Let $k$ be an integer greater than 1 and let $N$ be a positive integer. The space of cusp forms of weight $2k$ for $\Gamma_0(N)$ is denoted by $S_{2k}(N)$. Throughout this paper let $p = 1$ or a prime number. For $\kappa \in \mathbb{Z} + \frac{1}{2}$ we denote by $M_{\kappa}^!(\Gamma_0(4p))$ the space of weakly holomorphic modular forms of weight $\kappa$ on $\Gamma_0(4p)$. As usual, $M_{\kappa}(\Gamma_0(4p))$ (resp. $S_{\kappa}(\Gamma_0(4p))$) stands for the space of weight $\kappa$ modular forms (resp. cusp forms) on $\Gamma_0(4p)$. Let $H_\kappa(\Gamma_0(4p))$ be the space of weight $\kappa$ harmonic weak Maass forms on $\Gamma_0(4p)$. Let $M_{\kappa}^!(p)$ (resp. $H_{2-\kappa}(\Gamma_0(4p))$) denote the subspace of $M_{\kappa}^!(\Gamma_0(4p))$ (resp. $H_{2-\kappa}(\Gamma_0(4p))$), in which each form satisfies Kohnen’s plus space condition, that is, its Fourier expansion is supported only on those $n \in \mathbb{Z}$ for which $(-1)^{\kappa - \frac{3}{2}} n \equiv 0 \pmod{4p}$. Let $\kappa = k + \frac{1}{2}$ and $M_{\kappa}(p)$ (resp. $S_{\kappa}(p)$) denote the subspace of $M_{\kappa}(\Gamma_0(4p))$ (resp. $S_{\kappa}(\Gamma_0(4p))$), in which each form satisfies Kohnen’s plus space condition.

Let $\Delta(\tau) \in S_{12}(1)$ be the Ramanujan’s Delta function. The famous Lehmer’s conjecture states that the Fourier coefficients of $\Delta(\tau)$ never vanish. Concerning this conjecture, Ono [1] related the algebraicity of Fourier coefficients of weight $-10$ mock modular form whose shadow is $\Delta(\tau)$ to the vanishing of Fourier coefficients of $\Delta(\tau)$. Generalizing Ono’s results, Boylan [2] related the algebraicity of Fourier coefficients of weight $2-2k$ mock modular forms to the vanishing of Fourier coefficients of their shadows when $\dim S_{2k}(1) = 1$. In this paper we will extend their works to the half-integral weight case. In the following we recall some known facts.

**Fact 1.** By Shimura correspondence [3, Proposition 1] we have

$$\dim S_{k+\frac{1}{2}}(1) = \dim S_{2k}(1),$$

which implies that

$$\dim S_{k+\frac{1}{2}}(1) = 1 \iff k = 6, 8, 9, 10, 11, 13 \iff 1 - k = -5, -7, -8, -9, -10, -12.$$
Now we assume that \( k \in \{6, 8, 9, 10, 11, 13\} \). For \( \kappa > 2 \) there is an antilinear differential operator \( \xi_{2-\kappa} : H_{2-\kappa}(\Gamma_0(4p)) \to S_\kappa(\Gamma_0(4p)) \) defined by

\[
\xi_{2-\kappa}(f)(\tau) := 2iy^{2-\kappa} \cdot \frac{df}{d\tau}.
\]

**Fact 2.** For \( k \in \{6, 8, 9, 10, 11, 13\} \), it follows from \([4, \text{Theorem 1.1-(iii)} \text{ and Lemma 4.2-(c)}]\) that

\[
\xi_{\frac{3}{2}-k} : H_{\frac{3}{2}-k}(1) \to S_{k+\frac{3}{2}}(1) \quad \text{is surjective.}
\]

For any \( \kappa \in \mathbb{Z} + \frac{1}{2} \), the Duke-Jenkins basis \([5]\) for \( \mathcal{M}_\kappa := \mathcal{M}_\kappa^1(1) \) is constructed as follows. Let \( 2\kappa - 1 = 12\ell_\kappa + k' \) with uniquely determined \( \ell_\kappa \in \mathbb{Z} \) and \( k' \in \{0, 4, 6, 8, 10, 14\} \). If \( A_\kappa \) denotes the maximal order of a non-zero \( f \in \mathcal{M}_\kappa \) at \( i\infty \), then by the Shimura correspondence \([3]\) one has

\[
A_\kappa = \begin{cases} 
2\ell_\kappa - (-1)^{\kappa-1/2} & \text{if } \ell_\kappa \text{ is odd}, \\
2\ell_\kappa & \text{otherwise}.
\end{cases}
\]

A basis for \( \mathcal{M}_\kappa^1 \) then consists of functions of the form

\[
f_{\kappa,m}(\tau) = q^{-m} + \sum_{n \in \mathcal{A}_\kappa} a_\kappa(m, n) q^n,
\]

where \( m \geq -A_\kappa \) satisfies \((-1)^{\kappa-3/2} m \equiv 0, 1 \pmod{4} \). Using (1) and (2) we deduce the following facts.

**Fact 3.** For \( \kappa = \frac{3}{2} - k \) with \( k \in \{6, 8, 9, 10, 11, 13\} \), the maximal order \( A_\kappa \) of a non-zero \( f \in \mathcal{M}_\kappa \) at \( i\infty \) is given by \( A_\kappa = -4 \). Thus for each \( m \geq 4 \) satisfying \((-1)^k m \equiv 0, 1 \pmod{4} \), there exist unique modular forms \( f_{\frac{3}{2}-k,m}(\tau) \in \mathcal{M}_{\frac{3}{2}-k} \) with Fourier development

\[
f_{\frac{3}{2}-k,m}(\tau) = q^{-m} + \sum_{n \geq -3} a_{\frac{3}{2}-k}(m, n) q^n,
\]

which form a basis for the space \( \mathcal{M}_{\frac{3}{2}-k}^1 \).

**Fact 4.** For \( \kappa = k + \frac{1}{2} \) with \( k \in \{6, 8, 9, 10, 11, 13\} \), the space \( S_\kappa \) is spanned by

\[
f_{k} := f_{\kappa, -\alpha}
\]

where \( \alpha \) is given by

\[
\alpha = \alpha_k = A_\kappa = A_{k+\frac{1}{2}} = \begin{cases} 
1, & \text{if } k \text{ is even}, \\
3, & \text{if } k \text{ is odd},
\end{cases}
\]

and \( f_k \) has the form \( q^\alpha + O(q^k) \).

**Fact 5.** Let \( \kappa = k + \frac{1}{2} \) and \( f(z) \in H_{\kappa}(\Gamma_0(4p)) \) with Fourier expansion \( f(\tau) = \sum_{n \in \mathbb{Z}} c(y/n)n^\frac{\alpha}{2} \) where \( \tau = x + iy \).

For each prime \( l \) with \( \gcd(1, 4p) = 1 \), the \( l^2 \)-th Hecke operator is defined by

\[
f_l T(l^2)(\tau) = \sum_{n \in \mathbb{Z}} \left( c(y/l^2) + \frac{(-1)^k n}{l} \right) \left( l^{k-1} c(y/n) + l^{2k-1} c(l^2 \cdot y/l^2) \right) e^{2\pi i n x}.
\]

Then for each \( \mathcal{M} \in \mathcal{M}_{2-\kappa}(p) \), we obtain from \([6, (2.6)]\) or \([7, (7.2)]\) that for \( \kappa > 2 \),

\[
\xi_{2-\kappa}(\mathcal{M}|T(l^2)) = l^{2\kappa-2} \xi_{2-\kappa}(\mathcal{M}|T(l^2)).
\]

As a corollary of Fact 5, one has that if \( f(z) = \sum_{n \geq 1} a_f(n) q^n \in \mathcal{S}_{k+\frac{1}{2}} \), then

\[
f_{l^{k+\frac{1}{2}} T(l^2)} = \sum_{n \geq 1} \left( a_f(l^2 n) + \frac{(-1)^k n}{l} l^{k-1} a_f(n) + l^{2k-1} a_f(n^2) \right) q^n \in \mathcal{S}_{k+\frac{1}{2}}.
\]
(or see [3, Theorem 1(ii)].)

Let \((V, Q)\) be a non-degenerate rational quadratic space of signature \((b^+, b^-)\) and \(L\) an even lattice with dual \(L'\). Denote the standard basis elements of the group algebra \(\mathbb{C}[L'/L]\) by \(e_\gamma\) for \(\gamma \in L'/L\). Let \(\text{Mp}_2(\mathbb{Z})\) denote the integral metaplectic group, which consists of pairs \((\gamma, \phi)\), where \(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})\) and \(\phi: \mathbb{R} \to \mathbb{C}\) is a holomorphic function with \(\phi(\tau)^2 = c\tau + d\). It is well known that \(\text{Mp}_2(\mathbb{Z})\) is generated by \(S = ((0 \ 1 \ -1 \ 0), \sqrt{7})\) and \(T = ((\frac{1}{2} \ \frac{1}{2}), 1)\). Then there is a unitary representation \(\rho_L\) of the group \(\text{Mp}_2(\mathbb{Z})\) on \(\mathbb{C}[L'/L]\), the so-called Weil representation, which is defined by

\[
\rho_L(T)(e_\gamma) := e(Q(\gamma))e_\gamma,
\]

\[
\rho_L(S)(e_\gamma) := e((b^* - b^+)/\sqrt{|L'/L|}) \sum_{\delta \in L'/L} e(-\gamma, \delta)e_\delta,
\]

where \(e(z) := e^{2\pi iz}\) and \((X, Y) := Q(X + Y) - Q(X) - Q(Y)\) is the associated bilinear form. One has the relations

\[
S^2 = (ST)^3 = Z \quad (Z = ((-1 \ 0 \ -1 ), i))
\]

from which we note that

\[
\rho_L(Z)e_\gamma = b^* - b^- c_{-\gamma}.
\]

We write \(<\cdot, \cdot>\) for the standard scalar product on \(\mathbb{C}[L'/L]\), i.e.

\[
< \sum_{\gamma \in L'/L} \lambda_\gamma e_\gamma, \sum_{\gamma' \in L'/L} \mu_{\gamma'} e_{\gamma'} > = \sum_{\gamma \in L'/L} \lambda_\gamma \overline{\mu_{\gamma'}}.
\]

For \(\gamma, \delta \in L'/L\) and \((M, \phi) \in \text{Mp}_2(\mathbb{Z})\) the coefficient \(\rho_{\gamma, \delta}(M, \phi)\) of the representation \(\rho_L\) is defined by

\[
\rho_{\gamma, \delta}(M, \phi) = <\rho_L(M, \phi)e_\delta, e_\gamma>.
\]

Following [9], for an integer \(r\) we denote by \(H_{r+1/2, \rho_\ell}\) (resp. \(M_{r+1/2, \rho_\ell}\)), the space of \(\mathbb{C}[L'/L]\)-valued harmonic weak Maass forms (resp. weakly holomorphic modular forms) of weight \(r + 1/2\) and type \(\rho_\ell\).

Let \(L_r\) be the lattice \(2p^r\mathbb{Z}\) of signature \((1, 0)\) (resp. \((0, 1)\)) when \(r\) is even (resp. odd) equipped with the quadratic form \(Q_r(x) = (-1)^r x^2/4p\). Then its dual lattice \(L'_r\) is equal to \(\mathbb{Z}\). For a vector valued modular form \(F = \sum \gamma F_\gamma e_\gamma\), we define a map \(\Phi\) by

\[
\Phi(F)(\tau) := \sum_{\gamma} F_\gamma(4p\tau).
\]

It then follows from [8, Theorem 1] that the map \(\Phi\) defines an isomorphism from \(H_{r+1/2, \rho_\ell}\) to \(\mathbb{H}_{r+1/2}(p)\) since \(\rho_{\ell_r} = 1\).

For a \(\mathbb{C}[L'/L]\)-valued function \(f\) and \((M, \phi) \in \text{Mp}_2(\mathbb{Z})\) we define the Petersson slash operator by

\[
(f|_{r+1/2}(M, \phi))(\tau) = \phi(\tau)^{-2r-1} \rho_L(M, \phi)^{-1} f(M\tau).
\]

Let \(L := L_k\) and \(Q := Q_k\). Following [9] we define the vector valued cuspidal Poincaré series \(P_{\beta, n}^L(\tau)\) as follows:

for each \(\beta \in \mathbb{Z}/2p\mathbb{Z}\) and \(n \in \mathbb{Z} + Q(\beta)\) with \(n > 0\),

\[
P_{\beta, n}^L(\tau) := \frac{1}{2} \sum_{(M, \phi) \in \text{Mp}_2(\mathbb{Z})} \epsilon_\beta(n\tau)|_{\kappa}(M, \phi).
\]

Then we know from [9] that \(P_{\beta, n}^L(\tau)\) belongs to the space \(S_{\kappa, \rho_\ell}\). Let \(\mathcal{D}_k\) denote the set of all integers \(D\) such that \((-1)^k D > 0\) and \(D\) is congruent to a square modulo \(4p\).

**Theorem 1.1** ([4, Theorem 1.1]). For an integer \(k > 2\) we let \(\kappa = k + \frac{1}{2}\) and \(L := L_k\). For each \(D \in \mathcal{D}_k\), we define

\[
P_D^{\beta} := \Phi(P_{\beta, \frac{k}{2}}^L),
\]

where \(\beta\) is an integer such that \(D \equiv \beta^2 \pmod{4p}\). Then the following assertions are true.

(i) \(P_D^{\beta} \in S_{\kappa}(p)\) and the definition of \(P_D^{\beta}\) does not depend on the choice of \(\beta\).
(ii) For each \( f = \sum_{n\geq 1} a_f(n)q^n \in \mathcal{S}_\kappa(p) \), we have

\[
(f, c_{k,D}P^n) = a_f(|D|)
\]

where \((\cdot, \cdot)\) denotes the Petersson inner product and

\[
c_{k,D} = \begin{cases} 
\frac{(4\pi |D|)^{\kappa-1}}{T(\kappa-1)} \cdot \frac{s(D)}{3}, & \text{if } p = 1 \\
\frac{(4\pi |D|)^{\kappa-1}}{T(\kappa-1)} \cdot \frac{s(D)}{4}, & \text{if } p = 2 \\
\frac{(4\pi |D|)^{\kappa-1}}{T(\kappa-1)} \cdot \frac{s(D)}{6}, & \text{if } p > 2 
\end{cases}
\]

with \( s(D) = \begin{cases} 
1, & \text{if } p | D, \\
2, & \text{otherwise.}
\end{cases} \)

(iii) The set \( \{ P^n_D \mid D \in \mathcal{D}_k \} \) spans the space \( \mathcal{S}_\kappa(p) \). Moreover, if we let \( t = \dim \mathcal{S}_\kappa(p) \) and \( \{ f_1, f_2, \ldots, f_t \} \) be a basis for \( \mathcal{S}_\kappa(p) \) satisfying \( f_i = q^{|D_i|} + O(q^{|D_i|+1}) \) for some \( D_i \in \mathcal{D}_k \) \( (i = 1, \ldots, t) \) and \( 0 < |D_1| < |D_2| < \cdots < |D_t| \), then the set

\[
\{ P^n_{D_1}, P^n_{D_2}, \ldots, P^n_{D_t} \}
\]

forms a basis for \( \mathcal{S}_\kappa(p) \).

(iv) Let \( I \) be a nonempty finite subset of \( \mathbb{N} \). Then the following two conditions are equivalent.

(a) \( \sum_{\alpha \in I} \alpha c(D^n_{P_{\alpha}}(\tau)) = 0 \) for some \( \alpha \in \mathbb{C} \) and \( D_{\alpha} \in \mathcal{D}_k \).

(b) There exists \( g \in M_{2-k}(\mathbb{R}) \) with the principal part \( \sum_{\alpha \in I} \alpha |D_{\alpha}|^{1-k} q^{-|D_{\alpha}|} \).

Remark 1.2. Let \( p = 1 \) and take

\[
D_k = \begin{cases} 
1, & \text{if } k \text{ is even} \\
-3, & \text{if } k \text{ is odd.}
\end{cases}
\]

Then in Theorem 1.1 one can choose \( \beta = 1 \) and

\[
P^n_{\beta} := \phi(P^n_{\frac{1}{2}, \frac{1}{2}}).
\]

We let \( \tilde{\Gamma}_\infty := \langle T \rangle \). We define for \( s \in \mathbb{C} \) and \( y \in \mathbb{R} - \{ 0 \} : \)

\[
\mathcal{M}_s(y) = y^{-(2-\kappa)/2} M_{-(2-\kappa)/2, s-1/2}(y) \quad (y > 0),
\]

\[
\mathcal{W}_s(y) = |y|^{-(2-\kappa)/2} W_{\frac{s-\kappa}{2}, \frac{s-1}{2}}(|y|)
\]

where \( M_{\kappa, \mu}(z) \) and \( W_{\kappa, \mu}(z) \) denote the usual Whittaker functions. Now we take \( \kappa = k + 1/2 > 2, L := L_{1-k}, \) and \( Q := Q_{1-k} \). For each \( \beta \in \mathbb{Z}/2p\mathbb{Z} \) and \( m \in \mathbb{Z} + Q(\beta) \) with \( m < 0 \), modifying the Poincaré series in [9, (1.35)] we define the vector valued Maass Poincaré series \( F_{\beta,m}^L \) of index \( (\beta, m) \) by

\[
F_{\beta,m}^L(\tau, s) := \frac{1}{2\Gamma(2s)} \sum_{(M, \phi) \in \tilde{\Gamma}_{\infty} \setminus \text{Mp}_L(Z)} [\mathcal{M}_s(4\pi |m|y) \epsilon_\beta(mx)] |2^{1-s}(M, \phi)|^{1-s}.
\]

where \( \tau = x + iy \in \mathbb{H} \) and \( s = \sigma + it \in \mathbb{C} \) with \( \sigma > 1 \). Indeed, since \( \mathcal{M}_s(4\pi |m|y) \epsilon_\beta(mx) \) is invariant under slash operator \( |2-\kappa|_T \), the Maass Poincaré series is well defined. This series has desirable properties as follows. As in Section 1.3 in [9] it converges normally for \( \tau \in \mathbb{H} \) and \( s = \sigma + it \in \mathbb{C} \) with \( \sigma > 1 \) and hence defines a \( \text{Mp}_L(Z) \) - invariant function on \( \mathbb{H} \) under the slash operator \( |2-\kappa|_T \). Moreover, \( F_{\beta,m}^L(\tau, s) \) is an eigenfunction of \( \Delta_{2-k} \) with an eigenvalue \( s(1-s) + \kappa(\kappa-2)/4 \). Since \( \epsilon_\beta(\tau)|2-k, Z = \epsilon_\beta \) by (3), the invariance of \( F_{\beta,m}^L \) under the action of \( Z \) implies \( F_{\beta,m}^L = F_{\beta,m}^L \).

Let \( \kappa = k + 1 \) and \( L = L_{1-k} \) with \( k \) an integer > 2. For each \( \beta \in \mathbb{Z}/2p\mathbb{Z} \) and \( m \in \mathbb{Z} + Q(\beta) \) with \( m < 0 \), we obtain from [4, Corollary 1.5] that \( F_{\beta,m}^L(\tau, \frac{\kappa}{2}) \) belongs to the space \( H_{2-k,\rho_L} \).

Let

\[
Q = Q(k; z) := \phi \left( F_{\beta,m}^L \left( \tau, \frac{\kappa}{2} \right) \right) = Q^+ + Q^-
\]
where \( Q^+ = Q^+(k; z) \) is the holomorphic part of \( Q(k; z) \) and \( Q^- = Q^-(k; z) \) is the nonholomorphic part of \( Q(k; z) \). Let \( Q(k; z) \) have the Fourier development as follows:

\[
Q(k; z) = 2q^{-\alpha} + c_0^+(0) + \sum_{\beta} c_0^+(n) q^n + \sum_{\beta} c_0^-(n) \Gamma(\kappa - 1, 4\pi ny) q^{-n}.
\]

Now we are ready to state our main results.

**Theorem 1.3.** With the same notations as above the following assertions are true.

1. Let

\[
f_k |_{k+\frac{1}{2}} T(I^2) = \lambda_k (I^2) f_k
\]

for some \( \lambda_k (I^2) \in \mathbb{C} \). Then one has

\[
\lambda_k (I^2) = a_k (I^2 \alpha) + \left( \frac{(-1)^{k} \alpha}{l} \right) t^{k-1}.
\]

2. We have

\[
Q |_{\frac{1}{2} - k} T(I^2) - l^{1-2k} \lambda_k (I^2) Q = Q^+ |_{\frac{1}{2} - k} T(I^2) - l^{1-2k} \lambda_k (I^2) Q^+ \in \mathbb{M}_{\frac{1}{2} - k}^1.
\]

**Theorem 1.4.** For an odd prime \( l \), the following assertions are true.

1. We have

\[
c_0^+(I^2 \beta_0) \in \mathbb{Z}[c_0^0(\beta_0)] \subseteq \mathbb{Q}(c_0^0(\beta_0)).
\]

2. Assume that \( c_0^0(\beta_0) \) is irrational. Then

\[
a_k (I^2 \alpha_k) = t^{k-1} \left( \frac{(-1)^{k-1} \beta_k}{l} - \frac{(-1)^{k} \alpha_k}{l} \right) \text{ if and only if } c_0^+(I^2 \beta_k) \in \mathbb{Q}.
\]

3. Assume that \( \left( \frac{-\beta_k}{l} \right) = \left( \frac{\alpha_k}{l} \right) \) and \( c_0^0(\beta_0) \) is irrational. Then

\[
a_k (I^2 \alpha_k) = 0 \text{ if and only if } c_0^+(I^2 \beta_k) \in \mathbb{Q}.
\]

**Remark 1.5.** For simplicity, we dealt with the case \( p = 1 \) in our main results. But we remark that they can be extended to higher level cases whenever \( \dim \mathcal{S}_{k+rac{1}{2}} (p) = 1 \).

## 2 Proof of Theorem 1.3

First we are in need of two lemmas and one more fact.

**Lemma 2.1 (Lemma 4.1).** Let \( \kappa = k + \frac{1}{2} \) for an integer \( k > 2 \) and let \( D \in \mathbb{D}_k \). Then the following assertions are true.

(a) For each \( G \in H_{2-\kappa, D_{k+rac{1}{2}}} \), we have

\[
(4p)^{\kappa-1} \Phi \circ \xi_{2-\kappa} (G) = \xi_{2-\kappa} \circ \Phi (G).
\]

(b) For each \( f = \sum_{n=1} \Phi (n) q^n \in \mathcal{S}_\kappa (p) \),

\[
(f, (4p)^{\kappa-1} \Phi \xi_{2-\kappa} (F^{l-\frac{1}{2}}_{\beta, \frac{-\alpha}{\pi}} (\tau, \frac{\kappa}{2}))) = \begin{cases} \frac{3}{4 \pi (D)} \cdot \Phi (|D|), & \text{if } \frac{p}{p} = 1 \\ \frac{3}{4 \pi (D)} \cdot \Phi (|D|), & \text{if } \frac{p}{p} = 2 \\ \frac{6}{4 \pi (D)} \cdot \Phi (|D|), & \text{if } \frac{p}{p} > 2. \end{cases}
\]
Lemma 2.2 ([4, Lemma 4.2]). With the same notations as in Lemma 2.1, we have the following assertions.

(a) For a vector valued function $h = h_\beta(\tau) e_\beta$, one has

$$
\xi_{2,-\kappa}(h|_{\kappa=\kappa)}^{|M, \phi}) = (\xi_{2,-\kappa}(h)|_{\kappa)}^{|M, \phi}).
$$

(b) $\xi_{2,-\kappa}(1, 4\pi ny) = -(4\pi n)^{\kappa-1} e^{-4\pi ny}$.

(c) Let $m = -\frac{\left|\tau\right|}{\pi}$. Then one has

$$
\xi_{2,-\kappa}(F_{\beta,m}^{|\kappa-\kappa/2}) = \left(\frac{(4\pi |m|)^{\kappa-1}}{\Gamma(\kappa-1)} p_{\beta,|m|}^{|\tau})\right).
$$

Fact 6. Let $\kappa = k + \frac{1}{2}$.

(1) It follows from Lemmas 2.1 and 2.2 that

$$
\Phi\left(\xi_{2,-\kappa}(F_{\kappa}^{|\kappa-\kappa/2})\right) = \Phi\left(\frac{(\pi |D_k|^\kappa-1}{\Gamma(\kappa-1)} p_{\kappa}^{|\kappa/2})\right) = \left(\frac{(\pi |D_k|^\kappa-1}{\Gamma(\kappa-1)} P_{D_k}^{|\tau})\in \mathbb{S}_{\kappa+\frac{1}{2}}.
$$

(2) We obtain from Theorem 1.1-(ii) that

$$
(f_k, c_{k,D_k} P_{D_k}^{|}) = a_{f_k}(|D_k|) = 1
$$

where $c_{k,D_k} = \frac{(4\pi |D_k|)^{\kappa-1}}{\Gamma(\kappa-1)} \in \mathbb{R}$. 

For $k \in \{6, 8, 9, 10, 11, 13\}$, it follows from Fact 3, Fact 4, and Theorem 1.1-(iii), (iv) that $P_{D_k}^{|}$ does not vanish and

$$
P_{D_k}^{|} = c_k f_k
$$

for some $c_k \in \mathbb{C}^\times$. Thus one has from Fact 6 (2) that

$$
1 = (f_k, c_{k,D_k} c_k f_k) = \overline{c_k c_{k,D_k} ||f_k||^2,
$$

which implies

$$
c_k = c_k|^{-1} ||f_k||^{-2}.
$$

We compute that

$$
\xi_{2,-\kappa}(Q(k,z)) = \xi_{2,-\kappa}(\Phi\left(F_{\kappa}^{|\kappa-\kappa/2})\right))
$$

by Lemma 2.1-(a)

$$
= \left(4^{\kappa-1} \frac{(4\pi |D_k|)^{\kappa-1}}{\Gamma(\kappa-1)} P_{D_k}^{|(|\tau})\right)
$$

by Fact 6 (1)

$$
= \left(4^{\kappa-1} \frac{(4\pi |D_k|)^{\kappa-1}}{\Gamma(\kappa-1)} c_{k,D_k}||f_k||^{-2} f_k
$$

$$
= 3 ||f_k||^{-2} f_k.
$$

Since

$$
f_k|_{\kappa+\frac{1}{2}} T(|\tau|) = \sum_{n=0}^{\infty} \left(a_{f_k}(i^2 n) + \left(\frac{(-1)^k n}{l} \right) t^{k-1} a_{f_k}(n) + t^{2k-1} a_{f_k}(n/l^2)\right) q^n
$$

$$
= \lambda_k(|\tau|) f_k,
$$

one has

$$
\lambda_k(|\tau|) = a_{f_k}(i^2) + \left(\frac{(-1)^k}{l} \right) t^{k-1},
$$
Indeed, we observe that which combined with (8) yields the second assertion. Thus we have which implies that

\[
\zeta_2(4) = \sum_{n \geq 0, 1} (4\pi n)^{\zeta - 1} c Q(n) q^n,
\]

which proves the first assertion. Hence for all \( n \geq 1 \) with \((-1)^k n \equiv 0, 1 \pmod{4}\)

\[
\left( a_k(T^2) + \left( \frac{-1}{4} \right) k^{n} \right) a_k(n) = a_k(T^2 n) + \left( \frac{-1}{4} \right) k^{n} a_k(n) + \left( \frac{-1}{4} \right) k^{n-1} a_k(n/4^k).
\]

It follows from (5) that

\[
3 || f_k ||^{-2} f_k(z) = \xi_{2-k}(Q(k; z)) - \sum_{n \equiv 0, 1} (4\pi n)^{\zeta - 1} c Q(n) q^n,
\]

which implies that

\[
c Q(n) = -3 || f_k ||^{-2} \cdot (4\pi n)^{\zeta - 1} \cdot a_k(n).
\]

Now we put \( d_k := -3 || f_k ||^{-2} \cdot (4\pi)^{\zeta - 1} \). We obtain that for all positive integers \( n \) with \((-1)^k n \equiv 0, 1 \pmod{4}\),

\[
\begin{align*}
n^{\zeta - 1} (c Q(n^2)(n^2)^{\zeta - 1} + \left( \frac{-1}{4} \right)^k n \Gamma k c Q(n) n^{\zeta - 1}) \\
= n^{\zeta - 1} d_k \left( a_k(n^2) + a_k(n^2)^{\zeta - 1} \cdot \left( \frac{-1}{4} \right)^k n, n^{\zeta - 1} \cdot a_k(n) \right) \quad \text{by (7)} \\
= n^{\zeta - 1} d_k \left( a_k(n^2) + \left( \frac{-1}{4} \right)^k n \right) a_k(n) \quad \text{by (6)} \\
= \lambda_k(n^2) c Q(n).
\end{align*}
\]

Thus we have

\[
\begin{align*}
Q^{-1} \left[ \zeta_{2-k}(T^2) \right] = & \sum_{n \in \mathbb{Z}} \left( c Q(n^2) + \left( \frac{-1}{4} \right)^k n \Gamma k c Q(n) + 1^{\zeta - 1} \cdot c Q(n^2)^{\zeta - 1} \right) \Gamma(\zeta - 1, 4\pi n) q^n \\
= & (1^{\zeta - 1} \sum_{n \in \mathbb{Z}} c Q(n^2) d_k(n) + \left( \frac{-1}{4} \right)^k n \Gamma k c Q(n) + c Q(n^2)) \Gamma(\zeta - 1, 4\pi n) q^n \\
= & (1^{\zeta - 1} \sum_{n \in \mathbb{Z}} d_k(n)^{\zeta - 1} \left( c Q(n^2)(n^2)^{\zeta - 1} \cdot a_k(n) + a_k(n^2)^{\zeta - 1} \cdot \left( \frac{-1}{4} \right)^k n \right) \Gamma(\zeta - 1, 4\pi n) q^n \\
= & \left( 1^{\zeta - 1} \sum_{n \in \mathbb{Z}} d_k(n)^{\zeta - 1} \left( a_k(n^2) \cdot \left( \frac{-1}{4} \right)^k n \right) a_k(n) \right) \Gamma(\zeta - 1, 4\pi n) q^n \\
= & (1^{\zeta - 1} \sum_{n \in \mathbb{Z}} \lambda_k(n^2) c Q(n)) \Gamma(\zeta - 1, 4\pi n) q^n \quad \text{since } f_k \left[ T_{k^{\zeta - 1}}(T^2) \right] = \lambda_k(n^2) f_k \\
= & (1^{\zeta - 1} \lambda_k(n^2)) Q^{-1} - \lambda_k(n^2) Q^{-1}.
\end{align*}
\]

We obtain that

\[
\begin{align*}
1^{\zeta - 1} \xi_{2-k}(Q_{2-k}(T^2)) &= \left( \xi_{2-k}(Q) \right) \lambda_k(T^2) \quad \text{by Fact 5} \\
= & (1^{\zeta - 1} \sum_{n \in \mathbb{Z}} c Q(n^2) d_k(n) + \left( \frac{-1}{4} \right)^k n \Gamma k c Q(n) + c Q(n^2)) \Gamma(\zeta - 1, 4\pi n) q^n \\
= & (1^{\zeta - 1} \sum_{n \in \mathbb{Z}} d_k(n)^{\zeta - 1} \left( c Q(n^2)(n^2)^{\zeta - 1} \cdot a_k(n) + a_k(n^2)^{\zeta - 1} \cdot \left( \frac{-1}{4} \right)^k n \right) \Gamma(\zeta - 1, 4\pi n) q^n \\
= & (1^{\zeta - 1} \sum_{n \in \mathbb{Z}} d_k(n)^{\zeta - 1} \left( a_k(n^2) \cdot \left( \frac{-1}{4} \right)^k n \right) a_k(n) \right) \Gamma(\zeta - 1, 4\pi n) q^n \\
= & (1^{\zeta - 1} \sum_{n \in \mathbb{Z}} \lambda_k(n^2) c Q(n)) \Gamma(\zeta - 1, 4\pi n) q^n \quad \text{since } f_k \left[ T_{k^{\zeta - 1}}(T^2) \right] = \lambda_k(n^2) f_k \\
= & (1^{\zeta - 1} \lambda_k(n^2)) Q^{-1} - \lambda_k(n^2) Q^{-1}.
\end{align*}
\]

Indeed, we observe that

\[
\lambda_k(n^2) = a_k(n^2) + \left( \frac{-1}{4} \right)^k n \Gamma k a_k(n) \in \mathbb{Z}.
\]

Thus we have

\[
\begin{align*}
1^{\zeta - 1} \xi_{2-k}(Q_{2-k}(T^2)) &= \lambda_k(n^2) Q \\
\lambda_k(T^2) &= \lambda_k(n^2) Q \quad \text{since } \lambda_k(n^2) \in \mathbb{R}.
\end{align*}
\]

which combined with (8) yields the second assertion.
3 Proof of Theorem 1.4

We observe that

\[ Q|_{\frac{1}{2} - k} T(I^2) - l^{1 - 2k} \lambda_k(I^2) Q = Q'|_{\frac{1}{2} - k} T(I^2) - l^{1 - 2k} \lambda_k(I^2) Q^* \]

\[ = \left( 2q^{-\alpha} + c_0^\beta(0) + \sum_{n \equiv 1 \pmod{4}} c_0^\beta(n) q^n \right) |_{\frac{1}{2} - k} T(I^2) \]

\[ - l^{1 - 2k} \lambda_k(I^2) \left( 2q^{-\alpha} + c_0^\beta(0) + \sum_{n \equiv 1 \pmod{4}} c_0^\beta(n) q^n \right) \]

\[ = 2l^{1 - 2k} q^{-\alpha} t^2 + 2 \left( \left( \frac{-1}{k} \right) \frac{t}{l} - l^{1 - 2k} \lambda_k(I^2) \right) q^{-\alpha} + c_0^\beta(0)(1 + l^{1 - 2k} - l^{1 - 2k} \lambda_k(I^2)) \]

\[ + \sum_{n \equiv 1 \pmod{4}} c_0^\beta(n) q^n \left( \left( \frac{-1}{k} \right) \frac{n}{l} - l^{1 - 2k} \lambda_k(I^2) c_0^\beta(n) \right) q^n \]

\[ = 2l^{1 - 2k} f_{\frac{1}{2} - k, \alpha, t}Q^* \text{ since } -\alpha \geq -3. \]

So we find that

\[ 2f_{\frac{1}{2} - k, \alpha, t} = l^{2k - 1} Q^* |_{\frac{1}{2} - k} T(I^2) - \lambda_k(I^2) Q^* \]

has integral coefficients and for all positive integers \( n \) with \((-1)^{k-1} n \equiv 0, 1 \pmod{4}, \)

\[ l^{2k - 1} c_0^\beta(I^2 n) + \left( \frac{(-1)^{k-1} n}{l} \right) l^{k-1} c_0^\beta(n) + c_0^\beta(n/l^2) - \lambda_k(I^2) c_0^\beta(n) \]

\[ = c_0^\beta(n) \left( \left( \frac{-1}{k} \right) \frac{n}{l} - l^{k-1} - a_0^\beta(I^2 n) \right) + l^{2k-1} c_0^\beta(I^2 n) + c_0^\beta(n/l^2) \in 2\mathbb{Z}. \]

Then for \( n = \beta_k \) with

\[ \beta_k = \begin{cases} 3, & k \text{ even,} \\ 1, & k \text{ odd,} \end{cases} \]

we obtain that

\[ c_0^\beta(\beta_k) \left( \left( \frac{-1}{k} \frac{\beta_k}{l} \right) - \left( \frac{-1}{k} \frac{\alpha}{l} \right) \right) l^{k-1} - a_0^\beta(I^2 \alpha) \right) + l^{2k-1} c_0^\beta(I^2 \beta_k) \in 2\mathbb{Z}. \]

As a consequence of the above identity we get the assertions.

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**References**


