

## Research Article

## Open Access

Blanka Baculikova\* and Jozef Dzurina

# The fourth order strongly noncanonical operators

<https://doi.org/10.1515/math-2018-0135>

Received May 22, 2018; accepted November 26, 2018

**Abstract:** It is shown that the strongly noncanonical fourth order operator

$$\mathcal{L}y = \left( r_3(t) \left( r_2(t) \left( r_1(t) y'(t) \right)' \right)' \right)'$$

can be written in essentially unique canonical form as

$$\mathcal{L}y = q_4(t) \left( q_3(t) \left( q_2(t) \left( q_1(t) (q_0(t)y(t))' \right)' \right)' \right)'$$

The canonical representation essentially simplifies examination of the fourth order strongly noncanonical equations

$$\left( r_3(t) \left( r_2(t) \left( r_1(t) y'(t) \right)' \right)' \right)' + p(t)y(\tau(t)) = 0.$$

**Keywords:** canonical operator, fourth order differential equations**MSC:** 34K11, 34C10

## 1 Introduction

In the paper, we consider the fourth order delay differential equation

$$\left( r_3(t) \left( r_2(t) \left( r_1(t) y'(t) \right)' \right)' \right)' + p(t)y(\tau(t)) = 0, \quad (E)$$

where  $r_i \in C^{(4-i)}(t_0, \infty)$ ,  $r_i(t) > 0$ ,  $i = 1, \dots, 3$ ,  $p > 0$ ,  $\tau(t) \leq t$ ,  $\tau'(t) > 0$  and  $\tau(t) \rightarrow \infty$  as  $t \rightarrow \infty$ .

Fourth-order differential equations naturally appear in models concerning physical, biological, and chemical phenomena, such as, for instance, problems of elasticity, deformation of structures, or soil settlement, for example, see [1]. In mechanical and engineering problems, questions concerning the existence of oscillatory solutions play an important role. During the past decades, there has been a constant interest in obtaining sufficient conditions for oscillatory properties of different class of fourth order differential equations with deviating argument, see [1–13].

**\*Corresponding Author: Blanka Baculikova:** Department of Mathematics, Faculty of Electrical Engineering and Informatics, Technical University of Košice, Letná 9, 042 00 Košice, Slovakia, E-mail: [blanka.baculikova@tuke.sk](mailto:blanka.baculikova@tuke.sk)

**Jozef Dzurina:** Department of Mathematics, Faculty of Electrical Engineering and Informatics, Technical University of Košice, Letná 9, 042 00 Košice, Slovakia, E-mail: [jozef.dzurina@tuke.sk](mailto:jozef.dzurina@tuke.sk)

As far as the oscillation theory of fourth-order differential equations is concerned, the problem of investigating ways of factoring disconjugated operators

$$\mathcal{L} y \equiv \left( r_3(t) \left( r_2(t) \left( r_1(t) y'(t) \right)' \right)' \right)', \tag{1.1}$$

which are crucial in studying the perturbed differential equations such as (E), has been of special interest. Motivated by the famous work of George Polya, Trench [2] showed that if operator  $\mathcal{L} y$  is *strongly noncanonical*, that is,

$$\int_t^\infty \frac{1}{r_i(s)} ds < \infty, \quad i = 1, 2, 3, \tag{1.2}$$

then it can be written in an essentially unique *canonical form* as

$$\mathcal{L} y = q_4(t) \left( q_3(t) \left( q_2(t) \left( q_1(t) (q_0(t)y(t))' \right)' \right)' \right)',$$

so that  $q_i \in C^{(4-i)}(t_0, \infty)$ ,  $q_i(t) > 0$ ,  $i = 0, \dots, 4$  and

$$\int_t^\infty \frac{1}{q_i(s)} ds = \infty, \quad i = 1, 2, 3.$$

However, the computation using the Lemmas 1 and 2 from [2] leading to canonical representation is very complicated and does not provide closed formulas for  $q_i(t)$ . This brings us to the question whether is it possible to establish a closed form formulas for  $q_i$ . The aim of this paper is to positively answer to this question, showing simultaneously the advantage of the result in the investigation of oscillatory properties of strongly noncanonical equations.

## 2 Main results

Throughout the paper we assume that (1.2) hold and so we can employ the notation

$$\pi_i(t) = \int_t^\infty \frac{1}{r_i(s)} ds, \quad \pi_{ij}(t) = \int_t^\infty \frac{1}{r_i(s)} \pi_j(s) ds$$

and

$$\pi_{ijk}(t) = \int_t^\infty \frac{1}{r_i(s)} \pi_{jk}(s) ds,$$

where  $i, j, k \in \{1, 2, 3\}$  are mutually different.

We start with the following auxiliary results which are elementary but very useful.

**Lemma 1.** *Let (1.2) hold. Then*

$$\pi_{ij}(t) + \pi_{ji}(t) = \pi_i(t)\pi_j(t).$$

*Proof.* Since

$$(\pi_i(t)\pi_j(t))' = -\frac{\pi_j(t)}{r_i(t)} - \frac{\pi_i(t)}{r_j(t)},$$

an integration of this equality from  $t$  to  $\infty$ , yields

$$\pi_i(t)\pi_j(t) = \int_t^\infty \frac{1}{r_i(s)} \pi_j(s) ds + \int_t^\infty \frac{1}{r_j(s)} \pi_i(s) ds = \pi_{ij}(t) + \pi_{ji}(t).$$

□

To simplify our notation, we denote

$$\Omega(t) = \pi_{12}(t)\pi_{23}(t) - \pi_2(t)\pi_{123}(t).$$

The following result provides an alternative formula for  $\Omega(t)$ .

**Lemma 2.** *Let (1.2) hold. Then*

$$\pi_{12}(t)\pi_{23}(t) - \pi_2(t)\pi_{123}(t) = \pi_{21}(t)\pi_{32}(t) - \pi_2(t)\pi_{321}(t). \tag{2.1}$$

*Proof.* Proof of this lemma is similar to that of Lemma 1 and so it can be omitted. □

Now, we are prepared to introduce the main result.

**Theorem 1.** *The strongly noncanonical operator  $\mathcal{L}y$  has the following unique canonical representation*

$$\mathcal{L}y = \frac{1}{\pi_{321}(t)} \left( \frac{r_3\pi_{321}^2}{\Omega} \left( \frac{r_2\Omega^2}{\pi_{321}\pi_{123}} \left( \frac{r_1\pi_{123}^2}{\Omega} \left( \frac{y}{\pi_{123}} \right)' \right)' \right)' \right)' (t). \tag{2.2}$$

*Proof.* Straightforward evaluation of  $\mathcal{L}y$ , with  $\mathcal{L}$  as defined by (2.2) yields

$$\mathcal{L}_1y = \frac{r_1\pi_{123}^2}{\Omega} \left( \frac{y}{\pi_{123}} \right)' = \frac{r_1y'\pi_{123} + y\pi_{23}}{\Omega}. \tag{2.3}$$

Employing (2.1), we see that

$$\begin{aligned} \mathcal{L}_2y &= \frac{\Omega^2}{r_2\pi_{321}\pi_{123}} (\mathcal{L}_1y)' \\ &= \frac{r_2 \left( r_1y' \right)' \Omega - r_1y' (\pi_{321} - \pi_1\pi_{32}) + y\pi_{32}}{\pi_{321}}. \end{aligned}$$

It follows from Lemma 2 that  $\pi_{12}(t) + \pi_{21}(t) = \pi_1(t)\pi_2(t)$  and so

$$(\mathcal{L}_2y)' = \frac{\left( r_2 \left( r_1y' \right)' \right)' \Omega - r_1y' \frac{1}{r_3}\pi_{12} - y \frac{1}{r_3}\pi_2}{\pi_{321}} + \frac{r_2 \left( r_1y' \right)' \Omega \pi_{21} + r_1y' (\pi_1\pi_{32}\pi_{21} - \pi_{321}\pi_{21}) + y\pi_{32}\pi_{21}}{r_3\pi_{321}^2}$$

On the other hand, by Lemma 1 and 2

$$\begin{aligned} r_1y' (\pi_1\pi_{32}\pi_{21} - \pi_{321}\pi_{21} - \pi_{12}\pi_{321}) &= r_1y' (\pi_1\pi_{32}\pi_{21} - \pi_{321}\pi_1\pi_2) \\ &= r_1y' \pi_1\Omega. \end{aligned}$$

Consequently,

$$(\mathcal{L}_2y)' = \frac{\left( r_2 \left( r_1y' \right)' \right)' \Omega}{\pi_{321}} + \frac{r_2 \left( r_1y' \right)' \Omega \pi_{21} + r_1y' \pi_1\Omega + y\Omega}{r_3\pi_{321}^2}.$$

Then

$$\mathcal{L}_3y = \frac{r_3\pi_{321}^2}{\Omega} (\mathcal{L}_2y)' = r_3 \left( r_2 \left( r_1y' \right)' \right)' \pi_{321} + r_2 \left( r_1y' \right)' \pi_{21} + r_1y' \pi_1 + y.$$

Finally

$$\frac{1}{\pi_{321}(t)} (\mathcal{L}_3y)' = \left( r_3(t) \left( r_2(t) \left( r_1(t)y'(t) \right)' \right)' \right)' = \mathcal{L}y,$$

which means that the operators (1.1) and (2.2) are equivalent.

Now we shall show that operator (2.2) is canonical. Direct computation shows that

$$\int_{t_0}^t \frac{1}{q_1(s)} ds = \int_{t_0}^t \frac{\Omega(s)}{r_1(s)\pi_{123}^2(s)} ds = \int_{t_0}^t \left( \frac{\pi_{12}(s)}{\pi_{123}(s)} \right)' ds = \frac{\pi_{12}(t)}{\pi_{123}(t)} - c_1.$$

Applying twice the L'Hospital rule, we get

$$\lim_{t \rightarrow \infty} \frac{\pi_{12}(t)}{\pi_{123}(t)} = \lim_{t \rightarrow \infty} \frac{1}{\pi_3(t)} = \infty$$

and so

$$\int_{t_0}^{\infty} \frac{1}{q_1(s)} ds = \infty.$$

Similarly

$$\int_{t_0}^t \frac{1}{q_3(s)} ds = \int_{t_0}^t \frac{\Omega(s)}{r_3(s)\pi_{321}^2(s)} ds = \left( \frac{\pi_{32}(t)}{\pi_{321}(t)} - c_3 \right) \rightarrow \infty \text{ as } t \rightarrow \infty.$$

To evaluate the last one integral it is useful to see that

$$\begin{aligned} \left( \frac{\pi_{321}(t)\pi_{23}(t)}{\pi_{32}(t)\Omega(t)} + \frac{\pi_3(t)}{\pi_{32}(t)} \right)' &= \left( \frac{\pi_3(t)\pi_{21}(t) - \pi_{321}(t)}{\Omega(t)} \right)' \\ &= \frac{\pi_{321}(t)\pi_{123}(t)}{r_2(t)\Omega^2(t)}. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{t_0}^t \frac{1}{q_2(s)} ds &= \int_{t_0}^t \frac{\pi_{321}(s)\pi_{123}(s)}{r_2(s)\Omega^2(s)} ds = \int_{t_0}^t \left( \frac{\pi_{321}(s)\pi_{23}(s)}{\pi_{32}(s)\Omega(s)} + \frac{\pi_3(s)}{\pi_{32}(s)} \right)' ds \\ &= \frac{\pi_{321}(t)\pi_{23}(t)}{\pi_{32}(t)\Omega(t)} + \frac{\pi_3(t)}{\pi_{32}(t)} - c_2. \end{aligned}$$

Moreover,

$$\lim_{t \rightarrow \infty} \frac{\pi_3(t)}{\pi_{32}(t)} = \lim_{t \rightarrow \infty} \frac{1}{\pi_2(t)} = \infty$$

and we conclude that the operator (2.2) is canonical. By Trench's result [2] there exists the only one canonical representation of  $\mathcal{L}$  (up to multiplicative constants with product 1) and so our canonical form is unique.  $\square$

We support our results with couple of illustrative examples.

**Example 1.** Let us consider the following operator

$$\mathcal{L} y = \left( t^\gamma \left( t^\beta \left( t^\alpha y'(t) \right)' \right)' \right)', \quad \alpha > 1, \quad \beta > 1, \quad \gamma > 1. \tag{2.4}$$

By Theorem 1, this operator can be rewritten in canonical form as

$$\mathcal{L} y = \frac{1}{t^{3-\alpha-\beta-\gamma}} \left( t^{2-\alpha} \left( t^{2-\beta} \left( t^{2-\gamma} \left( \frac{y(t)}{t^{3-\alpha-\beta-\gamma}} \right)' \right)' \right)' \right)'$$

**Example 2.** The operator

$$\mathcal{L} y = \left( e^{\gamma t} \left( e^{\beta t} \left( e^{\alpha t} y'(t) \right)' \right)' \right)', \quad \alpha > 0, \quad \beta > 0, \quad \gamma > 0.$$

can be represented in canonical form as

$$\mathcal{L} y = e^{(\alpha+\beta+\gamma)t} \left( e^{-\alpha t} \left( e^{-\beta t} \left( e^{-\gamma t} \left( e^{(\alpha+\beta+\gamma)t} y(t) \right)' \right)' \right)' \right)'$$

### 3 Applications to differential equations

Theorem 1 can be applied to study oscillatory properties of differential equations. We are going to outline one such application based on comparison principle.

**Corollary 1.** *Noncanonical differential equation (E) can be written in canonical form as*

$$\frac{1}{\pi_{321}(t)} \left( \frac{r_3 \pi_{321}^2}{\Omega} \left( \frac{r_2 \Omega^2}{\pi_{321} \pi_{123}} \left( \frac{r_1 \pi_{123}^2}{\Omega} \left( \frac{y}{\pi_{123}} \right)' \right)' \right)' \right)' (t) + p(t)y(\tau(t)) = 0.$$

Setting  $z(t) = y(t)/\pi_{123}(t)$  we get the following comparison result that reduces oscillation of strongly noncanonical equation to that of canonical equation.

**Corollary 2.** *Noncanonical equation (E) is oscillatory if and only if the canonical equation*

$$\left( \frac{r_3 \pi_{321}^2}{\Omega} \left( \frac{r_2 \Omega^2}{\pi_{321} \pi_{123}} \left( \frac{r_1 \pi_{123}^2}{\Omega} z' \right)' \right)' \right)' (t) + \pi_{321}(t)\pi_{123}(\tau(t))p(t)z(\tau(t)) = 0. \tag{E_c}$$

is oscillatory.

**Example 3.** *Let us consider the fourth order differential equation*

$$\left( t^\gamma \left( t^\beta \left( t^\alpha y'(t) \right)' \right)' \right)' + p(t)y(\tau(t)) = 0 \tag{3.1}$$

with  $\alpha > 1, \beta > 1, \gamma > 1$ . By Corollary 2, this equation is oscillatory if and only if the canonical equation

$$\left( t^{2-\alpha} \left( t^{2-\beta} \left( t^{2-\gamma} z'(t) \right)' \right)' \right)' + (t\tau(t))^{3-\alpha-\beta-\gamma} p(t)z(\tau(t)) = 0 \tag{3.2}$$

is oscillatory. It is more convenient to study oscillation of (3.12) instead of (3.1).

Now, we are ready to study the properties of (E) with the help of (E<sub>c</sub>). Without loss of generality, we can consider only with the positive solutions of (E<sub>c</sub>). The following result is a modification of Kiguradze’s lemma [3].

Let us denote

$$q_1(t) = \frac{r_1(t)\pi_{123}^2(t)}{\Omega(t)}, \quad q_2(t) = \frac{r_2(t)\Omega^2(t)}{\pi_{321}(t)\pi_{123}(t)}, \quad q_3(t) = \frac{r_3(t)\pi_{321}^2(t)}{\Omega(t)}$$

and

$$P(t) = \pi_{321}(t)\pi_{123}(\tau(t))p(t).$$

Then (E<sub>c</sub>) can be rewritten as

$$\left( q_3 \left( q_2 \left( q_1 z' \right)' \right)' \right)' (t) + P(t)z(\tau(t)) = 0.$$

**Lemma 3.** *Assume that z(t) is an eventually positive solution of (E<sub>c</sub>), then either*

$$z(t) \in \mathcal{N}_1 \iff z'(t) > 0, \quad (q_1 z')'(t) < 0, \quad (q_2 (q_1 z'))'(t) > 0$$

or

$$z(t) \in \mathcal{N}_3 \iff z'(t) > 0, \quad (q_1 z')'(t) > 0, \quad (q_2 (q_1 z'))'(t) > 0$$

Consequently, the set  $\mathcal{N}$  of all positive solutions of (E) has the decomposition

$$\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_3.$$

To obtain oscillation of studied equation (E), we need to eliminate both cases of possible non-oscillatory solutions.

Let us denote

$$Q_1(t) = \left( \frac{1}{q_2(t)} \int_t^\infty \frac{1}{q_3(u)} \int_u^\infty P(s) ds du \right) \int_{t_1}^{\tau(t)} \frac{1}{q_1(u)} du$$

and

$$Q_2(t) = P(t) \int_{t_1}^t \frac{1}{q_1(s_1)} \int_{t_1}^{s_1} \frac{1}{q_2(u)} \int_{t_1}^s \frac{1}{q_3(s)} ds du ds_1.$$

**Theorem 2.** *Let (1.2) hold. Assume that both first-order delay differential equations*

$$x'(t) + Q_1(t)x(\tau(t)) = 0 \quad (3.3)$$

and

$$x'(t) + Q_2(t)x(\tau(t)) = 0. \quad (3.4)$$

are oscillatory. Then (E) is oscillatory.

*Proof.* Assume that  $y(t)$  is an eventually positive solution of (E), say for  $t \geq t_1$ . Then by Corollary 2,  $z(t) = y(t)/\pi_{123}(t)$  is a solution of (E<sub>c</sub>).

It follows from Lemma 3 that either  $z(t) \in \mathcal{N}_1$  or  $z(t) \in \mathcal{N}_3$ . At first, we admit that  $z(t) \in \mathcal{N}_1$ . Noting that  $q_1(t)z'(t)$  is decreasing, we see that

$$z(t) \geq \int_{t_1}^t \frac{1}{q_1(u)} q_1(u) z'(u) du \geq q_1(t) z'(t) \int_{t_1}^t \frac{1}{q_1(u)} du. \quad (3.5)$$

Integrating (E<sub>c</sub>) from  $t$  to  $\infty$ , we have

$$q_3 \left( q_2 \left( q_1 z' \right)' \right)' (t) \geq \int_t^\infty P(s) z(\tau(s)) ds. \quad (3.6)$$

Taking into account that  $z(\tau(t))$  is increasing, the last inequality yields

$$\left( q_2 \left( q_1 z' \right)' \right)' (t) \geq z(\tau(t)) \frac{1}{q_3(t)} \int_t^\infty P(s) ds. \quad (3.7)$$

Integrating once more, we are led to

$$-\left( q_1 z' \right)' (t) \geq z(\tau(t)) \frac{1}{q_2(t)} \int_t^\infty \frac{1}{q_3(u)} \int_u^\infty P(s) ds du. \quad (3.8)$$

Combining the last inequality with (3.5), one gets

$$\begin{aligned} -\left( q_1 z' \right)' (t) &\geq \left( z'(\tau(t)) \frac{q_1(\tau(t))}{q_2(t)} \int_t^\infty \frac{1}{q_3(u)} \int_u^\infty P(s) ds du \right) \int_{t_1}^{\tau(t)} \frac{1}{q_1(u)} du \\ &= q_1(\tau(t)) z'(\tau(t)) Q_1(t). \end{aligned} \quad (3.9)$$

Thus, the function  $x(t) = q_1(t)z'(t)$  is a positive solution of the differential inequality

$$x'(t) + Q_1(t)x(\tau(t)) \leq 0.$$

Hence, by Philos theorem [4], we conclude that the corresponding differential equation (3.3) also has a positive solution, which contradicts the assumptions of the theorem.

Now, we shall assume that  $z(t) \in \mathcal{N}_3$ . Since  $q_3 \left( q_2 \left( q_1 z' \right)' \right)'$  is decreasing, we are led to

$$\begin{aligned} q_2 \left( q_1 z' \right)' (t) &\geq \int_{t_1}^t \frac{1}{q_3(s)} q_3(s) \left( q_2 \left( q_1 z' \right)' \right)' (s) ds \\ &\geq q_3 \left( q_2 \left( q_1 z' \right)' \right)' (t) \int_{t_1}^t \frac{1}{q_3(s)} ds. \end{aligned}$$

Integrating the above inequality, one can verify that

$$z'(t) \geq q_3 \left( q_2 \left( q_1 z' \right)' \right)' (t) \frac{1}{q_1(t)} \int_{t_1}^t \frac{1}{q_2(u)} \int_{t_1}^s \frac{1}{q_3(s)} ds du$$

Integrating once more, we see that  $x(t) = q_3 \left( q_2 \left( q_1 z' \right)' \right)' (t)$  satisfies

$$z(t) \geq x(t) \int_{t_1}^t \frac{1}{q_1(s_1)} \int_{t_1}^{s_1} \frac{1}{q_2(u)} \int_{t_1}^s \frac{1}{q_3(s)} ds du ds_1.$$

Setting the last estimate into  $(E_c)$ , we see that  $x(t)$  is a positive solution of the differential inequality

$$x'(t) + Q_2(t)x(\tau(t)) \leq 0,$$

which in view of Philos theorem in [4] guarantees that the corresponding differential equation (3.4) has also a positive solution. This is a contradiction and the proof is complete now. □

Applying suitable criteria for oscillation of (3.3) and (3.4), we immediately obtain the criteria for oscillation of (E). We use the one which is due to Ladde et al. [5].

**Corollary 3.** *Let (1.2) hold. Assume that for  $i = 1, 2$*

$$\liminf_{t \rightarrow \infty} \int_{\tau(t)}^t Q_i(s) ds > \frac{1}{e} \tag{3.10}$$

*hold. Then (E) is oscillatory.*

We support our results by another example.

**Example 4.** *Let us consider the general Euler delay differential equation*

$$\left( t^\gamma \left( t^\beta \left( t^\alpha y'(t) \right)' \right)' \right)' + a t^{\alpha+\beta+\gamma-4} y(\lambda t) = 0 \tag{3.11}$$

*with  $\alpha > 1, \beta > 1, \gamma > 1, a > 0$ , and  $0 < \lambda < 1$ . By Corollary 2 and Example 3, this equation is oscillatory if and only if the canonical equation*

$$\left( t^{2-\alpha} \left( t^{2-\beta} \left( t^{2-\gamma} z'(t) \right)' \right)' \right)' + a \lambda^{\alpha+\beta+\gamma-3} t^{\alpha+\beta+\gamma-4} z(\lambda t) = 0 \tag{3.12}$$

is oscillatory. The straightforward computation yields that

$$Q_1(t) \sim \frac{a\lambda^{2-\alpha-\beta}}{(\alpha + \beta + \gamma - 3)(\beta + \gamma - 2)(\gamma - 1)} t^{-1}$$

and

$$Q_2(t) \sim \frac{a\lambda^{3-\alpha-\beta-\gamma}}{(\alpha + \beta + \gamma - 3)(\alpha + \beta - 2)(\alpha - 1)} t^{-1}.$$

By Corollary 4 considered equation is oscillatory, provided that both conditions

$$\frac{a\lambda^{2-\alpha-\beta}}{(\alpha + \beta + \gamma - 3)(\beta + \gamma - 2)(\gamma - 1)} \ln\left(\frac{1}{\lambda}\right) > \frac{1}{e}$$

and

$$\frac{a\lambda^{3-\alpha-\beta-\gamma}}{(\alpha + \beta + \gamma - 3)(\alpha + \beta - 2)(\alpha - 1)} \ln\left(\frac{1}{\lambda}\right) > \frac{1}{e}$$

are satisfied.

## 4 Summary

In this paper we provided canonical representation for strongly noncanonical operator. This canonical transformation is easy and immediate. Moreover, we point out its application in the oscillation theory.

**Acknowledgement:** The paper has been supported by the grant project KEGA 035TUKE-4/2017.

## References

- [1] Bartusek M., Dosla Z., Asymptotic problems for fourth-order nonlinear differential equations. *Boundary Value Problems* 2013, 1–15
- [2] Trench W., Canonical forms and principal systems for general disconjugate equations, *Trans. Amer. Math. Soc.* 184, 1974, 319–327
- [3] Kiguradze I. T., Chanturia T. A., *Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equation*, Kluwer Acad. Publ., Dordrecht 1993
- [4] Philos Ch. G., On the existence of nonoscillatory solutions tending to zero at  $\infty$  for differential equations with positive delay, *Arch. Math.*, 36, 1981, 168–178
- [5] Ladde G. S., Lakshmikantham V., Zhang B. G., *Oscillation Theory of Differential Equations with Deviating Argument*, Marcel Dekker, New York, 1987
- [6] Dzurina J., Comparison theorems for nonlinear ODE's, *Math. Slovaca* 42, (199), 299–315
- [7] Dzurina J., Kotorova R., Zero points of the solutions of a differential equation, *Acta Electrotechnica et Informatica* 7, 2007, 26–29
- [8] Jadlovska, I., Application of Lambert W function in oscillation theory, *Acta Electrotechnica et Informatica* 14, 2014, 9–17
- [9] Koplatadze R., Kvinkadze G., Stavroulakis I. P., Properties A and B of n-th order linear differential equations with deviating argument, *Georgian Math. J.* 6, 1999, 553–566
- [10] Kitamura Y., Kusano T., Oscillations of first-order nonlinear differential equations with deviating arguments, *Proc. Amer. Math. Soc.*, 78, 1980, 64–68
- [11] Kusano T., Naito M., Comparison theorems for functional differential equations with deviating arguments, *J. Math. Soc. Japan* 3, 1981, 509–533
- [12] Zhang T. Li, Ch., Thandapani E., Asymptotic behavior of fourth-order neutral dynamic equations with noncanonical operators, *Taiwanese J. Math.* 18, 2014, 1003–1019
- [13] Mahfoud W. E., Comparison theorems for delay differential equations, *Pacific J. Math.* 83, 1979, No. 83, 187–197