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Research Article

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On the algebraicity of coefficients of half-integral weight mock modular forms

Abstract: Extending works of Ono and Boylan to the half-integral weight case, we relate the algebraicity of Fourier coefficients of half-integral weight mock modular forms to the vanishing of Fourier coefficients of their shadows.

Keywords: Weakly holomorphic modular forms

MSC: 11F11, 11F67, 11F37

1 Introduction and statement of results

Let $k$ be an integer greater than 1 and let $N$ be a positive integer. The space of cusp forms of weight $2k$ for $\Gamma_0(N)$ is denoted by $S_{2k}(N)$. Throughout this paper let $p = 1$ or a prime number. For $\kappa \in \mathbb{Z} + \frac{1}{2}$ we denote by $M_{\kappa}^{!}(\Gamma_0(4p))$ the space of weakly holomorphic modular forms of weight $\kappa$ on $\Gamma_0(4p)$. As usual, $M_{\kappa}(\Gamma_0(4p))$ (resp. $S_{\kappa}(\Gamma_0(4p))$) stands for the space of weight $\kappa$ modular forms (resp. cusp forms) on $\Gamma_0(4p)$. Let $H_{\kappa}(\Gamma_0(4p))$ be the space of weight $\kappa$ harmonic weak Maass forms on $\Gamma_0(4p)$. Let $M_{\kappa, n}^{\!}(p)$ (resp. $H_{2 - \kappa}(\Gamma_0(4p))$) denote the subspace of $M_{\kappa}^{!}(\Gamma_0(4p))$ (resp. $H_{2 - \kappa}(\Gamma_0(4p))$), in which each form satisfies Kohnen’s plus space condition, that is, its Fourier expansion is supported only on those $n \in \mathbb{Z}$ for which $(-1)^{k-1} n \equiv 0 \mod 4p$.

Let $\Delta(\tau) \in S_{12}(1)$ be the Ramanujan’s Delta function. The famous Lehmer’s conjecture states that the Fourier coefficients of $\Delta(\tau)$ never vanish. Concerning this conjecture, Ono [1] related the algebraicity of Fourier coefficients of weight $-10$ mock modular form whose shadow is $\Delta(\tau)$ to the vanishing of Fourier coefficients of $\Delta(\tau)$. Generalizing Ono’s results, Boylan [2] related the algebraicity of Fourier coefficients of weight $2 - 2k$ mock modular forms to the vanishing of Fourier coefficients of their shadows when $\dim S_{2k}(1) = 1$. In this paper we will extend their works to the half-integral weight case. In the following we recall some known facts.

Fact 1. By Shimura correspondence [3, Proposition 1] we have

$$\dim S_{\frac{k+1}{2}}(1) = \dim S_{2k}(1),$$

which implies that

$$\dim S_{\frac{k+1}{2}}(1) = 1 \iff k = 6, 8, 9, 10, 11, 13 \iff 1 - k = -5, -7, -8, -9, -10, -12.$$
Now we assume that $k \in \{6, 8, 9, 10, 11, 13\}$. For $\kappa > 2$ there is an antilinear differential operator $\xi_{2-\kappa} : H_{2-\kappa}(\Gamma_0(4p)) \to S_\kappa(\Gamma_0(4p))$ defined by

$$\xi_{2-\kappa}(f)(\tau) := 2iy^{2-\kappa} \frac{df}{d\tau}.$$  

**Fact 2.** For $k \in \{6, 8, 9, 10, 11, 13\}$, it follows from [4, Theorem 1.1-(iii) and Lemma 4.2-(c)] that

$$\xi_{\frac{k}{2}-k} : H_{\frac{k}{2}-1}(1) \to S_{k+\frac{1}{2}}(1)$$  

is surjective.

For any $\kappa \in \mathbb{Z} + \frac{1}{2}$, the Duke-Jenkins basis [5] for $\mathcal{M}_\kappa^! := \mathcal{M}_\kappa^!(1)$ is constructed as follows. Let $2\kappa - 1 = 12\ell_\kappa + k'$ with uniquely determined $\ell_\kappa \in \mathbb{Z}$ and $k' \in \{0, 4, 6, 8, 10, 14\}$. If $A_\kappa$ denotes the maximal order of a non-zero $f \in \mathcal{M}_\kappa^!$ at $i\infty$, then by the Shimura correspondence [3] one has

$$A_\kappa = \begin{cases} 
2\ell_\kappa - (1)^{\kappa-1/2} & \text{if } \ell_\kappa \text{ is odd,} \\
2\ell_\kappa & \text{otherwise.}
\end{cases} \quad (1)$$

A basis for $\mathcal{M}_\kappa^!$ then consists of functions of the form

$$f_{\kappa,m}(\tau) = q^{-m} + \sum_{n \in A_\kappa} a_\kappa(m, n) q^n,$$  

where $m \geq -A_\kappa$ satisfies $(-1)^{\kappa-3/2}m \equiv 0, 1 \pmod{4}$. Using (1) and (2) we deduce the following facts.

**Fact 3.** For $\kappa = \frac{\kappa}{2} - k$ with $k \in \{6, 8, 9, 10, 11, 13\}$, the maximal order $A_\kappa$ of a non-zero $f \in \mathcal{M}_\kappa^!$ at $i\infty$ is given by $A_\kappa = -4$. Thus for each $m \geq 4$ satisfying $(-1)^{\kappa} m \equiv 0, 1 \pmod{4}$, there exist unique modular forms $f_{\frac{k}{2} - k, m}(\tau) \in \mathcal{M}_{\frac{k}{2} - k}$ with Fourier development

$$f_{\frac{k}{2} - k, m}(\tau) = q^{-m} + \sum_{n \geq -3} a_{\frac{k}{2} - k}(m, n) q^n,$$

which form a basis for the space $\mathcal{M}_{\frac{k}{2} - k}$.

**Fact 4.** For $\kappa = k + \frac{1}{2}$ with $k \in \{6, 8, 9, 10, 11, 13\}$, the space $S_\kappa$ is spanned by

$$f_k := f_{\kappa, -\alpha}$$

where $\alpha$ is given by

$$\alpha = \alpha_k = A_\kappa = A_{k + \frac{1}{2}} = \begin{cases} 1, & k \text{ even,} \\
3, & k \text{ odd,}
\end{cases}$$

and $f_k$ has the form $q^\alpha + O(q^k)$.

**Fact 5.** Let $\kappa = k + \frac{1}{2}$ and $f(z) \in H_\kappa(\Gamma_0(4p))$ with Fourier expansion $f(\tau) = \sum_{n \in \mathbb{Z}} c(y; n) e^{2\pi i nx}$ where $\tau = x + iy$. For each prime $l$ with $\gcd(1, 4p) = 1$, the $l^2$-th Hecke operator is defined by

$$f|_l T(l^2)(\tau) = \sum_{n \in \mathbb{Z}} c(y/P; nl^2) + \left(\frac{-1}{l}\right) l^{k-1} c(y; n) + l^{2k-1} c(P^2 y; n/l^2) \right) e^{2\pi inx}.$$  

Then for each $\mathcal{M} \in H_{2-\kappa}(p)$, we obtain from [6, (2.6)] or [7, (7.2)] that for $\kappa > 2$,

$$\xi_{2-\kappa}(\mathcal{M})|_l T(l^2) = l^{2\kappa - 2} \xi_{2-\kappa}(\mathcal{M}|_{2-\kappa} T(l^2)).$$

As a corollary of Fact 5, one has that if $f(z) = \sum_{n \in \mathbb{Z}} a_f(n) q^n \in S_{k+\frac{1}{2}}$, then

$$f|_{k+\frac{1}{2}} T(l^2) = \sum_{n \in \mathbb{Z}} \left( a_f(l^2 n) + \left(\frac{-1}{l}\right) l^{k-1} a_f(n) + l^{2k-1} a_f(n/l^2) \right) q^n \in S_{k+\frac{1}{2}}.$$
(or see [3, Theorem 1-(i)].)

Let \( (V, Q) \) be a non-degenerate rational quadratic space of signature \((b^+, b^-)\) and \(L\) an even lattice with dual \(L^\perp\). Denote the standard basis elements of the group algebra \( \mathbb{C}[L^\perp/L] \) by \( e_\gamma \) for \( \gamma \in L^\perp/L \). Let \( \text{Mp}_2(\mathbb{Z}) \) denote the integral metaplectic group, which consists of pairs \((\gamma, \phi)\), where \( \gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \text{SL}_2(\mathbb{Z}) \) and \( \phi : \mathbb{H} \to \mathbb{C} \) is a holomorphic function with \( \phi(\tau)^2 = c\tau + d \). It is well known that \( \text{Mp}_2(\mathbb{Z}) \) is generated by \( S = \left( \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right), \sqrt{T} \) and \( T = \left( \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right) \). Then there is a unitary representation \( \rho_L \) of the group \( \text{Mp}_2(\mathbb{Z}) \) on \( \mathbb{C}[L^\perp/L] \), the so-called Weil representation, which is defined by

\[
\rho_L(T)(e_\gamma) := e(Q(\gamma))e_\gamma,
\]

\[
\rho_L(S)(e_\gamma) := \frac{e((b^- - b^+)/8)}{\sqrt{|L^\perp/L|}} \sum_{\delta L^\perp/L} e(-\gamma, \delta)e_\delta,
\]

where \( e(z) := e^{2\pi iz} \) and \((X, Y) := Q(X + Y) - Q(X) - Q(Y)\) is the associated bilinear form. One has the relations

\[
S^2 = (ST)^3 = Z \quad (Z = ((-1/0 -1), i))
\]

from which we note that

\[
\rho_L(Z)e_\gamma = (b^- - b^+)e_\gamma.
\]

We write \( \langle \cdot, \cdot \rangle \) for the standard scalar product on \( \mathbb{C}[L^\perp/L] \), i.e.

\[
\langle \sum_{\gamma L^\perp/L} \lambda_\gamma e_\gamma, \sum_{\gamma L^\perp/L} \mu_\gamma e_\gamma \rangle := \sum_{\gamma L^\perp/L} \lambda_\gamma \overline{\mu_\gamma}.
\]

For \( \gamma, \delta \in L^\perp/L \) and \((M, \phi) \in \text{Mp}_2(\mathbb{Z})\) the coefficient \( \rho_{\gamma, \delta}(M, \phi) \) of the representation \( \rho_L \) is defined by

\[
\rho_{\gamma, \delta}(M, \phi) = \langle \rho_L(M, \phi)e_\delta, e_\gamma \rangle.
\]

Following [9], for an integer \( r \) we denote by \( H_{r+1/2, \rho_L} \) (resp. \( M^r_{r+1/2, \rho_L} \)), the space of \( \mathbb{C}[L^\perp/L] \)-valued harmonic weak Maass forms (resp. weakly holomorphic modular forms) of weight \( r + 1/2 \) and type \( \rho_L \).

Let \( L_r \) be the lattice \( 2p\mathbb{Z} \) of signature \((1, 0)\) (resp. \((0, 1)\)) when \( r \) is even (resp. odd) equipped with the quadratic form \( Q_r(x) = (-1)^r x^2/4p \). Then its dual lattice \( L_r^\perp \) is equal to \( \mathbb{Z} \). For a vector valued modular form \( F = \sum F_\gamma e_\gamma \), we define a map \( \phi \) by

\[
\phi(F)(\tau) := \sum_{\gamma \in \mathbb{H}} F_\gamma(4p\tau).
\]

It then follows from [8, Theorem 1] that the map \( \Phi \) defines an isomorphism from \( H_{r+1/2, \rho_L} \) to \( \mathbb{H}_{r+1/2}(p) \) since \( \rho_L = \rho_{L_{r+1/2}} \).

For a \( \mathbb{C}[L^\perp/L] \)-valued function \( f \) and \((M, \phi) \in \text{Mp}_2(\mathbb{Z})\) we define the Petersson slash operator by

\[
(f^L_{r+1/2}(M, \phi))(\tau) = \phi(\tau)^{-2r-1}(M, \phi)^{-1}f(M\tau).
\]

Let \( L := L_k \) and \( Q := Q_k \). Following [9] we define the vector valued cuspidal Poincaré series \( P^L_{\beta, \eta}(\tau) \) as follows: for each \( \beta \in \mathbb{Z}/2p\mathbb{Z} \) and \( n \in \mathbb{Z} + Q(\beta) \) with \( n > 0 \),

\[
P^L_{\beta, \eta}(\tau) := \frac{1}{2} \sum_{(M, \phi) \in \text{Mp}_2(\mathbb{Z})} \epsilon_\beta(n\tau)|_{\kappa}(M, \phi).
\]

Then we know from [9] that \( P^L_{\beta, \eta}(\tau) \) belongs to the space \( S_{\kappa, \rho_L} \). Let \( \mathcal{D}_k \) denote the set of all integers \( D \) such that \((-1)^k D > 0 \) and \( D \) is congruent to a square modulo \( 4p \).

**Theorem 1.1** ([4, Theorem 1.1]). For an integer \( k > 2 \) we let \( \kappa = k + \frac{1}{2} \) and \( L := L_k \). For each \( D \in \mathcal{D}_k \), we define

\[
P_D^* := \Phi(P^L_{\beta, \frac{D}{4p}}),
\]

where \( \beta \) is an integer such that \( D \equiv \beta^2 \mod{4p} \). Then the following assertions are true.

(i) \( P_D^* \in \mathbb{S}_\kappa(p) \) and the definition of \( P_D^* \) does not depend on the choice of \( \beta \).
(ii) For each \( f = \sum_{n \geq 1} a_f(n)q^n \in \mathcal{S}_\kappa(p) \), we have

\[
(f, c_{k,D}P_D^n) = a_f(|D|)
\]

where \((\cdot, \cdot)\) denotes the Petersson inner product and

\[
c_{k,D} = \begin{cases} 
\frac{(4\pi|D|)^{-\frac{k-1}{2}}}{T(\kappa-1)} \cdot \frac{s(D)}{3}, & \text{if } p = 1 \\
\frac{(4\pi|D|)^{-\frac{k-1}{2}}}{T(\kappa-1)} \cdot \frac{s(D)}{4}, & \text{if } p = 2 \\
\frac{(4\pi|D|)^{-\frac{k-1}{2}}}{T(\kappa-1)} \cdot \frac{s(D)}{6}, & \text{if } p > 2
\end{cases}
\]

with \( s(D) = \begin{cases} 
1, & \text{if } p|D, \\
2, & \text{otherwise.}
\end{cases} \)

(iii) The set \( \{ P_D^n \mid D \in \mathcal{D}_k \} \) spans the space \( \mathcal{S}_\kappa(p) \). Moreover, if we let \( t = \dim \mathcal{S}_\kappa(p) \) and \( \{ f_1, f_2, \ldots, f_t \} \) be a basis for \( \mathcal{S}_\kappa(p) \) satisfying \( f_i = q^{|D_i|} + O(q^{|D_i|+1}) \) for some \( D_i \in \mathcal{D}_k \) \( (i = 1, \ldots, t) \) and \( 0 < |D_1| < |D_2| < \cdots < |D_t| \), then the set

\[
\{ P_{D_1}^n, P_{D_2}^n, \ldots, P_{D_t}^n \}
\]

forms a basis for \( \mathcal{S}_\kappa(p) \).

(iv) Let \( \mathcal{B} \) be a nonempty finite subset of \( \mathbb{N} \). Then the following two conditions are equivalent.

(a) \( \sum \alpha_i P_{D_i}^n(\tau) = 0 \) for some \( \alpha_i \in \mathbb{C} \) and \( D_i \in \mathcal{D}_k \).

(b) There exists \( g \in M_{2-k}(p) \) with the principal part \( \sum \alpha_i |D_i|^{1-k} q^{-|D_i|} \).

Remark 1.2. Let \( p = 1 \) and take

\[
D_k = \begin{cases} 
1, & \text{if } k \text{ is even} \\
-3, & \text{if } k \text{ is odd.}
\end{cases}
\]

Then in Theorem 1.1 one can choose \( \beta = 1 \) and

\[
P_{D_k}^n := \phi(P_{D_k}^n).
\]

We let \( \bar{T}_\infty := \langle T \rangle \). We define for \( s \in \mathbb{C} \) and \( y \in \mathbb{R} - \{ 0 \} : 

\[
\mathcal{M}_s(y) = y^{-(2-k)/2} M_{(2-k)/2,s-1/2}(y) \quad (y > 0), \\
\mathcal{W}_s(y) = \left| y^{-(2-k)/2} W_{\frac{2-k}{4},s-1/2}(y) \right| 
\]

where \( M_{r,m}(z) \) and \( W_{r,m}(z) \) denote the usual Whittaker functions. Now we take \( \kappa = k + 1/2 > 2 \), \( L := L_{1-k} \), and \( Q := Q_{1-k} \). For each \( \beta \in \mathbb{Z}/2p\mathbb{Z} \) and \( m \in \mathbb{Z} + Q(\beta) \) with \( m < 0 \), modifying the Poincaré series in [9, (1.35)] we define the vector valued Maass Poincaré series \( F_{\beta,m}^L \) of index \( (\beta, m) \) by

\[
F_{\beta,m}^L(\tau, s) := \frac{1}{2\Gamma(2s)} \sum_{(M,\phi)\in F_{\beta,m}(\mathbb{Z})} \left[ \mathcal{M}_s(4\pi|\phi|) \mathcal{W}_s(\phi) \right]_{k} \mathbb{H}(M, \phi)
\]

where \( \tau = x + iy \in \mathbb{H} \) and \( s = \sigma + it \in \mathbb{C} \) with \( \sigma > 1 \). Indeed, since \( \mathcal{M}_s(4\pi|\phi|) \mathcal{W}_s(\phi) \) is invariant under slash operator \( 2-k \cdot T \), the Maass Poincaré series is well defined. This series has desirable properties as follows. As in Section 1.3 in [9] it converges normally for \( \tau \in \mathbb{H} \) and \( s = \sigma + it \in \mathbb{C} \) with \( \sigma > 1 \) and hence defines a \( M_{2-k}(\mathbb{Z}) \)-invariant function on \( \mathbb{H} \) under the slash operator \( 2-k \cdot T \). Moreover, \( F_{\beta,m}^L(\tau, s) \) is an eigenfunction of \( \Delta_{2-k} \) with an eigenvalue \( s(1-s) + \kappa(\kappa - 2)/4 \). Since \( \mathcal{M}_s(\tau)_{2-k} \mathbb{Z} = e_{\beta} \) by (3), the invariance of \( F_{\beta,m}^L \) under the action of \( \mathcal{Z} \) implies \( F_{\beta,m}^L = F_{m-\beta,\beta}^L \).

Let \( \kappa = k + 1/2 \) and \( L = L_{1-k} \) with \( k \) an integer \( \geq 2 \). For each \( \beta \in \mathbb{Z}/2p\mathbb{Z} \) and \( m \in \mathbb{Z} + Q(\beta) \) with \( m < 0 \), we obtain from [4, Corollary 1.5] that \( F_{\beta,m}^L(\tau, \frac{\kappa}{2}) \) belongs to the space \( H_{2-k,p} \).

Let

\[
Q = Q(k; z) := \phi \left( F_{\beta,m}^L \left( \tau, \frac{\kappa}{2} \right) \right) = Q^+ + Q^- 
\]
where $Q^+ = Q^+(k; z)$ is the holomorphic part of $Q(k; z)$ and $Q^- = Q^-(k; z)$ is the nonholomorphic part of $Q(k; z)$. Let $Q(k; z)$ have the Fourier development as follows:

$$Q(k; z) = 2q^{-\alpha} + c_0(0) + \sum_{n \in \mathbb{N}} c_0^+(n) q^n + \sum_{n \in \mathbb{N}} c_0(n) \Gamma(\kappa - 1, 4\pi ny)q^{-n}.$$ 

Now we are ready to state our main results.

**Theorem 1.3.** With the same notations as above the following assertions are true.

(1) Let

$$f_k|_{k+\frac{1}{2}} T(l^2) = \lambda_k(l^2)f_k$$

for some $\lambda_k(l^2) \in \mathbb{C}$. Then one has

$$\lambda_k(l^2) = a_k(l^2\alpha) + \left(\frac{(-1)^k\alpha}{l}\right) l^{-1}.$$ 

(2) We have

$$Q|_{\frac{1}{2}-k} T(l^2) - l^{1-2k}\lambda_k(l^2) = Q^+|_{\frac{1}{2}-k} T(l^2) - l^{1-2k}\lambda_k(l^2) Q^+ \in \mathbb{M}_{\frac{1}{2}-k}.$$ 

**Theorem 1.4.** For an odd prime $l$, the following assertions are true.

(1) We have

$$c_0^+(l^2 \beta_k) \in \frac{\mathbb{Z}[c_0^+(\beta_k)]}{l^{2k-1}} \subseteq \mathbb{Q}(c_0^+(\beta_k)).$$

(2) Assume that $c_0^+(\beta_k)$ is irrational. Then

$$a_k(l^2\alpha_k) = l^{k-1} \left( \left(\frac{(-1)^{k-1}\beta_k}{l}\right) - \left(\frac{(-1)^k\alpha_k}{l}\right) \right) \text{ if and only if } c_0^+(l^2 \beta_k) \in \mathbb{Q}.$$ 

(3) Assume that $\left(\frac{-\beta_k}{l}\right) = \left(\frac{\alpha_k}{l}\right)$ and $c_0^+(\beta_k)$ is irrational. Then

$$a_k(l^2\alpha_k) = 0 \text{ if and only if } c_0^+(l^2 \beta_k) \in \mathbb{Q}.$$ 

**Remark 1.5.** For simplicity, we dealt with the case $p = 1$ in our main results. But we remark that they can be extended to higher level cases whenever $\dim \mathbb{S}_{k+\frac{1}{2}}(p) = 1$. 

### 2 Proof of Theorem 1.3

First we are in need of two lemmas and one more fact.

**Lemma 2.1 ([4, Lemma 4.1]).** Let $\kappa = k + \frac{1}{2}$ for an integer $k > 2$ and let $D \in \mathbb{D}_k$. Then the following assertions are true.

(a) For each $G \in H_{2-k, \mu_{k+1}}$, we have

$$(4p)^{k-1} \Phi \circ \xi_{2-k}(G) = \xi_{2-k} \circ \Phi(G).$$

(b) For each $f = \sum_{n \geq 1} c_f(n) q^n \in \mathbb{S}_\kappa(p)$,

$$(f, (4p)^{k-1} \xi_{2-k} F_{\beta, \nu \frac{k}{2}}^a (\tau \frac{\kappa}{2})) = \begin{cases} \frac{1}{4\pi(D)} \cdot c_f(|D|), & \text{if } p = 1 \\ \frac{1}{3\pi(D)} \cdot c_f(|D|), & \text{if } p = 2 \\ \frac{6}{7\pi(D)} \cdot c_f(|D|), & \text{if } p > 2. \end{cases}$$
Lemma 2.2 ([4, Lemma 4.2]). With the same notations as in Lemma 2.1, we have the following assertions.
(a) For a vector valued function \( h = \mathbf{h}(\tau) \in \mathbb{C} \), we have
\[
\bar{\xi}_{2-\kappa}(\mathbf{h}_{2-\kappa}(M, \phi)) = (\xi_{2-\kappa}(h))_{2-\kappa}(M, \phi).
\]
(b) \( \xi_{2-\kappa}(\Gamma(\kappa - 1, 4\pi ny)) = (4\pi n)^{\kappa-1} e^{-4\pi ny} \).
(c) Let \( m = -\frac{\lfloor m \rfloor}{y} \). Then one has
\[
\xi_{2-\kappa}(P_{\beta,m}^\kappa(\tau, \frac{\kappa}{2})) = \left(\frac{(4\pi m \kappa-1)}{\Gamma(\kappa - 1)}\right) P_{\beta,m}^\kappa(\tau) \in \mathbb{C}^{k+\frac{1}{2}}.
\]

Fact 6. Let \( p = 1 \) and \( \kappa = k + \frac{1}{2} \).
(1) It follows from Lemmas 2.1 and 2.2 that
\[
\phi \left( \bar{\xi}_{2-\kappa} \left( P_{\beta,m}^{k+\frac{1}{2}} \left( \frac{\kappa}{2} \right) \right) \right) = \phi \left( \left( \frac{\pi |D_k|^{\kappa-1}}{\Gamma(\kappa - 1)} \right) P_{\beta,m}^k(\tau) \right) = \left( \frac{\pi |D_k|^{\kappa-1}}{\Gamma(\kappa - 1)} \right) \bar{P}_{\beta,m}^k(\tau) \in \mathbb{C}^{k+\frac{1}{2}}.
\]
(2) We obtain from Theorem 1.1-(ii) that
\[
(f_k, c_{k,D_k} P_{D_k}^+) = 1
\]
where \( c_{k,D_k} = \frac{\left(\frac{\pi |D_k|^{\kappa-1}}{\Gamma(\kappa - 1)}\right)}{\bar{P}_{\beta,m}} \in \mathbb{R} \).

For \( k \in \{6, 8, 9, 10, 11, 13\} \), it follows from Fact 3, Fact 4, and Theorem 1.1-(iii), (iv) that \( P_{D_k}^+ \) does not vanish and
\[
P_{D_k}^+ = c_k f_k
\]
for some \( c_k \in \mathbb{C}^* \). Thus one has from Fact 6 (2) that
\[
1 = (f_k, c_{k,D_k} c_k f_k) = \bar{c_k} c_{k,D_k} ||f_k||^2,
\]
which implies
\[
c_k = c_{k,D_k} ||f_k||^{-2}.
\]

We compute that
\[
\xi_{2-\kappa}(Q(k; z)) = \xi_{2-\kappa}(\Phi \left( P_{\beta,m}^{k+\frac{1}{2}} \left( \frac{\kappa}{2} \right) \right)) = \left(\frac{\pi |D_k|^{\kappa-1}}{\Gamma(\kappa - 1)}\right) \bar{P}_{\beta,m}^k(\tau) \text{ by Lemma 2.1-(a)} = \left(\frac{\pi |D_k|^{\kappa-1}}{\Gamma(\kappa - 1)}\right) c_k f_k = 3 ||f_k||^{-2} f_k.
\]

Since
\[
f_k|_{k+\frac{1}{2}} T(l^2) = \sum_{n \in (\mathbb{Z}^* \cup \{0, 1\})_{(\kappa)}} \left( a_{k,l^2}(n) + \left(\frac{(-1)^k n}{l}\right) l^{k-1} a_{k,l}(n) \right) q^n = 3 ||f_k||^{-2} f_k.
\]

one has
\[
\lambda_k(l^2) = a_{k,l^2} + \left(\frac{(-1)^k}{l}\right) l^{k-1},
\]
Indeed, we observe that which combined with (8) yields the second assertion.

\[
\left(a_f(T^\alpha) + \left(\frac{(-1)^k\alpha}{l}\right)t^{l-1}\right) a_f(n) = a_f(T^n) + \left(\frac{(-1)^k\alpha}{l}\right)t^{l-1} a_f(n) + t^{2k-1} a_f(n/l^2).
\]

It follows from (5) that

\[
3||f_k||^{-2} f_k(z) = \xi_{2-\kappa}(Q(k; z)) = -\sum_{\lambda \in \mathbb{Z}} (4\pi n)^{\kappa-1} c_Q(n) q^n,
\]

which implies that

\[
c_Q(n) = -3||f_k||^{-2} \cdot (4\pi)^{1-\kappa} \cdot a_f(n).
\]

Now we put \(d_k := 3||f_k||^{-2} \cdot (4\pi)^{1-\kappa}\). We obtain that for all positive integers \(n \) with \((-1)^k n \equiv 0, 1 \pmod{4}\),

\[
n^{1-\kappa} \left( c_Q(n/l^2) (n/l^2)^{\kappa-1} + t^{2k-1} c_Q(n/l^2) (n/l^2)^{\kappa-1} + \left(\frac{(-1)^k\alpha}{l}\right) t^{l-1} c_Q(n) n^{\kappa-1} \right)
\]

\[
= n^{1-\kappa} d_k \left( a_f(n/l^2) + a_f(n/l^2)^{2k-1} + \left(\frac{(-1)^k\alpha}{l}\right) t^{l-1} a_f(n) \right) \quad \text{by (7)}
\]

\[
= n^{1-\kappa} d_k \left( a_f(T^\alpha) + \left(\frac{(-1)^k\alpha}{l}\right) t^{l-1} a_f(n) \right) \quad \text{by (6)}
\]

\[
= \lambda_k(T^\alpha) c_Q(n).
\]

Thus we have

\[
Q^{-1} T_k^{-1}(I^2) = \sum_{n \in \mathbb{Z}} \left( c_Q(n/l^2) + \left(\frac{(-1)^k\alpha}{l}\right) t^{l-1} c_Q(n) (n/l^2) \right) \Gamma(\kappa-1, 4\pi ny) q^n
\]

\[
= t^{l-2k} \sum_{n \in \mathbb{Z}} \left( c_Q(n/l^2) (n/l^2)^{\kappa-1} + \left(\frac{(-1)^k\alpha}{l}\right) t^{l-1} c_Q(n) + c_Q(n/l^2) \right) \Gamma(\kappa-1, 4\pi ny) q^n
\]

\[
= t^{l-2k} \sum_{n \in \mathbb{Z}} d_k n^{1-\kappa} \left( c_Q(n/l^2) (n/l^2)^{\kappa-1} + \left(\frac{(-1)^k\alpha}{l}\right) t^{l-1} c_Q(n) n^{\kappa-1} \right) \Gamma(\kappa-1, 4\pi ny) q^n
\]

\[
= t^{l-2k} \sum_{n \in \mathbb{Z}} \lambda_k(T^\alpha) c_Q(n) \Gamma(\kappa-1, 4\pi ny) q^n \quad \text{since} \ f_k|T_k^{-1/2} (I^2) = \lambda_k(T^\alpha) f_k
\]

\[
= t^{l-2k} \lambda_k(T^\alpha) Q^{-1}.
\]

We obtain that

\[
I^{2\kappa-2} \xi_{2-\kappa}(Q|_{2-\kappa} (T^2)) = (\xi_{2-\kappa}(Q)|_{\kappa} T(I^2)) \quad \text{by Fact 5}
\]

\[
= 3||f_k||^{-2} f_k|_{\kappa} (T(I^2)) \quad \text{by (5)}
\]

\[
= 3||f_k||^{-2} \lambda_k(T^\alpha) f_k
\]

\[
= \xi_{2-\kappa}(\lambda_k(T^\alpha) Q) \quad \text{since} \ \lambda_k(T^\alpha) \in \mathbb{R}.
\]

Indeed, we observe that

\[
\lambda_k(T^\alpha) = a_f(T^\alpha) + \left(\frac{(-1)^k\alpha}{l}\right) t^{l-1} \in \mathbb{Z}.
\]

Thus we have

\[
I^{2\kappa-2} Q|_{2-\kappa} (T(I^2)) - \lambda_k(T^\alpha) Q \in \mathbb{M}_{2-\kappa},
\]

which combined with (8) yields the second assertion.
3 Proof of Theorem 1.4

We observe that
\[ Q|_{l^{-k}} T(l^2) - l^{-1-2k} \lambda_k(l^2) Q = Q|_{l^{-k}} T(l^2) - l^{-1-2k} \lambda_k(l^2) Q^+ \]
\[ = \left( 2q^{-\alpha} + c_Q^0(0) + \sum_{n \equiv 0 \pmod{2} \land (n, l) = 1} c_Q^0(n) q^n \right) |_{l^{-k}} T(l^2) \]
\[ - l^{-1-2k} \lambda_k(l^2) \left( 2q^{-\alpha} + c_Q^0(0) + \sum_{n \equiv 0 \pmod{2} \land (n, l) = 1} c_Q^0(n) q^n \right) \]
\[ = 2l^{1-2k} q^{-\alpha} + 2 \left( \left( \frac{-1}{l} \right)^{k^{-1}} - l^{-1-2k} \lambda_k(l^2) \right) q^{-\alpha} + c_Q^0(0)(1 + l^{-1-2k} - l^{-1-2k} \lambda_k(l^2)) \]
\[ + \sum_{n \equiv 0 \pmod{2} \land (n, l) = 1} (c_Q^0(l^2 n) + \left( \frac{-1}{l} \right)^{k^{-1}}) l^{-k} c_Q^0(n) + l^{-1-2k} c_Q^0(n/l^2) - l^{-1-2k} \lambda_k(l^2) c_Q^0(n) \]
\[ = 2l^{1-2k} f_{l^{-k}, \alpha}^0 \text{ since } -\alpha \geq -3. \]

So we find that
\[ 2f_{l^{-k}, \alpha}^0 = l^{2k-1} Q^+ |_{l^{-k}} T(l^2) - \lambda_k(l^2) Q^+ \]
has integral coefficients and for all positive integers \( n \) with \( (-1)^{k^{-1}} n \equiv 0, 1 (\text{mod } 4), \)
\[ l^{k-1} c_Q^0(l^2 n) + \left( \frac{-1}{l} \right)^{k^{-1}} l^{-k} c_Q^0(n) + c_Q^0(n/l^2) - \lambda_k(l^2) c_Q^0(n) \]
\[ = c_Q^0(n) \left( \left( \frac{-1}{l} \right)^{k^{-1}} - \left( \frac{-1}{l} \right)^{k} \right) l^{-k-1} - a_h(l^2, \alpha) + l^{2k-1} c_Q^0(l^2 n) + c_Q^0(n/l^2) \in 2\mathbb{Z}. \]

Then for \( n = \beta_k \) with
\[ \beta_k = \begin{cases} 3, & \text{k even,} \\ 1, & \text{k odd,} \end{cases} \]
we obtain that
\[ c_Q^0(\beta_k) \left( \left( \frac{-1}{l} \right)^{k^{-1}} \beta_k \right) - \left( \frac{-1}{l} \right)^{k} \beta_k \left( \left( \frac{-1}{l} \right)^{k} \right) l^{-k-1} - a_h(l^2, \alpha) \in 2\mathbb{Z}. \]

As a consequence of the above identity we get the assertions.

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