Research Article

Eric H. Liu and Wenjing Du*

Arithmetic properties for Andrews’ \((48, 6)\)- and \((48, 18)\)-singular overpartitions

https://doi.org/10.1515/math-2019-0026
Received November 3, 2018; accepted February 20, 2019

Abstract: Singular overpartition functions were defined by Andrews. Let \(C_{k,i}(n)\) denote the number of \((k, i)\)-singular overpartitions of \(n\), which counts the number of overpartitions of \(n\) in which no part is divisible by \(k\) and only parts \(\equiv \pm i \pmod{k}\) may be overlined. A number of congruences modulo 3, 9 and congruences modulo powers of 2 for \(C_{k,i}(n)\) were discovered by Ahmed and Baruah, Andrews, Chen, Hirschhorn and Sellers, Naika and Gireesh, Shen and Yao for some pairs \((k, i)\). In this paper, we prove some congruences modulo powers of 2 for \(C_{3,1}(n)\) and \(C_{3,1,18}(n)\).

Keywords: congruence, singular overpartition, theta function

MSC: 11P83, 05A17

1 Introduction

In a recent work, Andrews [1] defined combinatorial objects that he called singular overpartitions. Moreover, Andrews proved that these singular overpartitions, which depend on two parameters \(k\) and \(i\), can be enumerated by the function \(C_{k,i}(n)\) which denotes the number of overpartitions of \(n\) in which no part is divisible by \(k\) and only parts \(\equiv \pm i \pmod{k}\) may be overlined. Andrews also established the generating function for \(C_{k,i}(n)\). For \(k \geq 3\) and \(1 \leq i \leq \lfloor \frac{k}{2} \rfloor\), the generating function for \(C_{k,i}(n)\) is given by

\[
\sum_{n=0}^{\infty} C_{k,i}(n)q^n = \frac{(-q^i; q^k)_{\infty}(-q^{k-i}; q^k)_{\infty}(q^k; q^k)_{\infty}}{(q; q)_{\infty}},
\]

where

\[
(a; q)_{\infty} = \prod_{n=0}^{\infty}(1 - a q^n).
\]

Furthermore, by elementary generating function manipulations, Andrews [1] proved that for all \(n \geq 0\),

\[
C_{3,1}(9n + 3) \equiv C_{3,1}(9n + 6) \equiv 0 \pmod{3}.
\]

Since then, a number of congruences modulo 2, 3, 4, 8, 9, 16, 32 and 64 for \(C_{k,i}(n)\) have been discovered for some pairs \((k, i)\), see for example, Ahmed and Baruah [2], Chen [3] Chen, Hirschhorn and Sellers [4], Naika and Gireesh [5], Shen [6] and Yao [7].

In this paper, we prove some new congruences modulo powers of 2 for \((48, 6)\)- and \((48, 18)\)-singular overpartitions.

Eric H. Liu: School of Statistics and Information, Shanghai University of International Business and Economics, Shanghai, 201620, P. R. China, E-mail: liuhai@suibe.edu.cn
*Corresponding Author: Wenjing Du: Wenbo College, East China University of Political Science and Law, Shanghai 201620, P. R. China, E-mail: duwenjing-16@163.com

Open Access. © 2019 Liu and Du, published by De Gruyter. This work is licensed under the Creative Commons Attribution alone 4.0 License.
2 Congruences modulo 2, 8 and 16 for $\overline{C}_{48,6}(n)$

In this section, we prove some congruences modulo 2, 8 and 16 for $\overline{C}_{48,6}(n)$. It should be noted that congruences for $\overline{C}_{48,6}(n)$ and $\overline{C}_{48,18}(n)$ were not discovered before.

**Theorem 2.1.** For $n \geq 0$,
\[
\overline{C}_{48,6}(16n + 8) \equiv 0 \pmod{8},
\]
and
\[
\overline{C}_{48,6}(16n + 12) \equiv 0 \pmod{8},
\]
and
\[
\overline{C}_{48,6}(32n + 28) \equiv 0 \pmod{16}.
\]

**Theorem 2.2.** For $n, k \geq 0$,
\[
\overline{C}_{48,6} \left( 2^{2k+3}n + \frac{31 \times 2^{2k} - 10}{3} \right) \equiv 0 \pmod{2},
\]
\[
\overline{C}_{48,6} \left( 2^{2k+4}n + \frac{17 \times 2^{2k+1} - 10}{3} \right) \equiv 0 \pmod{2},
\]
\[
\overline{C}_{48,6} \left( 2^{2k+4}n + \frac{23 \times 2^{2k+1} - 10}{3} \right) \equiv 0 \pmod{2},
\]
\[
\overline{C}_{48,6} \left( 2^{2k+4}n + \frac{49 \times 2^{2k} - 10}{3} \right) \equiv 0 \pmod{2}.
\]

**Theorem 2.3.** For $n, k \geq 0$,
\[
\overline{C}_{48,6} \left( 4^{k+2}n + \frac{25 \times 4^k - 10}{3} \right) \equiv \begin{cases} 
1 \pmod{2}, & \text{if } 2n + 1 = \frac{m(3m - 1)}{2} \text{ for some integer } m, \\
0 \pmod{2}, & \text{otherwise}.
\end{cases}
\]

**Proofs of Theorems 2.1–2.3.** First, we introduce some notations. Throughout this paper, for any positive integer $k$, $f_k$ is defined by
\[
f_k := (q^k; q^k)_\infty = \prod_{n=1}^\infty (1 - q^{kn}),
\]
and $S(q)$ and $T(q)$ are the Göllnitz-Gordon functions defined by
\[
S(q) = \frac{1}{(q; q^8)_\infty(q^4; q^8)_\infty(q^7; q^8)_\infty}
\]
and
\[
T(q) = \frac{1}{(q^3; q^8)_\infty(q^5; q^8)_\infty(q^9; q^8)_\infty}.
\]
By (1.1), we have
\[
\sum_{n=0}^\infty \overline{C}_{48,6}(n)q^n = \frac{f_{12}T(-q^6)}{f_1f_6}
\]
and
\[
\sum_{n=0}^\infty \overline{C}_{48,18}(n)q^n = \frac{f_{12}S(-q^6)}{f_1f_6}.
\]
It follows from (3.15) and (3.16) in [8] that

\[ T(-q^2) = \frac{1}{2q} \left( \frac{f_3^3}{f_1 f_6^2} - \frac{f_1}{f_6} \right) \]  

(2.10)

and

\[ S(-q^2) = \frac{1}{2} \left( \frac{f_3^2}{f_1 f_6^2} + \frac{f_1}{f_6} \right). \]  

(2.11)

Substituting (2.10) into (2.8) yields

\[ 2 \sum_{n=0}^{\infty} C_{48,6}(n)q^{n+3} = \frac{1}{f_1 f_3} \cdot f_3^2 \cdot f_3 \cdot \frac{f_{12}}{f_6}. \]  

(2.12)

It follows from Lemma 2.6 in [9] that

\[ \frac{f_3}{f_1} = \frac{f_6 f_1 f_6 f_{24}}{f_3^2 f_3 f_{12} f_{24}} + q \frac{f_6 f_8 f_{24}}{f_3 f_2 f_{12} f_{24}}. \]  

(2.13)

Xia and Yao [10] proved the following 2-dissection formula for \( \frac{1}{f_1 f_3} \):

\[ \frac{1}{f_1 f_3} = \frac{f_6^2 f_{12}^2}{f_2 f_{12} f_{12}^2} + q \frac{f_6^2 f_{24}^2}{f_2 f_{12}^2 f_{24}}. \]  

(2.14)

By the binomial theorem, for any positive integer \( k \) and any prime \( p \),

\[ f_1^k = f_p^{k-1} \pmod{p^k}. \]  

(2.15)

Substituting (2.14) and (2.13) into (2.12) and employing (2.15), we have

\[ 2 \sum_{n=0}^{\infty} C_{48,6}(2n)q^{n+1} = \frac{1}{f_1^2} \cdot f_3^2 f_{12}^2 - \frac{1}{f_1^2} \cdot f_6 f_3 f_{12} \]  

(2.16)

and

\[ 2 \sum_{n=0}^{\infty} C_{48,6}(2n+1)q^{n+2} = \frac{f_3^2 f_6^2}{f_1^2 f_3 f_{12}^2} - \frac{f_3 f_{12} f_{12}^2}{f_1^2 f_{12} f_{24}} \]
\[ = \frac{f_3^2}{f_1^2} \cdot \frac{f_{12}^2}{f_1 f_{12}} - \frac{1}{f_1^2} \cdot f_3 f_{12} f_{24} \pmod{4}. \]  

(2.17)

From Lemma 2.1 in [11],

\[ \frac{1}{f_1} = \frac{f_3}{f_3^2} + 2q \frac{f_2 f_3^2}{f_2 f_6} \]  

(2.18)

and

\[ \frac{1}{f_1} = \frac{f_4}{f_2^2 f_6} + 4q f_2^2 f_6 \]  

(2.19)

Substituting (2.18) and (2.19) into (2.16), we obtain

\[ \sum_{n=0}^{\infty} C_{48,6}(4n)q^n = 2 \cdot \frac{1}{f_1} \cdot \frac{1}{f_1 f_3} \cdot f_3 f_{12}^2 - \frac{1}{f_1} \cdot \frac{f_3}{f_1} \cdot f_3^2 f_{12} f_{12} \]  

(2.20)

and

\[ 2 \sum_{n=0}^{\infty} C_{48,6}(4n + 2)q^{n+1} = \frac{f_3^2 f_6^2}{f_1^2 f_3 f_{12}^2} - \frac{f_3 f_{12} f_{12}^2}{f_1 f_{12} f_{24}}. \]  

(2.21)
Substituting (2.19), (2.14) and (2.13) into (2.20) and employing (2.15), we deduce that

\[
\sum_{n=0}^{\infty} \bar{C}_{A,6}(8n)q^n = \frac{f_3^2 f_6^2}{f_1 f_4} - \frac{f_3 f_6 f_{12}}{f_4 f_{16}} + 4q \frac{f_6 f_{24}}{f_1 f_{12}} + 4q f_6 f_{30} (\mod 8)
\]

(2.22)

and

\[
\sum_{n=0}^{\infty} \bar{C}_{A,6}(8n + 4)q^n = \frac{f_3^2 f_2 f_{12}}{f_1 f_4 f_6} + 8f_2 f_6 f_{12} - \frac{f_2^2 f_6 f_{12}}{f_4 f_{16}} - 4f_2 f_6 f_{30} (\mod 16).
\]

(2.23)

Substituting (2.18) and (2.19) into (2.22) and utilizing (2.15), we see that

\[
\sum_{n=0}^{\infty} \bar{C}_{A,6}(16n + 8)q^n = 4q f_3 f_6 - 4f_2 f_6 f_{12} + 4q f_2 f_6 f_{30} (\mod 8),
\]

(2.24)

where

\[
F(q) = \frac{f_3^3}{f_1^2} \cdot \frac{f_2^2 f_6}{f_{12}} - \frac{f_3 f_6 f_{12}}{f_4}.
\]

(2.25)

Hirschhorn, Garvan and Borwein [12] proved that

\[
\frac{f_3^3}{f_1^2} = \frac{f_2^2 f_6}{f_{12}} + q \frac{f_3 f_{12}}{f_4}.
\]

(2.26)

Substituting (2.26) into (2.25), we find that

\[
F(q) = 0.
\]

(2.27)

Congruence (2.1) follows from (2.24) and (2.27).

Substituting (2.18) and (2.19) into (2.23) and using (2.15) yields

\[
\sum_{n=0}^{\infty} \bar{C}_{A,6}(16n + 12)q^n = 4f_2 f_6 f_{12} + 4f_2 f_6 f_{30} + \left( \frac{f_{12}}{f_3 f_6} + f_3 \right) (\mod 16).
\]

(2.28)

Define

\[
\sum_{n=0}^{\infty} b(n)q^n = \frac{f_6}{f_1 f_2}.
\]

(2.29)

Replacing \(q\) by \(-q\) in (2.29) and employing (2.15) and (2.30), and using the fact that

\[
(-q; -q)_\infty = \frac{f_3^3}{f_1 f_4},
\]

(2.30)

we have

\[
\sum_{n=0}^{\infty} (-1)^n b(n)q^n = \frac{f_1 f_2 f_6}{f_2} \equiv f_1 (\mod 4).
\]

(2.31)
Hence,
\[
\frac{f_4}{f_1 f_2} + f_1 \equiv \sum_{n=0}^{\infty} (1 + (-1)^n)b(n)q^n = 2 \sum_{n=0}^{\infty} b(2n)q^{2n} \pmod{4} \tag{2.32}
\]
and
\[
\frac{f_4}{f_1 f_2} - f_1 \equiv \sum_{n=0}^{\infty} (1 - (-1)^n)b(n)q^n = 2 \sum_{n=0}^{\infty} b(2n + 1)q^{2n+1} \pmod{4}. \tag{2.33}
\]

In view of (2.15), (2.28) and (2.32),
\[
\sum_{n=0}^{\infty} \mathcal{C}_{48,6}(16n + 12)q^n \equiv 8 \frac{f_4^2 f_6}{f_1 f_3} \sum_{n=0}^{\infty} b(2n)q^{6n} = 8 \frac{f_4^2}{f_2} \sum_{n=0}^{\infty} b(2n)q^{6n} \pmod{16}. \tag{2.34}
\]

Congruences (2.2) and (2.3) follow from (2.34).

From (3.29) in [10],
\[
\frac{f_4^2}{f_2} = \frac{f_4^2 f_6}{f_2 f_6} + 2q f_4 f_6 + f_4^2 f_6 \tag{2.35}
\]
Substituting (2.18) and (2.35) into (2.17) and employing (2.15) yields
\[
2 \sum_{n=0}^{\infty} \mathcal{C}_{48,6}(4n + 1)q^{n+1} \equiv \frac{f_4^2 f_6}{f_1 f_4 f_12} - \frac{f_4^2}{f_1 f_2} \pmod{4} \tag{2.36}
\]
and
\[
\sum_{n=0}^{\infty} \mathcal{C}_{48,6}(4n + 3)q^{n+1} \equiv \frac{f_4 f_6 f_1}{f_2 f_6} f_6 f_{12} - \frac{f_4 f_1^2 f_2}{f_2^2 f_1} \equiv \frac{f_3^3}{f_1} \cdot \frac{f_4^2}{f_2} - \frac{f_2^2}{f_2} \pmod{2}. \tag{2.37}
\]
Substituting (2.26) into (2.37) and using (2.15), we deduce that
\[
\sum_{n=0}^{\infty} \mathcal{C}_{48,6}(4n + 3)q^{n+1} \equiv \frac{f_4 f_6 f_1}{f_2 f_6} f_6 f_{12} - \frac{f_4 f_1^2 f_2}{f_2^2 f_1} \equiv q^2 \frac{f_3^2}{f_2} \pmod{2},
\]
which implies that for \( n \geq 0, \)
\[
\mathcal{C}_{48,6}(8n + 7) \equiv 0 \pmod{2}. \tag{2.38}
\]
By (2.15) and (2.36),
\[
2 \sum_{n=0}^{\infty} \mathcal{C}_{48,6}(4n + 1)q^{n+1} \equiv \frac{1}{f_1} + \frac{f_4 f_6^2}{f_1 f_2} - \frac{f_4^2 f_6}{f_2^2 f_1} \pmod{4}. \tag{2.39}
\]
Substituting (2.18) into (2.39) and employing (2.15), we have
\[
2 \sum_{n=0}^{\infty} \mathcal{C}_{48,6}(8n + 5)q^{n+1} \equiv \frac{f_4 f_6 f_1}{f_1 f_6} f_1 f_{12} - \frac{f_4 f_1^2 f_2}{f_2^2 f_1} \equiv \frac{f_4}{f_2} f_6 f_1 (f_6 f_1 f_2 - f_1) \pmod{4}. \tag{2.40}
\]
Thanks to (2.33) and (2.40),
\[
\sum_{n=0}^{\infty} \mathcal{C}_{48,6}(8n + 5)q^n \equiv \frac{f_4^2}{f_6} \sum_{n=0}^{\infty} b(2n + 1)q^{2n}.
\]
Therefore, for \( n \geq 0 \),
\[
\overline{C}_{48,6}(16n + 13) \equiv 0 \pmod{2} \tag{2.41}
\]
and
\[
\overline{C}_{48,6}(16n + 5) \equiv b(2n + 1) \pmod{2}. \tag{2.42}
\]
It is well-known that
\[
f_1 = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n+1)}{2}}. \tag{2.43}
\]
Combining (2.43) and (2.31) yields
\[
b(n) \equiv \begin{cases} 
1 \pmod{2}, & \text{if } n = \frac{m(3m-1)}{2} \text{ for some integer } m, \\
0 \pmod{2}, & \text{otherwise}. 
\end{cases} \tag{2.44}
\]
Thanks to (2.15) and (2.21),
\[
2 \sum_{n=0}^{\infty} \overline{C}_{48,6}(4n + 2)q^{n+1} \equiv \frac{f_3^3}{f_1^2} - \frac{f_5\mu_2}{f_1^2} \pmod{4}. \tag{2.45}
\]
Based on (2.15), (2.12) and (2.45), we see that for \( n \geq 0 \),
\[
\overline{C}_{48,6}(4n + 10) \equiv C_{48,6}(n) \pmod{2}. \tag{2.46}
\]
By (2.46) and mathematical induction, for \( n, k \geq 0 \),
\[
\overline{C}_{48,6} \left( 4^k n + \frac{10(4^k - 1)}{3} \right) \equiv \overline{C}_{48,6}(n) \pmod{2}. \tag{2.47}
\]
Replacing \( n \) by \( 8n + 7, 16n + 8, 16n + 12, 16n + 13 \) in (2.47) and using (2.38), (2.1), (2.2), (2.41) respectively, we obtain the congruences stated in Theorem 2.2.

It follows from (2.42) and (2.44) that
\[
\overline{C}_{48,6}(16n + 5) \equiv \begin{cases} 
1 \pmod{2}, & \text{if } 2n + 1 = \frac{m(3m-1)}{2} \text{ for some integer } m, \\
0 \pmod{2}, & \text{otherwise}. 
\end{cases} \tag{2.48}
\]
Theorem 2.3 follows from (2.47) and (2.48). This completes the proof.

### 3 Congruences modulo 2, 8 and 16 for \( \overline{C}_{48,18}(n) \)

We prove several congruences modulo 2, 8 and 16 for \( \overline{C}_{48,18}(n) \) in this section.

**Theorem 3.1.** For \( n \geq 0 \),
\[
\overline{C}_{48,18}(16n + 11) \equiv 0 \pmod{8}, \tag{3.1}
\]
\[
\overline{C}_{48,18}(16n + 15) \equiv 0 \pmod{8}, \tag{3.2}
\]
\[
\overline{C}_{48,18}(32n + 15) \equiv 0 \pmod{16}. \tag{3.3}
\]
Theorem 3.2. For \( n, k \geq 0 \),
\[
\mathcal{C}_{48,18} \left( 2^{2k+3}n + \frac{7 \times 2^k - 1}{3} \right) \equiv 0 \pmod{2},
\]
(3.4)
\[
\mathcal{C}_{48,18} \left( 2^{2k+4}n + \frac{25 \times 2^k - 1}{3} \right) \equiv 0 \pmod{2},
\]
(3.5)
\[
\mathcal{C}_{48,18} \left( 2^{2k+4}n + \frac{17 \times 2^{k+1} - 1}{3} \right) \equiv 0 \pmod{2},
\]
(3.6)
\[
\mathcal{C}_{48,18} \left( 2^{2k+4}n + \frac{23 \times 2^{k+1} - 1}{3} \right) \equiv 0 \pmod{2}.
\]
(3.7)

Theorem 3.3. For \( n, k \geq 0 \),
\[
\mathcal{C}_{48,18} \left( 4^{k+2}n + \frac{4^k - 1}{3} \right) \equiv \begin{cases} 1 \pmod{2}, & \text{if } 2n = \frac{m(3m-1)}{2} \text{ for some integer } m, \\ 0 \pmod{2}, & \text{otherwise}. \end{cases}
\]
(3.8)

Proofs of Theorems 3.1–3.3. In view of (2.9) and (2.11),
\[
2 \sum_{n=0}^{\infty} \mathcal{C}_{48,18}(n)q^n = \frac{1}{f_1 f_3} \cdot f_6^2 + \frac{f_3}{f_1} \cdot \frac{f_12}{f_6}. 
\]
(3.9)
Substituting (2.14) and (2.13) into (3.9) yields
\[
2 \sum_{n=0}^{\infty} \mathcal{C}_{48,18}(n)q^n = \frac{f_2 f_1}{f_2 f_1 f_3 f_2} + \frac{f_3 f_6}{f_2 f_3 f_2} + q^2 \frac{f_3 f_2}{f_2 f_3 f_2} + \frac{f_2}{f_3 f_2} f_1 f_2 f_4 f_6. 
\]
Therefore,
\[
2 \sum_{n=0}^{\infty} \mathcal{C}_{48,18}(2n)q^n = \frac{f_2 f_1}{f_2 f_1 f_3 f_2} + \frac{f_3 f_6}{f_2 f_3 f_2} 
\]
(3.10)
and
\[
2 \sum_{n=0}^{\infty} \mathcal{C}_{48,18}(2n+1)q^n = \frac{1}{f_2} \cdot \frac{f_3 f_2}{f_2} + \frac{1}{f_1} \cdot \frac{f_2 f_6}{f_3 f_2}. 
\]
(3.11)
Substituting (2.18) and (2.19) into (3.11) yields
\[
2 \sum_{n=0}^{\infty} \mathcal{C}_{48,18}(4n+1)q^n = \frac{f_2 f_1 f_6}{f_1 f_3 f_6} + 2 \cdot \frac{1}{f_1 f_3} \cdot \frac{1}{f_3} \cdot f_1 f_6. 
\]
(3.12)
and
\[
\sum_{n=0}^{\infty} \mathcal{C}_{48,18}(4n+3)q^n = \frac{1}{f_1} \cdot \frac{f_2 f_1 f_6}{f_1 f_3 f_6} + 2 \cdot \frac{1}{f_1 f_3} \cdot \frac{1}{f_3} \cdot \frac{f_2}{f_1} \cdot f_6. 
\]
(3.13)
Substituting (2.19), (2.14) and (2.13) into (3.13) and employing (2.15), we have
\[
\sum_{n=0}^{\infty} \mathcal{C}_{48,18}(8n+3)q^n = \frac{f_1^3 f_3 f_6}{f_1^2 f_3^2 f_6^2} + 2 \cdot \frac{f_1^3 f_6^2}{f_1^2 f_3^2 f_6^2} + 4 q^2 \frac{f_3 f_6}{f_1 f_3 f_6^2} + 8 q^2 \frac{f_1^3 f_6^2}{f_1 f_3 f_6^2} 
\equiv \frac{1}{f_1} \cdot \frac{f_2 f_1 f_6}{f_1 f_3 f_6} + 2 \cdot \frac{1}{f_1 f_3} \cdot \frac{f_2}{f_1} \cdot f_6 + 4 q^2 \frac{f_3 f_6}{f_1 f_3 f_6^2} \pmod{8}. 
\]
(3.14)
and

\[
\sum_{n=0}^{\infty} \mathcal{C}_{48,18}(8n + 7)q^n = \frac{f_{12}^2 f_{6}^2 f_{24}^2}{f_{12}^2 f_{6} f_{12}} + 4 f_{3} f_{6}^2 f_{6} f_{12} + 2 f_{2} f_{2}^2 f_{12}^2 + 8 f_{2} f_{2} f_{6} \frac{f_{6}^2}{f_{24}} \frac{f_{12}^2}{f_{12}} \frac{f_{6}^2}{f_{24}} \frac{f_{6}^2}{f_{24}}
\]
\[
\equiv \left( \frac{1}{f_{1}^3} \right)^3 \frac{f_{12}^2 f_{6}^2 f_{24}^2}{f_{12}^2 f_{6} f_{12}} + 4 f_{3} f_{6}^2 f_{6} f_{12} + 2 f_{2} f_{2}^2 f_{12}^2 + 8 f_{2} f_{2} f_{6} \frac{f_{6}^2}{f_{24}} \frac{f_{12}^2}{f_{12}} \frac{f_{6}^2}{f_{24}} \frac{f_{6}^2}{f_{24}} \quad \text{(mod 16)}. \tag{3.15}
\]

Substituting (2.18) and (2.19) into (3.14) yields

\[
\sum_{n=0}^{\infty} \mathcal{C}_{48,18}(16n + 11)q^n \equiv 4 f_{1}^0 f_{6}^2 f_{2}^2 \frac{f_{6}^2}{f_{2} f_{6} f_{6}} - 4 f_{2}^0 f_{3} f_{12} \frac{f_{6}^2}{f_{2} f_{6} f_{6}}
\]
\[
\equiv 4 f_{2} f_{2} f_{12} \frac{f_{12}^2}{f_{12}^2 f_{6} f_{6}} \left( \frac{f_{12}^2}{f_{3} f_{6} f_{6}} - f_{3} \right) \quad \text{(by 2.15)}
\]
\[
\equiv 8 \frac{f_{2} f_{2} f_{12}^2}{f_{12}^2 f_{6}} \sum_{n=0}^{\infty} b(2n + 1)q^{6n+3} \quad \text{(mod 16)} \quad \text{(by 2.33)}
\]
\[
\equiv 8 \frac{f_{2} f_{12} f_{12}^2}{f_{12}^2 f_{6}} \sum_{n=0}^{\infty} b(2n + 1)q^{6n+3} \quad \text{(mod 16)} \quad \text{(by 2.15)}. \tag{3.17}
\]

Congruences (3.2) and (3.3) follow from (3.17).

Based on (2.15) and (3.10),

\[
2 \sum_{n=0}^{\infty} \mathcal{C}_{48,18}(2n)q^n = \frac{f_{2}^2}{f_{1}^2} \frac{f_{2}^2}{f_{2} f_{2} f_{6}} + \frac{f_{2}^2}{f_{1}^2} + \frac{f_{2} f_{6} f_{12}^2}{f_{2} f_{12} f_{24}} \quad \text{(mod 4)}. \tag{3.18}
\]

Substituting (2.18) and (2.35) into (3.18) and employing (2.15), we deduce

\[
2 \sum_{n=0}^{\infty} \mathcal{C}_{48,18}(4n)q^n = \frac{f_{2}^2 f_{6}^2}{f_{12} f_{6} f_{12}} + \frac{f_{2}^2 f_{6}^2}{f_{12} f_{6} f_{12}} + \frac{1}{f_{1}^2} \frac{f_{2} f_{6} f_{12}^2}{f_{24}} \quad \text{(mod 4)}. \tag{3.19}
\]

and

\[
\sum_{n=0}^{\infty} \mathcal{C}_{48,18}(4n + 2)q^n = \frac{f_{2}^2 f_{6} f_{12}^2}{f_{12} f_{6} f_{12}} + \frac{f_{2}^2 f_{6}^2 f_{6}^2}{f_{12}^2 f_{6}^2} \equiv \frac{f_{3}^2 f_{3} f_{3}^2}{f_{1}^2} + f_{6} \frac{f_{6}^2}{f_{2}} \quad \text{(mod 2)}. \tag{3.20}
\]

Substituting (2.26) into (3.20) and employing (2.15), we deduce that

\[
\sum_{n=0}^{\infty} \mathcal{C}_{48,18}(4n + 2)q^n \equiv \frac{f_{2} f_{6} f_{12}^2}{f_{12} f_{6} f_{12}} + f_{2} f_{6}^2 \frac{f_{6}^2}{f_{2}} + q f_{2} f_{12}^2 \equiv q f_{2} f_{12}^2 \quad \text{(mod 2)}. \tag{3.21}
\]

It follows from (3.21) that for \( n \geq 0 \),

\[
\mathcal{C}_{48,18}(8n + 2) \equiv 0 \quad \text{(mod 2)}. \tag{3.22}
\]
Substituting (2.18) into (3.19) and utilizing (2.15) and (2.32), we see that, modulo 4,

\[ 2 \sum_{n=0}^{\infty} \mathcal{C}_{48,18}(8n)q^n = \frac{f_2 f_3 f_5}{f_1 f_2 f_6} \left( \frac{f_7}{f_1 f_2} + f_1 \right) = 2 \frac{f_2^2}{f_6} \sum_{n=0}^{\infty} b(2n)q^{2n} \equiv 2 \sum_{n=0}^{\infty} b(2n)q^{2n}, \]

which implies that for \( n \geq 0, \)

\[ \mathcal{C}_{48,18}(16n + 8) \equiv 0 \pmod{2} \]  
(3.23)

and

\[ \mathcal{C}_{48,18}(16n) \equiv b(2n) \pmod{2}. \]  
(3.24)

By (2.15) and (3.12),

\[ 2 \sum_{n=0}^{\infty} \mathcal{C}_{48,18}(4n + 1)q^n \equiv \frac{f_1 f_2}{f_1 f_6} + \frac{f_2^2}{f_1 f_3} \pmod{4}. \]  
(3.25)

Thanks to (3.9) and (3.25), we see that for \( n \geq 0, \)

\[ \mathcal{C}_{48,18}(4n + 1) \equiv \mathcal{C}_{48,18}(n) \pmod{2}. \]  
(3.26)

By (3.26) and mathematical induction, we deduce that for \( n, k \geq 0, \)

\[ \mathcal{C}_{48,18} \left( 4^k n + \frac{4^k - 1}{3} \right) \equiv \mathcal{C}_{48,18}(n) \pmod{2}. \]  
(3.27)

Replacing \( n \) by \( 8n + 2, 16n + 8, 16n + 11, 16n + 15 \) in (3.27) and using (3.22), (3.23), (3.1) and (3.2) respectively, we arrive at the congruences stated in Theorem 3.2. It follows from (2.44) and (3.24) that

\[ \mathcal{C}_{48,18}(16n) \equiv \begin{cases} 
1 \pmod{2}, & \text{if } 2n = \frac{m(3m - 1)}{2} \text{ for some integer } m, \\
0 \pmod{2}, & \text{otherwise.} 
\end{cases} \]  
(3.28)

Theorem 3.3 follows from (3.27) and (3.28). This completes the proof.

Acknowledgments. This work was supported by the National Science Foundation of China (grant no. 11701362).

References