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Xia Zhang* and Yunyan Zhou

On categorical aspects of $S$-quantales

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Abstract: $S$-quantales are characterized as injective objects in the category of $S$-posets with respect to a certain class of homomorphisms that are order-preserving mappings. This paper is devoted to exhibitions of categorical structures on $S$-quantales.

Keywords: Pomonoid, $S$-poset, Complete lattice, $S$-quantale, Adjoint situation

MSC: 06F05, 20M30, 20M50

Dedicated to Professor Ulrich Knauer on his 75th Birthday.

1 Introduction

The term quantale was suggested by C.J. Mulvey at the Oberwolfach Category Meeting ([1]) as a “quantization” of the term locale ([2]). An important moment in the development of the theory of quantales was the realization that quantales give a semantics for propositional linear logic in the same way as Boolean algebras give a semantics for classical propositional logic ([3, 4]). Quantales arise naturally as lattices of ideals, subgroups, or other suitable substructures of algebras ([5, 6]).

Algebraic investigations on quantale-like structures, such as quantales, quantale modules, sup-algebras, $S$-quantales, etc. have been studied in [5], [7], [8], and [9], respectively. Some categorical considerations are also taken into account ([10], [6]). $S$-quantales were firstly introduced by Zhang and Laan in [11], which have been shown to play an important role in the theory of injectivity on the category of $S$-posets. The current paper is devoted to the study of categorical properties of $S$-quantales.

In this work, $S$ is always a pomonoid, that is, a monoid $S$ equipped with a partial order $\leq$ such that $ss' \leq tt'$ whenever $s \leq t, s' \leq t'$ in $S$. A poset $(A, \leq)$ together with a mapping $A \times S \rightarrow A$ (under which a pair $(a, s)$ maps to an element of $A$ denoted by $as$) is called an $S$-poset, denoted by $A_S$, if for any $a, b \in A_S, s, t \in S$,

1. $a(st) = (as)t$,
2. $a1 = a$,
3. $a \leq b, s \leq t$ imply that $as \leq bt$.

$S$-poset morphisms are order-preserving mappings which also preserve the $S$-action. We denote the category of $S$-posets with $S$-poset morphisms by $\text{Pos}_S$. An $S$-subposet of an $S$-poset $A_S$ is an action-closed subset of $A_S$ whose partial order is the restriction of the order from $A_S$.

*Corresponding Author: Xia Zhang: School of Mathematical Sciences, South China Normal University, 510631 Guangzhou, China, E-mail: xzhang@m.scnu.edu.cn
Yunyan Zhou: School of Mathematical Sciences, South China Normal University, 510631 Guangzhou, China, E-mail: 1173842289@qq.com

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Clearly, $S$-posets are generalizations of $S$-acts, whose relying category is denoted by $\text{Act}_S$.

Recall that an $S$-poset $A_S$ is an $S$-quantale (11) if

1. the poset $A$ is a complete lattice;
2. $(\vee M)s = \vee \{ ms \mid m \in M \}$ for each subset $M$ of $A$ and each $s \in S$.

An $S$-quantale morphism is a mapping between $S$-quantales which preserves both $S$-actions and arbitrary joins. An $S$-subquantale of an $S$-quantale $A_S$ is exactly the relative $S$-subposet of $A_S$ which is closed under arbitrary joins.

We denote the category of $S$-quantales with $S$-quantale morphisms by $\text{Quant}_S$. This work is devoted to the presentation of categorical aspects in $\text{Quant}_S$. We explore limits and colimits, monomorphisms and epimorphisms, respectively, and exhibit adjoint situations accordingly.

**Lemma 1.1.** The bottom of an $S$-quantale is a zero element.

**Proof.** The result follows by the fact that for the bottom $\bot A_S$ of an $S$-quantale $A_S$, and any $s \in S$,

$$\bot A_Ss = (\vee \emptyset)s = \vee \emptyset s = \bot = \bot A_S.$$  

Since an $S$-quantale morphism $f : A_S \to B_S$ preserves all joins, it follows by the adjoint functor theorem that it has a right adjoint $f^* : B_S \to A_S$, satisfying

$$f(a) \leq b \iff a \leq f^*(b),$$  

for all $a \in A_S$, $b \in B_S$.

**Lemma 1.2.** Let $f : A_S \to B_S$ be an $S$-quantale morphism. Then $f$ preserves the bottom.

**Proof.** Denote by $\bot A_S$ the bottom of $A_S$. Then $\bot A_S \leq f^*(b)$, for every $b \in B_S$. By (1), we have $f(\bot A_S) \leq b$.  

## 2 Limits and colimits in $\text{Quant}_S$

### Products and coproducts

**Proposition 2.1.** The product of a family of $S$-quantales is their cartesian product with componentwise action, and order.

**Proposition 2.2.** The coproduct of a family of $S$-quantales $\{X_i\}_{i \in I}$ is $(\prod_{i \in I} X_i, (\mu_j)_{j \in I})$, where $\mu_j : X_j \to \prod_{i \in I} X_i$, $j \in I$, is defined by

$$\mu_j(x) = (\overline{x}_i)_{i \in I}, \text{ where } \overline{x}_i = \begin{cases} x & i = j, \\ \bot_{X_i} & i \neq j. \end{cases}$$  

**Proof.** Clearly, $\mu_j$ is an $S$-quantale morphism for every $j \in I$. Let $f_j : X_j \to Q_S$, $j \in I$, be $S$-quantale morphisms. Define a mapping $\psi : \prod_{i \in I} X_i \to Q_S$ by

$$\psi((x_i)_{i \in I}) = \bigvee_{i \in I} f_i(x_i),$$

for any $(x_i)_{i \in I} \in \prod_{i \in I} X_i$. It is easy to see that $\psi$ preserves $S$-actions. For arbitrary indexed set $K$, we have

$$\psi\left(\bigvee_{k \in K} (x_i)_{i \in I}\right) = \psi\left(\bigvee_{k \in K} (x_i)\right)_{i \in I} = \bigvee_{i \in I} f_i\left(\bigvee_{k \in K} x_i\right) = \bigvee_{k \in K} \psi((x_i)_{i \in I}).$$

Moreover, by Lemma 1.2, $f_i(\bot_{X_i}) = \bot_{Q_S}$, for each $i \in I$. Hence

$$\psi(\mu_j(x)) = \psi((\overline{x}_i)_{i \in I}) = \bigvee_{i \in I} f_i(\overline{x}_i) = f_j(x).$$
for any \( j \in I, \ x \in X_j \).

Finally, suppose that there exists an \( S \)-quantale morphism \( \phi : \prod_{i \in I} X_i \rightarrow Q_S \) such that \( \phi \mu_i = f_i \), for every \( i \in I \). Then, for each \( (x_i)_{i \in I} \in \prod_{i \in I} X_i \), one gets that

\[
\phi((x_i)_{i \in I}) = \phi \left( \bigvee_{i \in I} \mu_i(x_i) \right) = \bigvee_{i \in I} \phi \mu_i(x_i) = \bigvee_{i \in I} f_i(x_i) = \psi((x_i)_{i \in I}),
\]

and hence \( \phi = \psi \) as needed.

\[ \Box \]

**Equalizers, coequalizers, pullbacks, and pushouts**

**Proposition 2.3.** Let \( f, g : A_S \rightarrow B_S \) be morphisms of \( S \)-quantales. The equalizer of \( f \) and \( g \) is given by \( E = \{ a \in A_S \mid f(a) = g(a) \} \), with action and order inherited from \( A_S \).

**Proof.** Clearly, \( E \) is an \( S \)-poset, and a complete lattice. So it is an \( S \)-quantale by the fact that \( f \) and \( g \) preserve arbitrary joins. Let \( \iota : E \rightarrow A \) be the inclusion mapping. For any morphism \( e : E \rightarrow A \) with \( fe = ge \), since \( e(E) \subseteq E \), it follows that \( \overline{e} \), which is the codomain restriction of \( e \), is the unique morphism fulfilling \( \iota \overline{e} = e \).

By [12] Theorem 12.3, we immediately get that \( \text{Quant}_S \) is complete.

**Proposition 2.4.** The category \( \text{Quant}_S \) is complete.

Let \( \rho \) be a congruence on \( S \)-quantale \( A_S \). In a natural way, the quotient \( A/\rho \) constitutes an \( S \)-quantale equipped with the order defined by a \( \rho \)-chain, where the joins in \( A/\rho \) are

\[
\bigvee_{i \in I} [a_i]_\rho = \left[ \bigvee_{i \in I} a_i \right]_\rho,
\]

and the canonical mapping \( \pi : A_S \rightarrow (A/\rho)_S \) becomes an \( S \)-quantale morphism, provided that \( \rho = \ker \pi \) ([9]).

For \( H \subseteq A_S \times A_S \), the corresponding \( S \)-quantale congruence generated by \( H \), will be denoted by \( \theta(H) \).

**Proposition 2.5.** Let \( f, g : A_S \rightarrow B_S \) be morphisms of \( S \)-quantales. The coequalizer of \( f \) and \( g \) is the quotient \( \left( B/\theta(H) \right)_S \), where \( H = \{ (f(a), g(a)) \mid a \in A_S \} \).

**Proof.** Let \( f, g : A_S \rightarrow B_S \) be morphisms of \( S \)-quantales, \( H = \{ (f(a), g(a)) \mid a \in A_S \} \), \( \pi \) be the canonical mapping from \( B_S \) to \( (B/\theta(H))_S \). Clearly, \( \pi f = \pi g \). For any \( S \)-quantale morphism \( h : B_S \rightarrow C_S \) satisfying \( hf = hg \), we obtain that \( \ker \pi \subseteq \ker h \), since \( (f(a), g(a)) \in \ker h \), for \( a \in A_S \).

Now define a mapping \( \bar{h} : (B/\theta(H))_S \rightarrow C_S \) by

\[
\bar{h}([b]_{\theta(H)}) = h(b),
\]

for \([b]_{\theta(H)} \in (B/\theta(H))_S \). Clearly \( \bar{h} \) is an \( S \)-act morphism and preserves arbitrary joins by (3). It is quite routine to check that \( \bar{h} \) is the unique morphism satisfying \( \bar{h} \pi = h \).

\[ \Box \]

**Proposition 2.6.** Let \( f : A_S \rightarrow C_S, g : B_S \rightarrow C_S \) be morphisms of \( S \)-quantales. The pullback of \( f \) and \( g \) is the \( S \)-subposet \( P = \{ (a, b) \in (A \times B)_S \mid f(a) = g(b) \} \) of \( (A \times B)_S \), together with the restricted projections of \( P_S \) into \( A_S \) and \( B_S \).

**Proof.** It is known that \( P_S \) is an \( S \)-quantale. For any \( S \)-quantale \( Q_S \) and an pair of morphisms \( f_1 : Q_S \rightarrow A_S, f_2 : Q_S \rightarrow B_S \) with \( f_1 = gf_2 \), one has that \( (f_1(q), f_2(q)) \in P_S \), for any \( q \in Q_S \). Now define a mapping \( \varphi : Q_S \rightarrow P_S \) by

\[
\varphi(q) = (f_1(q), f_2(q)),
\]

for \( q \in Q_S \). One gets that

\[
\varphi(q)s = (f_1(q), f_2(q))s = (f_1(q)s, f_2(q)s) = (f_1(qs), f_2(qs)) = \varphi(qs),
\]

for any natural number \( s \). Hence,
for each \( q \in Q_S \), \( s \in S \), and
\[
\varphi \left( \bigvee_{i \in I} q_i \right) = \left( f_1 \left( \bigvee_{i \in I} q_i \right), f_2 \left( \bigvee_{i \in I} q_i \right) \right) = \left( \bigvee_{i \in I} f_1(q_i), \bigvee_{i \in I} f_2(q_i) \right) = \bigvee_{i \in I} \varphi(q_i),
\]
for all \( q_i \in Q_S \), \( i \in I \). If \( \pi_A : P_S \to A_S \) and \( \pi_B : P_S \to B_S \) are the restricted projections, then \( f \pi_A = g \pi_B \). Straightforward checking shows that \( \varphi \) is the unique morphism satisfying \( \pi_A \varphi = f_1 \) and \( \pi_B \varphi = f_2 \).

**Proposition 2.7.** Let \( f : A_S \to B_1 \), \( g : A_S \to B_2 \) be morphisms of \( S \)-quantales. The pushout of \( f \) and \( g \) is \((B_1 \times B_2)/\theta(H)_S\), together with \( \mu_1 \) and \( \mu_2 \), where \( \mu_i : (B_1 \times B_2)_S \to (B_1 \times B_2)/\theta(H)_S \), \( i = 1, 2 \), are defined as in Proposition 2.2. \( \pi \) is the canonical mapping, \( H = \{ (\mu_1 f(a), \mu_2 g(a)) \mid a \in A_S \} \).

**Proof.** Since \((B_1 \times B_2)_S, (\mu_1, \mu_2))\) is the coproduct of \((B_1, B_2)\) by Proposition 2.2, the coequalizer of \( \mu_1 f \) and \( \mu_2 g \) is the quotient \((B_1 \times B_2)/\theta(H)_S\), where \( H = \{ (\mu_1 f(a), \mu_2 g(a)) \mid a \in A_S \} \), by Proposition 2.5. The result follows immediately by [12] Remark 11.31. \( \square \)

## 3 Monomorphisms

This section contributes to the presentation of several kinds of monomorphisms in the category \( \text{Quant}_S \). It is shown that different from the case of \( S \)-posets (see [13]), monomorphisms in \( \text{Quant}_S \) coincide with order-embeddings, which are precisely injective morphisms. It thus leads to the strengthening results that these classes of monomorphisms are also in accordance with those labeled regular and extremal in \( \text{Quant}_S \), which are exactly the category-theoretic embeddings when \( \text{Quant}_S \) is considered as a concrete category over \( \text{Set} \), \( \text{Act}_S \), and \( \text{Pos}_S \), respectively.

**Proposition 3.1.** Let \( f : A_S \to B_S \) be a morphism of \( S \)-quantales. Then the following statements are equivalent:

1. \( f \) is a monomorphism;
2. \( f \) is injective;
3. \( f \) is an order-embedding.

**Proof.** It is enough to show the implications \((1) \Rightarrow (2)\) and \((1) \Rightarrow (3)\) hold.

Let \( f : A_S \to B_S \) be a monomorphism of \( S \)-quantales. Consider \( S \)-subquantale \( \ker f \) of the product \((A \times A)_S\), and the restricted projection mappings \( h_i : \ker f \to A \), \( i = 1, 2 \). For any \((x, y) \in \ker f\), equalities
\[
fh_1(x, y) = f(x) = f(y) = fh_2(x, y)
\]
imply that \( fh_1 = fh_2 \) and hence \( h_1 = h_2 \) by assumption. Therefore, \( x = h_1(x, y) = h_2(x, y) = y \), and hence \( f \) is injective as needed.

It remains to prove that \( f \) is an order-embedding whenever it is a monomorphism. Suppose that \( f(a_1) \leq f(a_2) \) for \( a_1, a_2 \in A_S \). Then
\[
f(a_2) = f(a_1) \lor f(a_2) = f(a_1 \lor a_2).
\]
According to the above result of \( f \) being injective, we soon obtain that \( a_1 \leq a_2 \), and thus \( f \) is an order-embedding. \( \square \)

**Lemma 3.2.** Each inclusion mapping in \( \text{Quant}_S \) is a regular monomorphism.

**Proof.** Suppose that \( A_S \) is an \( S \)-subquantale of \( B_S \). Let \((B \times B)_S, (\mu_1, \mu_2)\) be the coproduct of \((B_S, B_S)\), described as in Proposition 2.2. Write
\[
R = \{ ((a, \bot), (\bot, a)) \mid a \in A_S \},
\]
where \( \bot \) is the bottom element of \( B_S \). Then the relation \( \rho \), which is defined by
\[
\rho = \left\{ (x \lor a, y \lor b), (x' \lor a', y' \lor b') \mid x, y, x', y' \in B_S, a, b, a', b' \in A_S, x \lor b = x' \lor b', y \lor a = y' \lor a' \right\},
\]
is the smallest congruence relation on $B \times B$ containing $R$. So $((B \times B)/\rho)_S$ becomes an $S$-quantale equipped with a suitable order defined by a $\rho$-chain, and the canonical mapping $\pi: (B \times B)_S \to ((B \times B)/\rho)_S$ given by $\pi(x, y) = [(x, y)]_\rho$, for each $(x, y) \in (B \times B)_S$, is a morphism.

Next we show that the inclusion mapping $\iota_A : A \to B$ is the equalizer of $\pi_{\mu_1} = \pi_{\mu_2}$. Suppose that $h : E_S \to B_S$ is any monomorphism satisfying $\pi_{\mu_1} h = \pi_{\mu_2} h$. Then for any $e \in E_S$, the equalities

$$[(h(e), \bot)]_\rho = \pi(h(e), \bot) = \pi_{\mu_1} h(e) = \pi_{\mu_2} h(e) = \pi(\bot, h(e)) = [(\bot, h(e))]_\rho$$

indicate that $((h(e), \bot), (\bot, h(e))) \in \rho$. According to the definition of $\rho$, we deduce that $(h(e), \bot) = (x \vee a, y \vee b)$ and $(\bot, h(e)) = (x' \vee a', y' \vee b')$ for some $x, y, x', y' \in B_S, a, a', b, b' \in A_S$. So $y = b = \bot, x = a' = \bot$, and correspondingly,

$$a = \bot \vee a = y \vee a = y' \vee a' = y' = \bot,$$

and

$$x = x' \bot \vee b' = b'.$$

Therefore, we have $h(e) = x \vee a = b' \vee a \in A_S$, i.e., $h(E) \subseteq A_S$. As a consequence, $\overline{h} : E \to A$ is the unique morphism satisfying $\iota_A \overline{h} = h$. 

**Theorem 3.3.** Let $f : A_S \to B_S$ be a morphism of $S$-quantales. Then the following assertions are equivalent:

1. $f$ is a regular monomorphism;
2. $f$ is an extremal monomorphism;
3. $f$ is a monomorphism;
4. $f$ is a $\text{Quant}_S$-embedding over $\text{Set}$;
5. $f$ is a $\text{Quant}_S$-embedding over $\text{Act}_S$;
6. $f$ is a $\text{Quant}_S$-embedding over $\text{Pos}_S$.

**Proof.** (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are general category-theoretic results.

(3) $\Rightarrow$ (4). Suppose that $f : A_S \to B_S$ is a monomorphism. Let $g : C_S \to A_S$ be a mapping with $fg : C_S \to B_S$ being an $S$-quantale morphism. Then $g$ preserves arbitrary joins by the fact that for $a_i \in C_S, i \in I$,

$$fg\left(\bigvee_{i \in I} a_i\right) = \bigvee_{i \in I} fg(a_i) = f\left(\bigvee_{i \in I} g(a_i)\right),$$

and $f$ being injective by Proposition 3.1. Similarly, we get that $g$ preserves $S$-actions. Thus $f$ is initial and then an $S$-quantale embedding over $\text{Set}$.

(4) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) are clear.

(6) $\Rightarrow$ (4). Let $f : A_S \to B_S$ be a $\text{Quant}_S$-embedding over $\text{Pos}_S$. $g : C_S \to A_S$ a mapping provided that $fg : C_S \to B_S$ is a morphism in $\text{Quant}_S$. We are going to show that $g$ is an $S$-poset morphism. This is the case since

$$fg(as) = fg(a)s = f(g(a)s),$$

for any $a \in A_S, s \in S$, and

$$fg(a_2) = fg(a_1 \vee a_2) = fg(a_1) \vee fg(a_2) = f\left(g(a_1) \vee g(a_2)\right),$$

for $a_1 \leq a_2$ in $A_S$. Note that the monomorphisms in $\text{Pos}_S$ are just the $S$-poset morphisms with injective underlying mappings, we immediately achieve that $g(as) = g(a)s$ and $g(a_1) \leq g(a_2)$. Therefore, $g$ is an $S$-poset morphism as required.

(3) $\Rightarrow$ (1). This follows by [12] Proposition 7.53 (2) and Lemma 3.2. 

## 4 Epimorphisms

Dual to discussions on monomorphisms studied in Section 3, this section is intended to motivate our investigation on relationships between various type of epimorphisms in $\text{Quant}_S$. However, the characterization of
epimorphisms in $\text{Quant}_S$ is quite complicated. So we merely cite the result and the reader is suggested to find complete illustrations in [14].

**Proposition 4.1** ([14] Th. 4.2). *Epimorphisms in $\text{Quant}_S$ are exactly onto morphisms.*

**Theorem 4.2.** For a morphism $f : A_S \to B_S$ of $S$-quantales, the following statements are equivalent:

1. $f$ is a regular epimorphism;
2. $f$ is an extremal epimorphism;
3. $f$ is an epimorphism;
4. $f$ is a $\text{Quant}_S$-quotient morphism over $\text{Set}$;
5. $f$ is a $\text{Quant}_S$-quotient morphism over $\text{Act}_S$;
6. $f$ is a $\text{Quant}_S$-quotient morphism over $\text{Pos}_S$.

**Proof.** (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are clear.

(3) $\Rightarrow$ (4). Let $g : B_S \to C_S$ be a mapping between $S$-quantales such that $gf$ is an $S$-quantale morphism. Let us verify that $g$ is an $S$-quantale morphism, as well. It is easy to see that $g$ is an $S$-poset morphism. Since $f$ is an epimorphism, it is onto by Proposition 4.1. Hence we may assume that for any $M \subseteq B_S$, $\forall M = f(a)$ for some $a \in A_S$. By the reason that $f$ preserves arbitrary joins, we have

$$f(a) = \bigvee M = \bigvee_{x \in f^{-1}(M)} f(x) = \left( \bigvee_{x \in f^{-1}(M)} x \right).$$

Consequently,

$$g(\bigvee M) = gf(a) = g\left( \bigvee_{x \in f^{-1}(M)} x \right) = \bigvee_{x \in f^{-1}(M)} g(x) = \bigvee_{m \in M} g(m).$$

(4) $\Rightarrow$ (3), (4) $\Rightarrow$ (5) $\Rightarrow$ (6) are clear.

(6) $\Rightarrow$ (2). Let $f : A_S \to B_S$ be a $\text{Quant}_S$-quotient morphism over $\text{Pos}_S$. Suppose that $g : A_S \to C_S$ and $h : C_S \to B_S$ are $S$-quantale morphisms such that $f = hg$ and $h$ is a monomorphism. Then $h$ is injective by Proposition 3.1. Note that $f$ is a $\text{Pos}_S$-epimorphism by hypotheses, and hence is surjective. So $h$ is surjective, as well, and thus bijective. Now, considering the inverse mapping $h^{-1}$ with $g = h^{-1}f$, we remain to show that $h^{-1}$ is an $S$-poset morphism. In fact, $f$ being onto indicates that $h^{-1}$ is action-preserving. Observe that

$$h\left( h^{-1}(b) \bigvee h^{-1}(b') \right) = hh^{-1}(b) \bigvee hh^{-1}(b') = b \bigvee b' = hh^{-1}(b'),$$

for any $b \preceq b'$ in $B_S$. Thus $h^{-1}(b) \bigvee h^{-1}(b') = h^{-1}(b')$, which expresses that $h^{-1}$ is an $S$-poset morphism, and hereby an $S$-quantale morphism by assumption. 

\[\square\]

### 5 Adjoint situations

The final part is devoted to observation on the adjoint situation between $\text{Pos}$ and $\text{Quant}_S$. By a *free $S$-quantale on a poset $P$* we mean an $S$-quantale $Q_S$ together with a monotone mapping $\psi : P \to Q_S$ with the universal property that given any $S$-quantale $A_S$ and a monotone mapping $f : P \to A_S$, there exists a unique $S$-quantale morphism $\tilde{f} : Q_S \to A_S$ such that $f$ can be factored through.

**Lemma 5.1** ([13] Th.10). *For a given poset $P$ and a pomonoid $S$, the free $S$-poset on $P$ is given by $P \times S$, with componentwise order and the action $(x, s) t = (x, st)$, for every $x \in P$, $s, t \in S$.*

Let $(P \times S)_S$ be the free $S$-poset presented in Lemma 5.1. Write

$$\Omega(P \times S) = \{ D \subseteq P \times S \mid D = D1 \},$$
where $D_i$ is the down-set of $D$ for $D \subseteq P \times S$, more precisely,

$$D_i = \{(p, s) \in P \times S \mid (p, s) \leq (p_1, s_1) \text{ for some } (p_1, s_1) \in D\}.$$ 

Note that $(p \downarrow \times s_1) \downarrow = p \downarrow \times s_1$ provides that $p \downarrow \times s_1 \in \Omega(P \times S)$ for every element $p \in P$, $s \in S$. Define an action * on $\Omega(P \times S)$ by

$$D \ast t := \{(p, s) \in P \times S \mid (p, s) \leq (p_1, s_1 t) \text{ for some } (p_1, s_1) \in D\},$$

for $t \in S$. Then it is clear that $D \ast t = (Dt) \downarrow$. We claim that $(\Omega(P \times S)_S, \ast, \subseteq)$ is the free object in Quant$_S$.

**Proposition 5.2.** Let $S$ be a pomonoid, $P$ be a poset. Then $(\Omega(P \times S)_S, \ast, \subseteq)$ is an $S$-quantale.

**Proof.** Observe first that

$$(D \ast t_1) \ast t_2 = \{(p, s) \in P \times S \mid (p, s) \leq (p_1, s_1 t_2) \text{ for some } (p_1, s_1) \in D \ast t_1\}$$

$$= \{(p, s) \in P \times S \mid (p, s) \leq (p_1, s_1 t_2), (p_1, s_1) \leq (p_2, s_2 t_1) \text{ for some } (p_2, s_2) \in D\}$$

$$= \{(p, s) \in P \times S \mid (p, s) \leq (p_1, s_1 t_2), (p_2, s_2) \in D\}$$

$$= D \ast (t_1 t_2)$$

for any $t_1, t_2 \in S$, $D \in \Omega(P \times S)_S$, and $D \ast 1 = (D1) \downarrow = D$. This shows that $(\Omega(P \times S)_S, \ast)$ is an $S$-act. Clearly, $D \ast s \subseteq D \ast t$, whenever $D_1 \subseteq D_2$ in $\Omega(P \times S)_S$, and $s \leq t$ in $S$. So $\Omega(P \times S)_S$ is an $S$-poset. It is straightforward to check that $(\bigcup_{i \in I} D_i) \ast t = \bigcup_{i \in I} (D_i \ast t)$ for every $D_i \in \Omega(P \times S)_S$, $i \in I$, $t \in S$. \hfill \Box

Lemma 5.3 comes true directly by the definition of $\Omega(P \times S)_S$.

**Lemma 5.3.** Let $S$ be a pomonoid, $P$ be a poset. Then $D = \bigcup_{(p, s) \in D} (p \downarrow \times s_1)$ for every $D \in \Omega(P \times S)_S$.

**Lemma 5.4.** Let $S$ be a pomonoid, $P$ be a poset. Then $p \downarrow \times t_1 = (p \downarrow \downarrow \downarrow 1) \ast t$ holds in $\Omega(P \times S)_S$ for every $p \in P$, $t \in S$.

**Proof.** It is clear that $(q, s) \in (p \downarrow \times t_1) \ast t$ for every $(q, s) \in p \downarrow \times t_1$, since $(q, s) \leq (p, t)$. On the other hand, for any $(q, s) \in (p \downarrow \times t_1) \ast t$, $(q, s) \leq (p_1, s_1 t) = (p_1, s_1) t$ for some $(p_1, s_1) \in p \downarrow \times t_1$, it follows that $(q, s) \leq (p, 1) t = (p, t)$. Hence $(q, s) \in p \downarrow \times t_1$. \hfill \Box

**Theorem 5.5.** Let $S$ be a pomonoid, $P$ be a poset. Then the free $S$-quantale on $P$ is given by the $S$-quantale $\Omega(P \times S)_S$.

**Proof.** Define a mapping $\tau : P \rightarrow \Omega(P \times S)_S$ by $\tau(p) = p \downarrow \downarrow \downarrow 1$ for every $p \in P$. Obviously, $\tau$ is order-preserving. Let $Q_S$ be an $S$-quantale, $f : P \rightarrow Q_S$ be any monotone mapping. Define a mapping $\bar{f} : \Omega(P \times S)_S \rightarrow Q_S$ by

$$\bar{f}(D) = \bigvee \{f(p)s \mid (p, s) \in D\},$$

for every $D \in \Omega(P \times S)_S$. We claim that $\bar{f}$ is the unique $S$-quantale morphism with the property that $\bar{f} \tau = f$.

It is clear that $\bar{f}$ preserves $S$-actions. Take $D_i \in \Omega(P \times S)_S$, $i \in I$, then equalities

$$\bar{f}
\left(\bigcup_{i \in I} D_i\right)
= \bigvee \left\{f(p)s \mid (p, s) \in \bigcup_{i \in I} D_i\right\}
= \bigvee_{i \in I} \left\{f(p)s \mid (p, s) \in D_i\right\}
= \bigvee_{i \in I} \bar{f}(D_i)$$

indicate that $\bar{f}$ preserves arbitrary joins. Evidently, for any $p \in P$,

$$\bar{f} \tau(p) = \bar{f}(p \downarrow \downarrow \downarrow 1) = \bigvee \{f(q)s \mid (q, s) \in p \downarrow \downarrow \downarrow 1\} \leq f(p),$$

while the fact that $f(p)$ being one of the terms in the sup that defines $\bar{f} \tau(p)$ guarantees the opposite implication. Suppose that $\bar{f} : \Omega(P \times S)_S \rightarrow Q_S$ is an $S$-quantale morphism such that $\bar{f} \tau = f$. Then by Lemma 5.3 and Lemma 5.4, we achieve that

$$\bar{f}^\prime(D) = \bar{f}
\left(\bigcup_{(p, s) \in D} (p \downarrow \times s_1)\right)
= \bigvee \left\{\bar{f}^\prime(p \downarrow \times s_1) \mid (p, s) \in D\right\}
= \bigvee \left\{\bar{f}^\prime((p \downarrow \times 1) \ast s) \mid (p, s) \in D\right\}$$
\[
\bigvee_{(p,s)\in D} f'(p) \times 1_\downarrow \\sigma = \bigvee_{(p,s)\in D} f'(\tau)(p)s = \bigvee_{(p,s)\in D} f(p)s = \bar{f}(D),
\]
for every \(D \in \Omega(P \times S)_S\), which finishes our proof. \(\square\)

**Corollary 5.6.** The category \(\text{Quant}_S\) has a separator.

**Proof.** Let \(f, g : A_S \to B_S\) be a pair of morphisms in \(\text{Quant}_S\) with \(f \neq g\). Then there exists \(a \in A_S\) such that \(f(a) \neq g(a)\). Let \(P\) be a poset. Define a mapping \(k : P \to A_S\) by \(k(p) = a, \forall p \in P\). We are aware that \(k\) is a morphism in \(\text{Pos}\). Hence there is a unique \(S\)-quantale morphism \(\tilde{k} : \Omega(P \times S)_S \to A_S\) with \(\tilde{k} \tau = k\), where \(\tau : P \to \Omega(P \times S)_S\) is defined as in Theorem 5.5. This yields that \(f\tilde{k} \neq g\tilde{k}\), and consequently gives that \(\Omega(P \times S)_S\) is a separator. \(\square\)

We thereby obtain a free functor from the category of posets into the category of \(S\)-quantales, which is shown to be left adjoint to the forgetful functor.

**Proposition 5.7.** There is a free functor \(F : \text{Pos} \to \text{Quant}_S\) given by

\[
\begin{array}{ccc}
P & \longrightarrow & FP \\
f \downarrow & & \downarrow Ff \\
Q & \longrightarrow & FQ,
\end{array}
\]

where \(FP = \Omega(P \times S)_S\), and

\[
Ff(D) = \{(x, y) \in Q \times S \mid (x, y) \leq (f(p), s)\text{ for some } (p, s) \in D\},
\]
for any monotone mapping \(f : P \to Q\) and \(D \in FP\).

**Theorem 5.8.** The free functor \(F : \text{Pos} \to \text{Quant}_S\) is left adjoint to the forgetful functor \([\_] : \text{Quant}_S \to \text{Pos}\).

**Proof.** Let us prove that \(\eta : \text{id}_{\text{Pos}} \to [\_]F\) with \(\eta_P : P \to [\Omega(P \times S)_S]_S\), where \(P\) is a \(\text{Pos}\)-object, \(\eta_P(p) = p \downarrow \times 1_\downarrow\), \(\forall p \in P\), is a natural transformation. Suppose that \(f : P \to P^{'\prime}\) is a morphism in \(\text{Pos}\). Then

\[
Ff \circ \eta_{P'}(p) = Ff(p \downarrow \times 1_\downarrow) = \{(x, y) \in P^{'\prime} \times S \mid (x, y) \leq (f(\tilde{p}), s)\text{ for some } (\tilde{p}, s) \in p \downarrow \times 1_\downarrow\}
\]
\[
= \{(x, y) \in P^{'\prime} \times S \mid (x, y) \leq (f(p), s) \leq (f(p), 1), (\tilde{p}, s) \in p \downarrow \times 1_\downarrow\},
\]
for \(p \in P\), and

\[
(\eta_{P'} \circ f)(p) = \eta_{P'}(f(p)) = f(p) \downarrow \times 1_\downarrow.
\]

It results in \(Ff \circ \eta_{P'} = \eta_{P'} \circ f\) as needed. Now, by Theorem 5.5 and [12] 19.4(2), we obtain that \(F\) is left adjoint to \([\_]\). \(\square\)

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