1) \(\Rightarrow\) (2) and (1) \(\Rightarrow\) (3) hold.

Let \( f : A_S \rightarrow B_S \) be a morphism of \( S \)-quantales. Consider \( S \)-subquantale ker\( f \) of the product \( (A \times A)_S \), and the restricted projection mappings \( h_i : \ker f \rightarrow A \), \( i = 1, 2 \). For any \((x, y) \in \ker f\), equalities
\[
fh_1(x, y) = f(x) = f(y) = fh_2(x, y)
\]
imply that \( fh_1 = fh_2 \) and hence \( h_1 = h_2 \) by assumption. Therefore, \( x = h_1(x, y) = h_2(x, y) = y \), and hence \( f \) is injective as needed.

It remains to prove that \( f \) is an order-embedding whenever it is a monomorphism. Suppose that \( f(a_1) \leq f(a_2) \) for \( a_1, a_2 \in A_S \). Then
\[
f(a_2) = f(a_1) \lor f(a_2) = f(a_1 \lor a_2).
\]
According to the above result of \( f \) being injective, we soon obtain that \( a_1 \leq a_2 \), and thus \( f \) is an order-embedding.

**Lemma 3.2.** Each inclusion mapping in \( \text{Quant}_S \) is a regular monomorphism.

**Proof.** Suppose that \( A_S \) is an \( S \)-subquantale of \( B_S \). Let \(((B \times B)_S, (\mu_1, \mu_2))\) be the coproduct of \((B_S, B_S)\), described as in Proposition 2.2. Write
\[
R = \{(a, b) | a \in A_S\},
\]
where \( \bot \) is the bottom element of \( B_S \). Then the relation \( \rho \), which is defined by
\[
\rho = \{(x \lor a, y \lor b) | (x \lor a', y \lor b') | x, y, a, b, a', b' \in B_S, x \lor b = x' \lor b', y \lor a = y' \lor a'\}
\]
is the smallest congruence relation on $B \times B$ containing $R$. So $((B \times B)/\rho)_S$ becomes an $S$-quantale equipped with a suitable order defined by a $\rho$-chain, and the canonical mapping $\pi : (B \times B)_S \to ((B \times B)/\rho)_S$ given by $\pi(x, y) = [(x, y)]_\rho$, for each $(x, y) \in (B \times B)_S$, is a morphism.

Next we show that the inclusion mapping $\iota_A : A \to B$ is the equalizer of $\pi \mu_1 = \pi \mu_2$. Suppose that $h : E_S \to B_S$ is any monomorphism satisfying $\pi \mu_1 h = \pi \mu_2 h$. Then for any $e \in E_S$, the equalities

$$[(h(e), \bot)]_\rho = \pi(h(e), \bot) = \pi \mu_1 h(e) = \pi \mu_2 h(e) = \pi(\bot, h(e)) = [(\bot, h(e))]_\rho$$

indicate that $((h(e), \bot), (\bot, h(e))) \notin \rho$. According to the definition of $\rho$, we deduce that $(h(e), \bot) = (x \lor a, y \lor b)$ and $(\bot, h(e)) = (x' \lor a', y' \lor b')$ for some $x, y, x', y' \in B_S$, $a, a', b, b' \in A_S$. So $y = b = \bot, x = a' = \bot$, and correspondingly,

$$a = \bot \lor a = y \lor a = y' \lor a' = y' \lor \bot = y',$$

and

$$x = x \lor \bot = \bot \lor b' = b'.
$$

Therefore, we have $h(e) = x \lor a = b' \lor a \in A_S$, i.e., $h(E) \subseteq A_S$. As a consequence, $\tilde{h} = h : E \to A$ is the unique morphism satisfying $\iota_A \tilde{h} = h$.

**Theorem 3.3.** Let $f : A_S \to B_S$ be a morphism of $S$-quantales. Then the following assertions are equivalent:

1. $f$ is a regular monomorphism;
2. $f$ is an extremal monomorphism;
3. $f$ is a monomorphism;
4. $f$ is a $\text{Quant}_S$-embedding over $\text{Set}$;
5. $f$ is a $\text{Quant}_S$-embedding over $\text{Act}_S$;
6. $f$ is a $\text{Quant}_S$-embedding over $\text{Pos}_S$.

**Proof.** (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are general category-theoretic results.

(3) $\Rightarrow$ (4). Suppose that $f : A_S \to B_S$ is a monomorphism. Let $g : C_S \to A_S$ be a mapping with $fg : C_S \to B_S$ being an $S$-quantale morphism. Then $g$ preserves arbitrary joins by the fact that for $a_i \in C_S$, $i \in I$,

$$fg \left( \bigvee_{i \in I} a_i \right) = \bigvee_{i \in I} fg(a_i) = f \left( \bigvee_{i \in I} g(a_i) \right),$$

and $f$ being injective by Proposition 3.1. Similarly, we get that $g$ preserves $S$-actions. Thus $f$ is initial and then an $\text{Quant}_S$-embedding over $\text{Set}$.

(4) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) $\Rightarrow$ (6) are clear.

(6) $\Rightarrow$ (4). Let $f : A_S \to B_S$ be a $\text{Quant}_S$-embedding over $\text{Pos}_S$, $g : C_S \to A_S$ a mapping provided that $fg : C_S \to B_S$ is a morphism in $\text{Quant}_S$. We are going to show that $g$ is an $S$-poset morphism. This is the case since

$$fg(as) = fg(a)s = f(g(a)s),$$

for any $a \in A_S, s \in S$, and

$$fg(a_2) = fg(a_1 \lor a_2) = fg(a_1) \lor fg(a_2) = f(g(a_1) \lor g(a_2)),$$

for $a_1 \leq a_2$ in $A_S$. Note that the monomorphisms in $\text{Pos}_S$ are just the $S$-poset morphisms with injective underlying mappings, we immediately achieve that $g(as) = g(a)s$ and $g(a_1) \leq g(a_2)$. Therefore, $g$ is an $S$-poset morphism as required.

(3) $\Rightarrow$ (1). This follows by [12] Proposition 7.53 (2) and Lemma 3.2.

**4 Epimorphisms**

Dual to discussions on monomorphisms studied in Section 3, this section is intended to motivate our investigation on relationships between various type of epimorphisms in $\text{Quant}_S$. However, the characterization of
epimorphisms in Quant$_S$ is quite complicated. So we merely cite the result and the reader is suggested to find complete illustrations in [14].

**Proposition 4.1** ([14] Th. 4.2). Epimorphisms in Quant$_S$ are exactly onto morphisms.

**Theorem 4.2.** For a morphism $f : A_S \to B_S$ of S-quantales, the following statements are equivalent:

1. $f$ is a regular epimorphism;
2. $f$ is an extremal epimorphism;
3. $f$ is an epimorphism;
4. $f$ is a Quant$_S$-quotient morphism over Set;
5. $f$ is a Quant$_S$-quotient morphism over Acts$_S$;
6. $f$ is a Quant$_S$-quotient morphism over Pos$_S$.

**Proof.** (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are clear.

(3) $\Rightarrow$ (1) follows by [14] Corollary 14.

(3) $\Rightarrow$ (4). Let $g : B_S \to C_S$ be a mapping between S-quantales such that $gf$ is an S-quantale morphism. Let us verify that $g$ is an S-quantale morphism, as well. It is easy to see that $g$ is an S-poset morphism. Since $f$ is an epimorphism, it is onto by Proposition 4.1. Hence we may assume that for any $M \subseteq B_S$, $\forall M = f(a)$ for some $a \in A_S$. By the reason that $f$ preserves arbitrary joins, we have

$$f(a) = \bigvee M = \bigvee_{x \in f^{-1}(M)} f(x) = f\left(\bigvee_{x \in f^{-1}(M)} x\right).$$

Consequently,

$$g(\bigvee M) = gf(a) = g\left(\bigvee_{x \in f^{-1}(M)} x\right) = \bigvee_{m \in M} g(m).$$

(4) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) $\Rightarrow$ (6) are clear.

(6) $\Rightarrow$ (2). Let $f : A_S \to B_S$ be a Quant$_S$-quotient morphism over Pos$_S$. Suppose that $g : A_S \to C_S$ and $h : C_S \to B_S$ are S-quantale morphisms such that $f = hg$ and $h$ is a monomorphism. Then $h$ is injective by Proposition 3.1. Note that $f$ is a Pos$_S$-epimorphism by hypotheses, and hence is surjective. So $h$ is surjective, as well, and thus bijective. Now, considering the inverse mapping $h^{-1}$ with $g = h^{-1}f$, we remain to show that $h^{-1}$ is an S-poset morphism. In fact, $f$ bing onto indicates that $h^{-1}$ is action-preserving. Observe that

$$h\left(h^{-1}(b) \lor h^{-1}(b')\right) = hh^{-1}(b) \lor hh^{-1}(b') = b \lor b' = hh^{-1}(b'),$$

for any $b \leq b'$ in $B_S$. Thus $h^{-1}(b) \lor h^{-1}(b') = h^{-1}(b')$, which expresses that $h^{-1}$ is an S-poset morphism, and hereby an S-quantale morphism by assumption.

\[\square\]

## 5 Adjoint situations

The final part is devoted to observation on the adjoint situation between Pos and Quant$_S$. By a free S-quantale on a poset $P$ we mean an S-quantale $Q_S$ together with a monotone mapping $\psi : P \to Q_S$ with the universal property that given any S-quantale $A_S$ and a monotone mapping $f : P \to A_S$, there exists a unique S-quantale morphism $f : Q_S \to A_S$ such that $f$ can be factored through.

**Lemma 5.1** ([13] Th.10). For a given poset $P$ and a pomonoid $S$, the free S-poset on $P$ is given by $P \times S$, with componentwise order and the action $(x, s) t = (x, st)$, for every $x \in P$, $s, t \in S$.

Let $(P \times S)_S$ be the free S-poset presented in Lemma 5.1. Write

$$\Omega(P \times S) = \{ D \subseteq P \times S \mid D = D\downarrow \},$$
where $D_i$ is the down-set of $D$ for $D \subseteq P \times S$, more precisely,

$$D_i = \{ (p, s) \in P \times S \mid (p, s) \leq (p_1, s_1) \text{ for some } (p_1, s_1) \in D \}.$$ 

Note that $(p_\downarrow \times s_\downarrow) \downarrow = p_\downarrow \times s_\downarrow$ provides that $p_\downarrow \times s_\downarrow \in \Omega(P \times S)$ for every element $p \in P$, $s \in S$. Define an action $\ast$ on $\Omega(P \times S)$ by

$$D \ast t := \{ (p, s) \in P \times S \mid (p, s) \leq (p_\downarrow, s_\downarrow t) \text{ for some } (p_1, s_1) \in D \},$$

for $t \in S$. Then it is clear that $D \ast t = (Dt) \downarrow$. We claim that $(\Omega(P \times S), \ast, \leq)$ is the free object in Quant$_S$.

**Proposition 5.2.** Let $S$ be a pomonoid, $P$ be a poset. Then $(\Omega(P \times S), \ast, \leq)$ is an $S$-quantale.

**Proof.** Observe first that

$$(D \ast t_1) \ast t_2 = \{ (p, s) \in P \times S \mid (p, s) \leq (p_\downarrow, s_\downarrow t_2) \text{ for some } (p_1, s_1) \in D \ast t_1 \}
= \{ (p, s) \in P \times S \mid (p, s) \leq (p_1, s_1 t_2), \text{ for some } (p_1, s_1) \in D \}
= \{ (p, s) \in P \times S \mid (p, s) \leq (p_\downarrow, s_\downarrow t_2), \text{ for some } (p_1, s_1) \in D \}
= D \ast (t_1 t_2)$$

for any $t_1, t_2 \in S$, $D \in \Omega(P \times S)$, and $D \ast 1 = (D1) \downarrow = D$. This shows that $(\Omega(P \times S), \ast)$ is an $S$-act. Clearly, $D_1 \ast s \subseteq D_2 \ast t$, whenever $D_1 \subseteq D_2$ in $\Omega(P \times S)$, and $s \leq t$ in $S$. So $\Omega(P \times S)$ is an $S$-poset. It is straightforward to check that $(\bigcup_{i \in I} D_i) \ast t = \bigcup_{i \in I} (D_i \ast t)$ for every $D_i \in \Omega(P \times S)$, $i \in I$, $t \in S$. □

Lemma 5.3 comes true directly by the definition of $\Omega(P \times S)$.

**Lemma 5.3.** Let $S$ be a pomonoid, $P$ be a poset. Then $D = \bigcup_{(p, s) \in D} (p_\downarrow \times s_\downarrow)$ for every $D \in \Omega(P \times S)$.

**Lemma 5.4.** Let $S$ be a pomonoid, $P$ be a poset. Then $p_\downarrow \times t_\downarrow = (p_\downarrow \times 1_\downarrow) \ast t$ holds in $\Omega(P \times S)_S$ for every $p \in P$, $t \in S$.

**Proof.** It is clear that $(q, s) \in (p_\downarrow \times 1_\downarrow) \ast t$ for every $(q, s) \in p_\downarrow \times t_\downarrow$, since $(q, s) \leq (p, t)$. On the other hand, for any $(q, s) \in (p_\downarrow \times 1_\downarrow) \ast t$, $(q, s) \leq (p_\downarrow, s_\downarrow t) = (p_1, s_1) t$ for some $(p_1, s_1) \in p_\downarrow \times 1_\downarrow$, it follows that $(q, s) \leq (p_\downarrow, 1_\downarrow) t = (p, t)$. Hence $(q, s) \in p_\downarrow \times t_\downarrow$. □

**Theorem 5.5.** Let $S$ be a pomonoid, $P$ be a poset. Then the free $S$-quantale on $P$ is given by the $S$-quantale $\Omega(P \times S)_S$.

**Proof.** Define a mapping $\tau : P \rightarrow \Omega(P \times S)_S$ by $\tau(p) = p_\downarrow \times 1_\downarrow$ for every $p \in P$. Obviously, $\tau$ is order-preserving. Let $Q_S$ be an $S$-quantale, $f : P \rightarrow Q_S$ be any monotone mapping. Define a mapping $\bar{f} : \Omega(P \times S)_S \rightarrow Q_S$ by

$$\bar{f}(D) = \bigvee \{ f(p) s \mid (p, s) \in D \},$$

for every $D \in \Omega(P \times S)_S$. We claim that $\bar{f}$ is the unique $S$-quantale morphism with the property that $\bar{f} \tau = f$.

It is clear that $\bar{f}$ preserves $S$-actions. Take $D_i \in \Omega(P \times S)_S$, $i \in I$, then equalities

$$\bar{f} \left( \bigcup_{i \in I} D_i \right) \bigwedge \bigwedge_{i \in I} f(p) s = (p, s) \in \bigcup_{i \in I} D_i \bigwedge \bigvee_{i \in I} f(p) s = \bigvee_{i \in I} f(p) s \bigwedge (p, s) \in D_i \bigwedge \bigvee_{i \in I} f(p) s = \bigvee_{i \in I} f(D_i)$$

indicate that $\bar{f}$ preserves arbitrary joins. Evidently, for any $p \in P$,

$$\bar{f}(p) = \bar{f}(p \downarrow \times 1_\downarrow) = \bigvee \{ f(q) s \mid (q, s) \in p_\downarrow \times 1_\downarrow \} \leq f(p),$$

while the fact that $f(p)$ being one of the terms in the sup that defines $\bar{f}(p)$ guarantees the opposite implication. Suppose that $\bar{f} : \Omega(P \times S)_S \rightarrow Q_S$ is an $S$-quantale morphism such that $\bar{f} \tau = f$. Then by Lemma 5.3 and Lemma 5.4, we achieve that

$$\bar{f}(D) = f \left( \bigcup_{(p, s) \in D} (p_\downarrow \times s_\downarrow) \right) = \bigvee_{(p, s) \in D} \bar{f}(p_\downarrow \times s_\downarrow) = \bigvee_{(p, s) \in D} \bar{f}(p_\downarrow \times s_\downarrow) = f(p_\downarrow \times 1_\downarrow) \ast s$$

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There is a free functor to be left adjoint to the forgetful functor.

Corollary 5.6. The category $\text{Quant}_S$ has a separator.

Proof. Let $f, g : A_S \to B_S$ be a pair of morphisms in $\text{Quant}_S$ with $f \neq g$. Then there exists $a \in A_S$ such that $f(a) \neq g(a)$. Let $P$ be a poset. Define a mapping $k : P \to A_S$ by $k(p) = a$, $\forall p \in P$. We are aware that $k$ is a morphism in $\text{Pos}$. Hence there is a unique $S$-quantale morphism $\bar{k} : \mathcal{Q}(P \times S)_S \to A_S$ with $\bar{k} \tau = k$, where $\tau : P \to \mathcal{Q}(P \times S)_S$ is defined as in Theorem 5.5. This yields that $f\bar{k} \neq g\bar{k}$, and consequently gives that $\mathcal{Q}(P \times S)_S$ is a separator.

We thereby obtain a free functor from the category of posets into the category of $S$-quantales, which is shown to be left adjoint to the forgetful functor.

Proposition 5.7. There is a free functor $F : \text{Pos} \to \text{Quant}_S$ given by

\[
\begin{array}{ccc}
P & \xrightarrow{\eta P} & FP \\
f \downarrow & & \downarrow Ff \\
Q & \xrightarrow{\eta Q} & FQ,
\end{array}
\]

where $FP = \mathcal{Q}(P \times S)_S$, and $Ff(D) = \{(x, y) \in Q \times S \mid (x, y) \leq (f(p), s) \text{ for some } (p, s) \in D\}$, for any monotone mapping $f : P \to Q$ and $D \in FP$.

Theorem 5.8. The free functor $F : \text{Pos} \to \text{Quant}_S$ is left adjoint to the forgetful functor $[\quad] : \text{Quant}_S \to \text{Pos}$.

Proof. Let us prove that $\eta : \text{id}_{\text{Pos}} \to [\quad]F$ with $\eta P : P \to [\mathcal{Q}(P \times S)_S]$, where $P$ is a $\text{Pos}$-object, $\eta P(p) = p \downarrow \times 1\downarrow$, $\forall p \in P$, is a natural transformation. Suppose that $f : P \to P'$ is a morphism in $\text{Pos}$. Then

\[
Ff \circ \eta P(p) = Ff(p \downarrow \times 1\downarrow) = \{(x, y) \in P' \times S \mid (x, y) \leq (f(\tilde{p}), s) \text{ for some } (\tilde{p}, s) \in p \downarrow \times 1\downarrow\}
\]

\[
= \{(x, y) \in P' \times S \mid (x, y) \leq (f(\tilde{p}), s) \leq (f(p), 1), (\tilde{p}, s) \in p \downarrow \times 1\downarrow\},
\]

for $p \in P$, and

\[
(\eta P' \circ f)(p) = \eta P'(f(p)) = f(p) \downarrow \times 1\downarrow.
\]

It results in $Ff \circ \eta P = \eta P' \circ f$ as needed. Now, by Theorem 5.5 and [12] 19.4(2), we obtain that $F$ is left adjoint to $[\quad]$.

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