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On categorical aspects of S-quantales

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Abstract: S-quantales are characterized as injective objects in the category of S-posets with respect to certain class of homomorphisms that are order-preserving mappings. This paper is devoted to exhibitions of categorical structures on S-quantales.

Keywords: Pomonoid, S-poset, Complete lattice, S-quantale, Adjoint situation

MSC: 06F05, 20M30, 20M50

Dedicated to Professor Ulrich Knauer on his 75th Birthday.

1 Introduction

The term quantale was suggested by C.J. Mulvey at the Oberwolfach Category Meeting ([1]) as a "quantization" of the term locale ([2]). An important moment in the development of the theory of quantales was the realization that quantales give a semantics for propositional linear logic in the same way as Boolean algebras give a semantics for classical propositional logic ([3, 4]). Quantales arise naturally as lattices of ideals, subgroups, or other suitable substructures of algebras ([5, 6]).

Algebraic investigations on qutale-like structures, such as quantales, quantale modules, sup-algebras, S-quantales, etc. have been studied in [5], [7], [8], and [9], respectively. Some categorical considerations are also taken into account ([10], [6]). S-quantales were firstly introduced by Zhang and Laan in [11], which have been shown to play an important role in the theory of injectivity on the category of S-posets. The current paper is devoted to the study of categorical properties of S-quantales.

In this work, S is always a *pomonoid*, that is, a monoid S equipped with a partial order $\leqslant$ such that $ss' \leqslant tt'$ whenever $s \leqslant t$, $s' \leqslant t'$ in S. A poset $(A, \leqslant)$ together with a mapping $A \times S \to A$ (under which a pair (a, s) maps to an element of A denoted by as) is called an *S-poset*, denoted by A_S, if for any $a, b \in A_S,\ s, t \in S$,

1. $a(st) = (as)t$,
2. $a1 = a$,
3. $a \leqslant b,\ s \leqslant t$ imply that $as \leqslant bt$.

S-poset morphisms are order-preserving mappings which also preserve the S-action. We denote the category of S-posets with S-poset morphisms by Pos_S. An *S-subposet* of an S-poset A_S is an action-closed subset of A_S whose partial order is the restriction of the order from A_S.

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Clearly, S-posets are generalizations of S-acts, whose relying category is denoted by Act_S.

Recall that an S-poset A_S is an S-*quantale* ([11]) if

(1) the poset A is a complete lattice;

(2) $(\bigvee M)s = \bigvee\{ms \mid m \in M\}$ for each subset M of A and each $s \in S$.

An *S-quantale morphism* is a mapping between S-quantales which preserves both S-actions and arbitrary joins. An *S-subquantale* of an S-quantale A_S is exactly the relative S-subposet of A_S which is closed under arbitrary joins.

We denote the category of S-quantales with S-quantale morphisms by Quant_S. This work is devoted to the presentation of categorical aspects in Quant_S. We explore limits and colimits, monomorphisms and epimorphisms, respectively, and exhibit adjoint situations accordingly.

Lemma 1.1. *The bottom of an S-quantale is a zero element.*

Proof. The result follows by the fact that for the bottom $\bot_{A_S}$ of an S-quantale A_S, and any $s \in S$,

$$\bot_{A_S} s = (\vee\varnothing)s = \vee_{a\in\varnothing}(as) = \vee\varnothing = \bot_{A_S}.$$ □

Since an S-quantale morphism $f : A_S \to B_S$ preserves all joins, it follows by the adjoint functor theorem that it has a right adjoint $f_* : B_S \to A_S$, satisfying

$$f(a) \leqslant b \Longleftrightarrow a \leqslant f_*(b), \tag{1}$$

for all $a \in A_S$, $b \in B_S$.

Lemma 1.2. *Let $f : A_S \to B_S$ be an S-quantale morphism. Then f preserves the bottom.*

Proof. Denote by $\bot_{A_S}$ the bottom of A_S. Then $\bot_{A_S} \leqslant f_*(b)$, for every $b \in B_S$. By (1), we have $f(\bot_{A_S}) \leqslant b$. □

2 Limits and colimits in Quant_S

Products and coproducts

Proposition 2.1. *The product of a family of S-quantales is their cartesian product with componentwise action, and order.*

Proposition 2.2. *The coproduct of a family of S-quantales $\{X_i\}_{i\in I}$ is $(\prod_{i\in I} X_i, (\mu_j)_{j\in I})$, where $\mu_j : X_j \to \prod_{i\in I} X_i$, $j \in I$, is defined by*

$$\mu_j(x) = (\widetilde{x_i})_{i\in I}, \text{ where } \widetilde{x_i} = \begin{cases} x & i = j, \\ \bot_{X_i} & i \neq j. \end{cases} \tag{2}$$

Proof. Clearly, μ_j is an S-quantale morphism for every $j \in I$. Let $f_j : X_j \to Q_S$, $j \in I$, be S-quantale morphisms. Define a mapping $\psi : \prod_{i\in I} X_i \to Q_S$ by

$$\psi((x_i)_{i\in I}) = \bigvee_{i\in I} f_i(x_i),$$

for any $(x_i)_{i\in I} \in \prod_{i\in I} X_i$. It is easy to see that ψ preserves S-actions. For arbitrary indexed set K, we have

$$\psi\left(\bigvee_{k\in K}(x_{i_k})_{i\in I}\right) = \psi\left(\left(\bigvee_{k\in K} x_{i_k}\right)_{i\in I}\right) = \bigvee_{i\in I} f_i\left(\bigvee_{k\in K} x_{i_k}\right) = \bigvee_{k\in K}\psi((x_{i_k})_{i\in I}).$$

Moreover, by Lemma 1.2, $f_i(\bot_{X_i}) = \bot_{Q_S}$, for each $i \in I$. Hence

$$\psi(\mu_j(x)) = \psi((\widetilde{x_i})_{i\in I}) = \bigvee_{i\in I} f_i(\widetilde{x_i}) = f_j(x),$$

for any $j \in I$, $x \in X_j$.

Finally, suppose that there exists an S-quantale morphism $\phi : \prod_{i\in I} X_i \to Q_S$ such that $\phi\mu_i = f_i$, for every $i \in I$. Then, for each $(x_i)_{i\in I} \in \prod_{i\in I} X_i$, one gets that

$$\phi((x_i)_{i\in I}) = \phi\left(\bigvee_{i\in I} \mu_i(x_i)\right) = \bigvee_{i\in I} \phi\mu_i(x_i) = \bigvee_{i\in I} f_i(x_i) = \psi((x_i)_{i\in I}),$$

and hence $\phi = \psi$ as needed. □

Equalizers, coequalizers, pullbacks, and pushouts

Proposition 2.3. *Let $f, g : A_S \to B_S$ be morphisms of S-quantales. The equalizer of f and g is given by $E = \{a \in A_S \mid f(a) = g(a)\}$, with action and order inherited from A_S.*

Proof. Clearly, E is an S-poset, and a complete lattice. So it is an S-quantale by the fact that f and g preserve arbitrary joins. Let $\iota : E \hookrightarrow A$ be the inclusion mapping. For any morphism $e : E^{'} \to A$ with $fe = ge$, since $e(E^{'}) \subseteq E$, it follows that $\overline{e}$, which is the codomain restriction of e, is the unique morphism fulfilling $\iota\overline{e} = e$. □

By [12] Theorem 12.3, we immediately get that Quant_S is complete.

Proposition 2.4. *The category* Quant_S *is complete.*

Let ρ be a congruence on S-quantale A_S. In a natural way, the quotient A/ρ constitutes an S-quantale equipped with the order defined by a ρ-chain, where the joins in A/ρ are

$$\bigvee_{i\in I} [a_i]_\rho = \left[\bigvee_{i\in I} a_i\right]_\rho, \tag{3}$$

and the canonical mapping $\pi : A_S \to (A/\rho)_S$ becomes an S-quantale morphism, provided that $\rho = \mathsf{ker}\pi$ ([9]). For $H \subseteq A_S \times A_S$, the corresponding S-quantale congruence generated by H, will be denoted by $\theta(H)$.

Proposition 2.5. *Let $f, g : A_S \to B_S$ be morphisms of S-quantales. The coequalizer of f and g is the quotient $(B/\theta(H))_S$, where $H = \{(f(a), g(a)) \mid a \in A_S\}$.*

Proof. Let $f, g : A_S \to B_S$ be morphisms of S-quantales, $H = \{(f(a), g(a)) \mid a \in A_S\}$, π be the canonical mapping from B_S to $(B/\theta(H))_S$. Clearly, $\pi f = \pi g$. For any S-quantale morphism $h : B_S \to C_S$ satisfying $hf = hg$, we obtain that $\mathsf{ker}\pi \subseteq \mathsf{ker}h$, since $(f(a), g(a)) \in \mathsf{ker}h$, for $a \in A_S$.

Now define a mapping $\bar{h} : (B/\theta(H))_S \to C_S$ by

$$\bar{h}\left([b]_{\theta(H)}\right) = h(b),$$

for $[b]_{\theta(H)} \in (B/\theta(H))_S$. Clearly $\bar{h}$ is an S-act morphism and preserves arbitrary joins by (3). It is quite routine to check that $\bar{h}$ is the unique morphism satisfying $\bar{h}\pi = h$. □

Proposition 2.6. *Let $f : A_S \to C_S$, $g : B_S \to C_S$ be morphisms of S-quantales. The pullback of f and g is the S-subposet $P = \{(a, b) \in (A \times B)_S \mid f(a) = g(b)\}$ of $(A \times B)_S$, together with the restricted projections of P_S into A_S and B_S.*

Proof. It is known that P_S is an S-quantale. For any S-quantale Q_S and an pair of morphisms $f_1 : Q_S \to A_S$, $f_2 : Q_S \to B_S$ with $ff_1 = gf_2$, one has that $(f_1(q), f_2(q)) \in P_S$, for any $q \in Q_S$. Now define a mapping $\varphi : Q_S \to P_S$ by

$$\varphi(q) = (f_1(q), f_2(q)),$$

for $q \in Q_S$. One gets that

$$\varphi(q)s = (f_1(q), f_2(q))s = (f_1(q)s, f_2(q)s) = (f_1(qs), f_2(qs)) = \varphi(qs),$$

for each $q \in Q_S,\ s \in S$, and

$$\varphi\left(\bigvee_{i\in I} q_i\right) = \left(f_1\left(\bigvee_{i\in I} q_i\right), f_2\left(\bigvee_{i\in I} q_i\right)\right) = \left(\bigvee_{i\in I} f_1(q_i), \bigvee_{i\in I} f_2(q_i)\right) = \bigvee_{i\in I}((f_1(q_i), f_2(q_i)) = \bigvee_{i\in I}\varphi(q_i),$$

for all $q_i \in Q_S,\ i \in I$. If $\pi_A : P_S \to A_S$ and $\pi_B : P_S \to B_S$ are the restricted projections, then $f\pi_A = g\pi_B$. Straightforward checking shows that φ is the unique morphism satisfying $\pi_A\varphi = f_1$ and $\pi_B\varphi = f_2$. □

Proposition 2.7. *Let $f : A_S \to B_1,\ g : A_S \to B_2$ be morphisms of S-quantales. The pushout of f and g is $((B_1 \times B_2)/\theta(H))_S$, together with $\pi\mu_1$ and $\pi\mu_2$, where $\mu_i : B_i \to (B_1 \times B_2)_S,\ i = 1, 2$, are defined as in Proposition 2.2, π is the canonical mapping, $H = \{(\mu_1 f(a),\ \mu_2 g(a)) \mid a \in A_S\}$.*

Proof. Since $((B_1 \times B_2)_S, (\mu_1,\ \mu_2))$ is the coproduct of $(B_1,\ B_2)$ by Proposition 2.2, the coequalizer of $\mu_1 f$ and $\mu_2 g$ is the quotient $((B_1 \times B_2)/\theta(H))_S$, where $H = \{(\mu_1 f(a),\ \mu_2 g(a)) \mid a \in A_S\}$, by Proposition 2.5. The result follows immediately by [12] Remark 11.31. □

3 Monomorphisms

This section contributes to the presentation of several kinds of monomorphisms in the category Quant_S. It is shown that deferent from the case of S-posets (see [13]), monomorphisms in Quant_S coincide with order-embeddings, which are precisely injective morphisms. It thus leads to the strengthening results that these classes of monomorphisms are also in accordance with those labeled regular and extremal in Quant_S, which are exactly the category-theoretic embeddings when Quant_S is considered as a concrete category over Set, Act_S, and Pos_S, respectively.

Proposition 3.1. *Let $f : A_S \to B_S$ be a morphism of S-quantales. Then the following statements are equivalent:*
(1) *f is a monomorphism;*
(2) *f is injective;*
(3) *f is an order-embedding.*

Proof. It is enough to show the implications (1)⇒ (2) and (1)⇒ (3) hold.

Let $f : A_S \to B_S$ be a monomorphism of S-quantales. Consider S-subquantale $\mathrm{ker}f$ of the product $(A \times A)_S$, and the restricted projection mappings $h_i : \mathrm{ker}f \to A,\ i = 1, 2$. For any $(x, y) \in \mathrm{ker}f$, equalities

$$fh_1(x, y) = f(x) = f(y) = fh_2(x, y)$$

imply that $fh_1 = fh_2$ and hence $h_1 = h_2$ by assumption. Therefore, $x = h_1(x, y) = h_2(x, y) = y$, and hence f is injective as needed.

It remains to prove that f is an order-embedding whenever it is a monomorphism. Suppose that $f(a_1) \leqslant f(a_2)$ for $a_1, a_2 \in A_S$. Then

$$f(a_2) = f(a_1) \bigvee f(a_2) = f\left(a_1 \bigvee a_2\right).$$

According to the above result of f being injective, we soon obtain that $a_1 \leqslant a_2$, and thus f is an order-embedding. □

Lemma 3.2. *Each inclusion mapping in Quant_S is a regular monomorphism.*

Proof. Suppose that A_S is an S-subquantale of B_S. Let $((B \times B)_S, (\mu_1, \mu_2))$ be the coproduct of (B_S, B_S), described as in Proposition 2.2. Write

$$R = \{((a, \bot),\ (\bot, a)) \mid a \in A_S\},$$

where $\bot$ is the bottom element of B_S. Then the relation ρ, which is defined by

$$\rho = \left\{\left((x \vee a, y \vee b), (x' \vee a', y' \vee b')\right) \middle|\ x, y, x', y' \in B_S,\ a, b, a', b' \in A_S,\ x \vee b = x' \vee b',\ y \vee a = y' \vee a'\right\}$$

is the smallest congruence relation on $B \times B$ containing R. So $((B \times B)/\rho)_S$ becomes an S-quantale equipped with a suitable order defined by a ρ-chain, and the canonical mapping $\pi : (B \times B)_S \to ((B \times B)/\rho)_S$ given by $\pi(x, y) = [(x, y)]_\rho$, for each $(x, y) \in (B \times B)_S$, is a morphism.

Next we show that the inclusion mapping $\iota_A : A \hookrightarrow B$ is the equalizer of $\pi\mu_1 = \pi\mu_2$. Suppose that $h : E_S \to B_S$ is any monomorphism satisfying $\pi\mu_1 h = \pi\mu_2 h$. Then for any $e \in E_S$, the equalities

$$[(h(e), \perp)]_\rho = \pi(h(e), \perp) = \pi\mu_1 h(e) = \pi\mu_2 h(e) = \pi(\perp, h(e)) = [(\perp, h(e))]_\rho$$

indicate that $((h(e), \perp), (\perp, h(e))) \in \rho$. According to the definition of ρ, we deduce that $(h(e), \perp) = (x \vee a, y \vee b)$ and $(\perp, h(e)) = (x^{'} \vee a^{'}, y^{'} \vee b^{'})$ for some $x, y, x^{'}, y^{'} \in B_S,\ a, a^{'}, b, b^{'} \in A_S$. So $y = b = \perp$, $x^{'} = a^{'} = \perp$, and correspondingly,

$$a = \perp \vee a = y \vee a = y^{'} \vee a^{'} = y^{'} \vee \perp = y^{'},$$

and

$$x = x \vee \perp = \perp \vee b^{'} = b^{'}.$$

Therefore, we have $h(e) = x \vee a = b' \vee a \in A_S$, i.e., $h(E) \subseteq A_S$. As a consequence, $\widetilde{h} = h : E \to A$ is the unique morphism satisfying $\iota_A \widetilde{h} = h$. □

Theorem 3.3. *Let $f : A_S \to B_S$ be a morphism of S-quantales. Then the following assertions are equivalent:*
(1) *f is a regular monomorphism;*
(2) *f is an extremal monomorphism;*
(3) *f is a monomorphism;*
(4) *f is a* Quant_S*-embedding over* Set*;*
(5) *f is a* Quant_S*-embedding over* Act_S*;*
(6) *f is a* Quant_S*-embedding over* Pos_S.

Proof. (1) ⇒ (2) ⇒ (3) are general category-theoretic results.

(3) ⇒ (4). Suppose that $f : A_S \to B_S$ is a monomorphism. Let $g : C_S \to A_S$ be a mapping with $fg : C_S \to B_S$ being an S-quantale morphism. Then g preserves arbitrary joins by the fact that for $a_i \in C_S,\ i \in I$,

$$fg\left(\bigvee_{i \in I} a_i\right) = \bigvee_{i \in I} fg(a_i) = f\left(\bigvee_{i \in I} g(a_i)\right),$$

and f being injective by Proposition 3.1. Similarly, we get that g preserves S-actions. Thus f is initial and then an S-quantale embedding over Set.

(4) ⇒ (3), (4) ⇒ (5) ⇒ (6) are clear.

(6) ⇒ (4). Let $f : A_S \to B_S$ be a Quant_S-embedding over Pos_S, $g : C_S \to A_S$ a mapping provided that $fg : C_S \to B_S$ is a morphism in Quant_S. We are going to show that g is an S-poset morphism. This is the case since

$$fg(as) = fg(a)s = f(g(a)s),$$

for any $a \in A_S, s \in S$, and

$$fg(a_2) = fg(a_1 \bigvee a_2) = fg(a_1) \bigvee fg(a_2) = f(g(a_1) \bigvee g(a_2)),$$

for $a_1 \leqslant a_2$ in A_S. Note that the monomorphisms in Pos_S are just the S-poset morphisms with injective underlying mappings, we immediately achieve that $g(as) = g(a)s$ and $g(a_1) \leqslant g(a_2)$. Therefore, g is an S-poset morphism as required.

(3) ⇒ (1). This follows by [12] Proposition 7.53 (2) and Lemma 3.2. □

4 Epimorphisms

Dual to discussions on monomorphisms studied in Section 3, this section is intended to motivate our investigation on relationships between various type of epimorphisms in Quant_S. However, the characterization of epimorphisms in Quant_S is quite complicated. So we merely cite the result and the reader is suggested to find complete illustrations in [14].

Proposition 4.1 ([14] Th. 4.2). *Epimorphisms in* Quant_S *are exactly onto morphisms.*

Theorem 4.2. *For a morphism $f : A_S \to B_S$ of S-quantales, the following statements are equivalent:*
(1) *f is a regular epimorphism;*
(2) *f is an extremal epimorphism;*
(3) *f is an epimorphism;*
(4) *f is a* Quant_S*-quotient morphism over* Set;
(5) *f is a* Quant_S*-quotient morphism over* Act_S;
(6) *f is a* Quant_S*-quotient morphism over* Pos_S.

Proof. (1) ⇒ (2) ⇒ (3) are clear.

(3) ⇒ (1) follows by [14] Corollary 14.

(3) ⇒ (4). Let $g : B_S \to C_S$ be a mapping between S-quantales such that gf is an S-quantale morphism. Let us verify that g is an S-quantale morphism, as well. It is easy to see that g is an S-poset morphism. Since f is an epimorphism, it is onto by Proposition 4.1. Hence we may assume that for any $M \subseteq B_S$, $\bigvee M = f(a)$ for some $a \in A_S$. By the reason that f preserves arbitrary joins, we have

$$f(a) = \bigvee M = \bigvee_{x \in f^{-1}(M)} f(x) = f\left(\bigvee_{x \in f^{-1}(M)} x\right).$$

Consequently,

$$g\left(\bigvee M\right) = gf(a) = gf\left(\bigvee_{x \in f^{-1}(M)} x\right) = \bigvee_{x \in f^{-1}(M)} gf(x) = \bigvee_{m \in M} g(m).$$

(4) ⇒ (3), (4) ⇒ (5) ⇒ (6) are clear.

(6) ⇒ (2). Let $f : A_S \to B_S$ be a Quant_S-quotient morphism over Pos_S. Suppose that $g : A_S \to C_S$ and $h : C_S \to B_S$ are S-quantale morphisms such that $f = hg$ and h is a monomorphism. Then h is injective by Proposition 3.1. Note that f is a Pos_S-epimorphism by hypotheses, and hence is surjective. So h is surjective, as well, and thus bijective. Now, considering the inverse mapping h^{-1} with $g = h^{-1}f$, we remain to show that h^{-1} is an S-poset morphism. In fact, f bing onto indicates that h^{-1} is action-preserving. Observe that

$$h\left(h^{-1}(b) \bigvee h^{-1}(b')\right) = hh^{-1}(b) \bigvee hh^{-1}(b') = b \bigvee b' = hh^{-1}(b'),$$

for any $b \leqslant b'$ in B_S. Thus $h^{-1}(b) \vee h^{-1}(b') = h^{-1}(b')$, which expresses that h^{-1} is an S-poset morphism, and hereby an S-quantale morphism by assumption. □

5 Adjoint situations

The final part is devoted to observation on the adjoint situation between Pos and Quant_S. By a *free S-quantale on a poset P* we mean an S-quantale Q_S together with a monotone mapping $\psi : P \to Q_S$ with the universal property that given any S-quantale A_S and a monotone mapping $f : P \to A_S$, there exists a unique S-quantale morphism $\bar{f} : Q_S \to A_S$ such that f can be factored through.

Lemma 5.1 ([13] Th.10). *For a given poset P and a pomonoid S, the free S-poset on P is given by $P \times S$, with componentwise order and the action $(x, s)t = (x, st)$, for every $x \in P$, $s, t \in S$.*

Let $(P \times S)_S$ be the free S-poset presented in Lemma 5.1. Write

$$\mathcal{Q}(P \times S) = \{D \subseteq P \times S \mid D = D{\downarrow}\},$$

where $D\downarrow$ is the down-set of D for $D \subseteq P \times S$, more precisely,

$$D\downarrow = \{(p,s) \in P \times S \mid (p,s) \leqslant (p_1, s_1) \text{ for some } (p_1, s_1) \in D\}.$$

Note that $(p\downarrow \times s\downarrow)\downarrow = p\downarrow \times s\downarrow$ provides that $p\downarrow \times s\downarrow \in \mathcal{Q}(P \times S)$ for every element $p \in P,\ s \in S$. Define an action $*$ on $\mathcal{Q}(P \times S)$ by

$$D * t := \{(p,s) \in P \times S \mid (p,s) \leqslant (p_1, s_1 t) \text{ for some } (p_1, s_1) \in D\},$$

for $t \in S$. Then it is clear that $D * t = (Dt)\downarrow$. We claim that $(\mathcal{Q}(P \times S)_S, *, \subseteq)$ is the free object in $\mathbf{Quant}_S$.

Proposition 5.2. *Let S be a pomonoid, P be a poset. Then $(\mathcal{Q}(P \times S)_S, *, \subseteq)$ is an S-quantale.*

Proof. Observe first that

$$\begin{aligned}(D * t_1) * t_2 &= \{(p,s) \in P \times S \mid (p,s) \leqslant (p_1, s_1 t_2) \text{ for some } (p_1, s_1) \in D * t_1\}\\ &= \{(p,s) \in P \times S \mid (p,s) \leqslant (p_1, s_1 t_2),\ (p_1, s_1) \leqslant (p_2, s_2 t_1) \text{ for some } (p_2, s_2) \in D\}\\ &= \{(p,s) \in P \times S \mid (p,s) \leqslant (p_2, s_2 t_1 t_2), \text{ for some } (p_2, s_2) \in D\}\\ &= D * (t_1 t_2)\end{aligned}$$

for any $t_1, t_2 \in S$, $D \in \mathcal{Q}(P \times S)$, and $D * 1 = (D1)\downarrow = D$. This shows that $(\mathcal{Q}(P \times S), *)$ is an S-act. Clearly, $D_1 * s \subseteq D_2 * t$, whenever $D_1 \subseteq D_2$ in $\mathcal{Q}(P \times S)$, and $s \leqslant t$ in S. So $\mathcal{Q}(P \times S)$ is an S-poset. It is straightforward to check that $(\bigcup_{i \in I} D_i) * t = \bigcup_{i \in I}(D_i * t)$ for every $D_i \in \mathcal{Q}(P \times S)$, $i \in I,\ t \in S$. □

Lemma 5.3 comes true directly by the definition of $\mathcal{Q}(P \times S)$.

Lemma 5.3. *Let S be a pomonoid, P be a poset. Then $D = \bigcup_{(p,s) \in D} (p\downarrow \times s\downarrow)$ for every $D \in \mathcal{Q}(P \times S)$.*

Lemma 5.4. *Let S be a pomonoid, P be a poset. Then $p\downarrow \times t\downarrow = (p\downarrow \times 1\downarrow) * t$ holds in $\mathcal{Q}(P \times S)_S$ for every $p \in P,\ t \in S$.*

Proof. It is clear that $(q,s) \in (p\downarrow \times 1\downarrow) * t$ for every $(q,s) \in p\downarrow \times t\downarrow$, since $(q,s) \leqslant (p,t)$. On the other hand, for any $(q,s) \in (p\downarrow \times 1\downarrow) * t$, $(q,s) \leqslant (p_1, s_1 t) = (p_1, s_1)t$ for some $(p_1, s_1) \in p\downarrow \times 1\downarrow$, it follows that $(q,s) \leqslant (p,1)t = (p,t)$. Hence $(q,s) \in p\downarrow \times t\downarrow$. □

Theorem 5.5. *Let S be a pomonoid, P be a poset. Then the free S-quantale on P is given by the S-quantale $\mathcal{Q}(P \times S)_S$.*

Proof. Define a mapping $\tau : P \to \mathcal{Q}(P \times S)_S$ by $\tau(p) = p\downarrow \times 1\downarrow$ for every $p \in P$. Obviously, τ is order-preserving. Let Q_S be an S-quantale, $f : P \to Q_S$ be any monotone mapping. Define a mapping $\bar{f} : \mathcal{Q}(P \times S)_S \to Q_S$ by

$$\bar{f}(D) = \bigvee\{f(p)s \mid (p,s) \in D\},$$

for every $D \in \mathcal{Q}(P \times S)_S$. We claim that $\bar{f}$ is the unique S-quantale morphism with the property that $\bar{f}\tau = f$.

It is clear that $\bar{f}$ preserves S-actions. Take $D_i \in \mathcal{Q}(P \times S)_S$, $i \in I$, then equalities

$$\bar{f}\left(\bigcup_{i \in I} D_i\right) = \bigvee\left\{f(p)s \,\middle|\, (p,s) \in \bigcup_{i \in I} D_i\right\} = \bigvee_{i \in I}\left\{\bigvee\{f(p)s \,\middle|\, (p,s) \in D_i\}\right\} = \bigvee_{i \in I}\bar{f}(D_i)$$

indicate that $\bar{f}$ preserves arbitrary joins. Evidently, for any $p \in P$,

$$\bar{f}\tau(p) = \bar{f}(p\downarrow \times 1\downarrow) = \bigvee\{f(q)s \mid (q,s) \in p\downarrow \times 1\downarrow\} \leqslant f(p),$$

while the fact that $f(p)$ being one of the terms in the sup that defines $\bar{f}\tau(p)$ guarantees the opposite implication. Suppose that $f' : \mathcal{Q}(P \times S)_S \to Q_S$ is an S-quantale morphism such that $f'\tau = f$. Then by Lemma 5.3 and Lemma 5.4, we achieve that

$$f'(D) = f'\left(\bigcup_{(p,s) \in D}(p\downarrow \times s\downarrow)\right) = \bigvee_{(p,s) \in D} f'(p\downarrow \times s\downarrow) = \bigvee_{(p,s) \in D} f'((p\downarrow \times 1\downarrow) * s)$$

$$= \bigvee_{(p,s)\in D} f^{'}(p\downarrow \times 1\downarrow)s = \bigvee_{(p,s)\in D} f^{'}\tau(p)s = \bigvee_{(p,s)\in D} f(p)s = \bar{f}(D),$$

for every $D \in \mathfrak{Q}(P \times S)_S$, which finishes our proof. □

Corollary 5.6. *The category* Quant_S *has a separator.*

Proof. Let $f, g : A_S \to B_S$ be a pair of morphisms in Quant_S with $f \neq g$. Then there exists $a \in A_S$ such that $f(a) \neq g(a)$. Let P be a poset. Define a mapping $k : P \to A_S$ by $k(p) = a$, $\forall p \in P$. We are aware that k is a morphism in Pos. Hence there is a unique S-quantale morphism $\bar{k} : \mathfrak{Q}(P \times S)_S \to A_S$ with $\bar{k}\tau = k$, where $\tau : P \to \mathfrak{Q}(P\times S)_S$ is defined as in Theorem 5.5. This yields that $f\bar{k} \neq g\bar{k}$, and consequently gives that $\mathfrak{Q}(P\times S)_S$ is a separator. □

We thereby obtain a free functor from the category of posets into the category of S-quantales, which is shown to be left adjoint to the forgetful functor.

Proposition 5.7. *There is a free functor* $F : \mathsf{Pos} \to \mathsf{Quant}_S$ *given by*

$$\begin{array}{ccc} P & \longrightarrow & FP \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Ff} \\ Q & \longrightarrow & FQ, \end{array}$$

where $FP = \mathfrak{Q}(P \times S)_S$, *and*

$$Ff(D) = \{(x, y) \in Q \times S \mid (x, y) \leqslant (f(p), s) \text{ for some } (p, s) \in D\},$$

for any monotone mapping $f : P \to Q$ *and* $D \in FP$.

Theorem 5.8. *The free functor* $F : \mathsf{Pos} \to \mathsf{Quant}_S$ *is left adjoint to the forgetful functor* $\lfloor\ \rfloor : \mathsf{Quant}_S \to \mathsf{Pos}$.

Proof. Let us prove that $\eta : \mathrm{id}_{\mathsf{Pos}} \to \lfloor\ \rfloor F$ with $\eta_P : P \to \lfloor\mathfrak{Q}(P \times S)_S\rfloor$, where P is a Pos-object, $\eta_P(p) = p\downarrow \times 1\downarrow$, $\forall p \in P$, is a natural transformation. Suppose that $f : P \to P^{'}$ is a morphism in Pos. Then

$$\begin{aligned} Ff \circ \eta_P(p) &= Ff(p\downarrow \times 1\downarrow) = \{(x, y) \in P^{'} \times S \mid (x, y) \leqslant (f(\tilde{p}), s) \text{ for some } (\tilde{p}, s) \in p\downarrow \times 1\downarrow\} \\ &= \{(x, y) \in P^{'} \times S \mid (x, y) \leqslant (f(\tilde{p}), s) \leqslant (f(p), 1), (\tilde{p}, s) \in p\downarrow \times 1\downarrow\}, \end{aligned}$$

for $p \in P$, and

$$(\eta_{P'} \circ f)(p) = \eta_{P'}(f(p)) = f(p)\downarrow \times 1\downarrow.$$

It results in $Ff \circ \eta_P = \eta_{P'} \circ f$ as needed. Now, by Theorem 5.5 and [12] 19.4(2), we obtain that F is left adjoint to $\lfloor\ \rfloor$. □

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