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On Diophantine approximation by unlike powers of primes

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Abstract: Suppose that $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ are nonzero real numbers, not all of the same sign, $\lambda_1/\lambda_2$ is irrational, $\lambda_2/\lambda_4$ and $\lambda_3/\lambda_5$ are rational. Let $\eta$ real, and $\varepsilon > 0$. Then there are infinitely many solutions in primes $p_j$ to the inequality $|\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^3 + \lambda_4 p_4^4 + \lambda_5 p_5^5 + \eta| < (\max p_j)^{-1/32+\varepsilon}$. This improves an earlier result under extra conditions of $\lambda_i$. 

Keywords: Diophantine approximation; primes; Davenport-Heilbronn method

MSC: 11D75; 11P32; 11P55

1 Introduction

Given $k \geq 1$ and non-zero real numbers $\lambda_1, \lambda_2, \cdots, \lambda_k$ (not all in rational ratio, not all in same sign), we write

$$F(p) = \sum_{j=1}^{s} \lambda_j p_j^k,$$

where $p = (p_1, p_2, \ldots, p_s)$ with each $p_j$ a prime. Various authors have considered the distribution of values of such forms, see [17, 18] for example.

For $k = 1$, Vaughan [17] first proved that for any real $\eta$, there are infinitely many solutions in primes $p_j$ to the inequality

$$|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < (\max p_j)^{-1/10+\varepsilon}$$

with $\xi = 1/10$. The exponent was subsequently improved by Baker and Harman [1] to $\xi = 1/6$, Harman [6] to $\xi = 1/5$ and Matomäki [14] to $\xi = 2/9$.

For $k = 2$, Baker and Harman [1] and Harman [7] showed that there are infinitely many solutions in primes $p_j$ to the inequality

$$|\lambda_1 p_1^2 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^2 + \lambda_5 p_5^9 + \eta| < (\max p_j)^{-1/8+\varepsilon}.$$

In 2011, Li and Wang [11] proved that there are infinitely many solutions in primes $p_j$ to the inequality

$$|\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^2 + \lambda_5 p_5^5 + \eta| < (\max p_j)^{-1/28+\varepsilon}.$$

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Later, Languasco and Zaccagnini [9], Liu and Sun [12], and Wang and Yao [20] replaced $1/28$ with $1/18$, $1/16$ and $1/14$, respectively.

For $k \geq 3$, Vaughan [18] first proved that there are infinitely many solutions in primes $p_j$ to the inequality

$$|\lambda_1 p_1^k + \lambda_2 p_2^k + \cdots + \lambda_5 p_5^k + \eta| < (\max p_j)^{-\sigma + \varepsilon}.$$  

In 2006, Cook and Harman [2] improved the exponent $\sigma$.

In 2016, the first author and the second author [3] first established that if $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ are nonzero real numbers, not all of the same sign and $\lambda_1/\lambda_2$ is irrational, there are infinitely many solutions in primes $p_j$ to the inequality

$$|\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^3 + \lambda_4 p_4^4 + \lambda_5 p_5^5 + \eta| < (\max p_j)^{-1/720 + \varepsilon}.$$  

Later, Mu [15], Liu [13], Mu and Qu [16] replaced $\frac{1}{720}$ in (1.1) with $1/180$, $5/288$ and $5/252$ respectively.

In this paper, under some extra conditions of $\lambda_j$, we get the following result.

**Theorem 1.1.** Suppose that $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ are nonzero real numbers, not all of the same sign, $\lambda_1/\lambda_2$ is irrational, $\lambda_3/\lambda_4$ and $\lambda_3/\lambda_5$ are rational. Let $\eta$ real, and $\varepsilon > 0$. Then there are infinitely many solutions in primes $p_j$ to the inequality

$$|\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^3 + \lambda_4 p_4^4 + \lambda_5 p_5^5 + \eta| < (\max p_j)^{-1/32 + \varepsilon}. $$

In the previous arguments, the key of this problem is the estimates for exponential sums over squares of primes (or for certain double sums if sieve methods are invoked). In [13], Liu used $S_2(\lambda_2) \ll P_2^{-1/8 + \varepsilon}$. In [16], Mu and Qu used sieve method of Harman [7], and got $S_2'(\lambda_2) \ll P_2^{-1/7 + \varepsilon}$. Using the method of Mu and Qu [16], even if one got the best estimation $S_2''(\lambda_2) \ll P_2^{-1/6 + \varepsilon}$, $5/252$ can only be replaced by $5/216$. But in this paper our method don't depend on the estimates of $S_2(\lambda_2)$.

**Notation:** Throughout the paper, the letter $\delta$ denotes a sufficiently small, fixed positive number. The letter $\varepsilon$ denotes an arbitrarily sufficiently small positive real number. Any statement in which $\varepsilon$ occurs holds for each fixed $\varepsilon > 0$. $c$ denotes an absolute constant, not necessarily the same in all occurrences. The letter $p$, with or without subscript, denotes a prime number. Constants, both explicit and implicit, in Vinogradov symbols may depend on $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$. We write $e(x) = \exp(2\pi ix)$.

## 2 Outline of the method

We use the Hardy-Littlewood circle method which first stated by Davenport-Heilbronn. Note that $\lambda_1/\lambda_2$ is irrational and $\lambda_2/\lambda_4$ is rational. Without loss of generality, we assume that $|\lambda_2/\lambda_4| \leq 1$. Let $a/q$ be a continued fraction convergent to $\lambda_1/\lambda_2$ and put $X = q^{12/5}$. Then $(\lambda_2 \alpha)/(\lambda_4 q) = a'/q'$ is a continued fraction convergent to $\lambda_1/\lambda_4$, where $(a', q') = 1$. Thus we have $q \asymp q'$. Suppose that $0 < \tau < 1$, and write $P_j = X^{1/j}$ and $\mathcal{J}_j = [\delta P_j, P_j]$ for $1 \leq j \leq 5$. We define

$$K_\tau(\alpha) = \left( \frac{\sin \pi \tau \alpha}{\pi \alpha} \right)^2, \quad S_j(\alpha) = \sum_{p \in \mathcal{J}_j} (\log p)e(ap').$$

Then we can easily get

$$K_\tau(\alpha) \ll \min(\tau^2, |\alpha|^{-2}), \quad \int_{\mathbb{R}} K_\tau(\alpha)e(ax)da = \max(0, \tau - |x|).$$

(2.1)

For any measurable subset $X$ of $\mathbb{R}$, we define

$$\mathcal{J}(X) := \int_X S_1(\lambda_1 \alpha)S_2(\lambda_2 \alpha)S_3(\lambda_3 \alpha)S_4(\lambda_4 \alpha)S_5(\lambda_5 \alpha)K_\tau(\alpha)e(\eta \alpha)da.$$
Then by (2.1), we have
\[
\beta(\mathbb{R}) = \sum_{p_j \in J} \frac{1}{(\log p_j^2)} \left( \sum_{p \leq p_j} \log(p) \right) \int e(\alpha(\lambda_1 p_1 + \cdots + \lambda_4 p_4^4 + \lambda_5 p_5^5 + \eta))K_\tau(d\alpha) \leq \tau(\log X)^5N(\eta, X),
\]
(2.3)
where \(N(\eta, X)\) is the number of solutions to the inequality
\[
|\lambda_1 p_1 + \cdots + \lambda_4 p_4^4 + \lambda_5 p_5^5 + \eta| < \tau,
\]
To estimate the integral \(\beta(\mathbb{R})\), we divide the real line into three parts: the major arc \(\mathcal{M}\), the minor arc \(\mathcal{m}\) and the trivial arc \(\mathcal{t}\), which are defined by
\[
\mathcal{M} = \{ \alpha : |\alpha| \leq 1 \}, \quad \mathcal{m} = \{ \alpha : 1 < |\alpha| \leq \xi \}, \quad \mathcal{t} = \{ \alpha : |\alpha| > \xi \},
\]
where \(\xi = \tau^{-2} X^{1/60+\epsilon}\). By the arguments of section 5 in [15], we have
\[
\beta(t) = o(\tau^2 X^{77/60}).
\]
(2.4)

3 Preliminary lemmas

Lemma 3.1. [19, Theorem 3.1] Suppose that \(N \geq 2\) and \(a\) satisfies
\[
|qa - a| \leq q^{-1}, \quad (a, q) = 1, \quad q \in \mathbb{N}, \quad a \in \mathbb{Z}.
\]
Then we have
\[
\sum_{p \leq N} \log(p) e(ap) \ll (\log N)^4(N^{1/2} q^{1/2} + N^4 + Nq^{-1/2}).
\]

Corollary 3.2. Suppose that \(X \geq Z \geq X^{1+\epsilon}\) and \(|S_1(a)| > Z\). Then there are coprime integers \(a, q\) satisfying
\[
1 \leq q \ll (X/Z)^2 X^{\epsilon}, \quad |qa - a| \ll (X/Z)^2 X^{\epsilon-1}.
\]
Proof. This follows from Lemma 3.1 immediately.

Lemma 3.3. [8, Theorem 3] Let \(k \geq 3\) and \(\sigma(k) = 1/(3 \cdot 2^{k-1})\). Suppose that \(N \geq 2\) and \(a\) satisfies
\[
|qa - a| \leq Q^{-1}, \quad (a, q) = 1, \quad q \in \mathbb{N}, \quad q \leq Q, \quad a \in \mathbb{Z},
\]
where \(Q = N^{(k^2 - 2k\sigma(k))/(2k-1)}\). Then, for any \(\epsilon > 0\),
\[
\sum_{p \leq N} \log(p) e(ap^k) \ll N^{1-\sigma(k)+\epsilon} + \frac{N^{1+\epsilon}}{(q + N^{(1/\sigma(k))})^{1/2}}.
\]

Corollary 3.4. Suppose that \(P_a \geq Z \geq P_a^{1-1/2k+\epsilon}\) and \(|S_a(a)| > Z\). Then there are coprime integers \(a, q\) satisfying
\[
1 \leq q \ll (P_a/Z)^2 P_a^{\epsilon}, \quad |qa - a| \ll (P_a/Z)^2 P_a^{\epsilon-4}.
\]
Proof. This follows from Lemma 3.3 immediately.
Lemma 3.5. [7, Lemma 3] Suppose that $N \geq 2$ and $a$ satisfies

$$|qa - a| \leq q^{-1}, \quad (a, q) = 1, \quad q \in \mathbb{N}, \quad a \in \mathbb{Z}.$$  

Then, for any $\varepsilon > 0$,

$$\sum_{p \leq N} (\log p) e(ap^2) \ll N^{1+\varepsilon} \left( \frac{1}{q} + \frac{1}{N^{1/2}} + \frac{q}{N^2} \right)^{1/4}.$$  

Corollary 3.6. [7, Corollary 1] Suppose that $P_2 \geq Z \geq P_2^{7/8+\varepsilon}$, and that $|S_2(a)| > Z$. Then there are coprime integers $a, q$ satisfying

$$1 \ll q \ll (P_2/Z)^4 P_2^5, \quad |qa - a| \ll (P_2/Z)^4 P_2^{-2}.$$  

Lemma 3.7. [16, Lemma 3.7] Suppose that

$$f(a) \in \{ S_1(\lambda_1 a)^2, S_2(\lambda_2 a)^2, S_3(\lambda_3 a)^4, S_4(\lambda_4 a)^2 S_5(\lambda_5 a)^2, S_2(\lambda_2 a)^2 S_4(\lambda_4 a)^4, S_2(\lambda_2 a)^2 S_5(\lambda_5 a)^6 \}.$$

Then we have

$$\int_{-1}^{1} |f(a)| da \ll f(0) X^{-1+\varepsilon}; \quad (3.1)$$

$$\int_{\mathbb{R}} |f(a)| K_\varepsilon(a) da \ll \tau f(0) X^{-1+\varepsilon}. \quad (3.2)$$

We define the multiplicative function $w_3(q)$ by taking

$$w_3(p^{3u+v}) = \begin{cases} \frac{3p^{-u-1/2}}{\varepsilon}, & \text{when } u \geq 0 \text{ and } v = 1; \\ \frac{p^{-u-1}}{\varepsilon}, & \text{when } u \geq 0 \text{ and } 2 \leq v \leq 3. \end{cases} \quad (3.3)$$

Lemma 3.8. [21, Lemma 2.3] If $a$ is a real number satisfying that there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a, q) = 1$, $1 \leq q \leq P^{1/4}$ and $|qa - a| \leq P^{-9/4}$, then one has

$$\sum_{p \leq x < 2P} e(x^3 a) \ll \frac{w_3(q)P}{1 + \varepsilon p|a - a/q|},$$

otherwise, one has $\sum_{p \leq x < 2P} e(x^3 a) \ll P^{1+\varepsilon}$.

Lemma 3.9. [21, Lemma 2.1] Let $c$ be a constant. For $Q \geq 2$, one has

$$\sum_{1 \leq q \leq Q} d(q)^c w_3(q)^2 \ll (\log Q)^A,$$

where $A$ is a positive constant, $d(q)$ is the divisor function.

4 The major arc

In this section, we give a low bound for the integral on the major arc $\mathcal{M}$. First, we consider the standard major arc $\mathcal{M}^* = \{ a : |a| \leq X^{-1+1/2+\varepsilon} \}$. Using the idea due to Harman [7], we get the following lemma (one can also see section 3 of Mu and Qu [16]). One may improve the standard major arc to $\{ a : |a| \leq X^{-1+2/15-\varepsilon} \}$ by using some ideas due to Languasco and Zaccagnini [10] (one can also see [5]). But there is no improvement for our result, because our improvement comes from the minor arc.
Lemma 4.1. We have
\[ \beta(\mathfrak{m}^*) \gg \tau^2 X^{77/60}. \] (4.1)

Lemma 4.2. We have
\[ \beta(\mathfrak{m} \setminus \mathfrak{m}^*) = o(\tau^2 X^{77/60}). \] (4.2)

Proof. For a given \( \alpha \), by Dirichlet’s theorem in Diophantine approximation, there exist integers \( a_1, a_2, q_1, q_2 \) depending on \( \alpha \) such that
\[ |q_1 \lambda_1 \alpha - a_1| \leq X^{-1+1/100}, \quad |q_2 \lambda_2 \alpha - a_2| \leq X^{-1+1/100} \]
with \( (a_1, q_1) = 1 \) and \( 1 \leq q_j \leq X^{1-1/100} \). Since \( \alpha \in \mathfrak{m} \setminus \mathfrak{m}^* \), we see that \( a_1a_2 \neq 0 \) and \( |a_1|/|\alpha| < q_j \). Now we assert that
\[ \max(q_1, q_2) \geq X^{1/100}. \] (4.3)

We will reason by absurdity. Suppose both \( q_1 \) and \( q_2 \) are less that \( X^{1/100} \). We have
\[ |a_2 q_1 \lambda_1 / \lambda_2 - a_1 q_2| = \left| \frac{a_2}{\lambda_2} (q_1 \lambda_1 \alpha - a_1) - \frac{a_1}{\lambda_2} (q_2 \lambda_2 \alpha - a_2) \right| \ll X^{-1+1/50}. \]
Since there is a convergent \( a/q \) to \( \lambda_1 / \lambda_2 \) with \( q = 5^{1/12} \). Thus we have
\[ |a_2 q_1 \lambda_1 / \lambda_2 - a_1 q_2| = o(q^{-1}). \] (4.4)

But
\[ |a_2 q_1| \ll q_1 q_2 \ll X^{1/50} = o(q). \] (4.5)

This contradicts the definition of \( q \) as the denominator of a convergent to \( \lambda_1 / \lambda_2 \) (see Lemma 9 of [1]). Thus one of \( q_1, q_2 \) is greater than \( X^{1/100} \). Then, by Lemmas 3.1 and 3.5, we have
\[ \min \left( |S_1(\lambda_1 \alpha_1)|, |S_2(\lambda_2 \alpha_2)|^2 \right) \ll X^{1-1/200+\epsilon}. \] (4.6)

Hence, by the arguments of Lemma 4.6 of [3], it is easy to get
\[ \beta(\mathfrak{m} \setminus \mathfrak{m}^*) = o(\tau^2 X^{77/60}). \]

5 The minor arc

First, we divide the minor arc \( m \) into four parts. Let \( m' = m_1 \cup m_2 \cup m_3 \), and \( m_0 = m \setminus m' \), where
\[ m_1 = \{ \alpha \in m : |S_1(\lambda_1 \alpha)| \leq X^{1-1/6\epsilon} \}, \]
\[ m_2 = \{ \alpha \in m : |S_1(\lambda_1 \alpha)| > X^{1-1/6\epsilon}; |S_2(\lambda_2 \alpha)| > X^{1/2-1/16\epsilon} \}, \]
\[ m_3 = \{ \alpha \in m : |S_1(\lambda_1 \alpha)| > X^{1-1/6\epsilon}; |S_4(\lambda_4 \alpha)| > X^{1/4-1/96\epsilon} \} \].

Now, we begin to estimate the integral on \( m_j \) respectively. First, it is easy to see that
\[ \beta(m_1) \ll \left( \max_{\alpha \in m_1} |S_1(\lambda_1 \alpha)| \right)^{3/16} \left( \int_{\mathbb{R}} |S_1(\lambda_1 \alpha)|^2 K_r(\alpha) d\alpha \right)^{13/32} \left( \int_{\mathbb{R}} |S_3(\lambda_3 \alpha)|^8 K_r(\alpha) d\alpha \right)^{3/32} \] (5.1)
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\[ \times \left( \int_\mathbb{R} |S_2(\lambda_2 a)^2 S_4(\lambda_4 a)^4 K(r(a))da \right)^{1/4} \times \left( \int_\mathbb{R} |S_2(\lambda_2 a)^2 S_5(\lambda_5 a)^6 K(r(a))da \right)^{1/8} \]

\[ \ll (X^{1-1/6+\varepsilon})^{3/16} (\tau X^{5/3+\varepsilon})^{3/32} (\tau X^{1+\varepsilon})^{1/4} (\tau X^{6/5+\varepsilon}) \ll \tau X^{77/60-1/32+\varepsilon}. \]

**Lemma 5.1.** We have

\[ \mathcal{B}(m_2) \ll \tau X^{77/60-1/32+\varepsilon}. \]  

**Proof.** We use the method of Harman [7]. We divide \( m_2 \) into disjoint sets such that for \( \alpha \in \mathcal{A}(Z_1, Z_2, y) \), we have

\[ Z_1 \leq |S_1(\lambda_1 a)| < 2Z_1 \quad \text{or} \quad Z_2 \leq |S_2(\lambda_2 a)| < 2Z_2 \quad \text{or} \quad y \leq |a| < 2y, \]

where \( Z_1 = X^{1-1/6+\varepsilon} 2^{t_1} \), \( Z_2 = X^{1/2-1/16+\varepsilon} 2^{t_2} \), \( y = 2^s \) for some positive integers \( t_1, t_2, s \). Thus, by Corollaries 3.2 and 3.6, there exist two pairs of coprime integers \((a_1, q_1), (a_2, q_2)\) with \( a_1 a_2 \neq 0 \) and

\[ 1 \leq q_1 \ll (X/Z_1)^2 X^\varepsilon, \quad |q_1 \lambda_1 a - a_1| \ll (X/Z_1)^2 X^{\varepsilon-1}; \]

\[ 1 \leq q_2 \ll (X^{1/2}/Z_2)^4 X^\varepsilon, \quad |q_2 \lambda_2 a - a_2| \ll (X^{1/2}/Z_2)^4 X^{\varepsilon-1}. \]

Then for any \( \alpha \in \mathcal{A}(Z_1, Z_2, y) \), we have \( |a_j/a| \ll q_j \).

Let \( \mathcal{A}' = \mathcal{A}(Z_1, Z_2, y, Q_1, Q_2) \) be the subset of \( \mathcal{A}(Z_1, Z_2, y) \) for which \( q_j \sim Q_j \). Then, by a familiar argument (see P. 147 of [17] for example),

\[ |a_2 q_1 \lambda_1 a_1 - a_1 q_2| = \frac{|a_2 q_1 \lambda_1 a - a_1|}{\lambda_2 a} \ll Q_2 (X/Z_1)^2 X^{\varepsilon-1} + Q_1 (X^{1/2}/Z_2)^4 X^{\varepsilon-1} \ll \frac{X^{3+2\varepsilon}}{Z_1^2 Z_2^2} \ll X^{-5/12-4\varepsilon}. \]

Also

\[ |a_2 q_1| \ll y Q_1 Q_2. \]

Note that \( q = X^{5/12} \). We have

\[ \left\| a_2 q_1 \frac{\lambda_1 a_1}{\lambda_2} \right\| \leq \frac{1}{4q}, \quad q_1 \sim Q_1, \quad a_2 \sim y Q_2, \]

\[ (5.3) \]

since \( X \) is sufficiently large. Then by the pigeon-hole principle and the Legendre’s law of best approximation for continued fractions, the above inequality (5.7) have \( \ll y Q_1 Q_2 q^{-1} \) solutions of \( |a_2 q_1| \) (see Lemma 9 of [1]). Clearly, each value of \( |a_2 q_1| \) corresponds to \( \ll X^\varepsilon \) values of \( a_1, a_2, q_1, q_2 \) by the well-known bound on the divisor function. Hence, we conclude that

\[ \mu(\mathcal{A}') \ll X^{5/12} y Q_1 Q_2 \min \left( \frac{(X/Z_1)^2 X^{\varepsilon-1} Q_1}{q}, \frac{(X^{1/2}/Z_2)^4 X^{\varepsilon-1} Q_2}{q} \right) \]

\[ \ll X^{5/12} y Q_1 Q_2 \frac{X^{1+\varepsilon}}{Z_1 Z_2^2 Q_1^{1/2} Q_2^{1/2}} \ll \frac{X^{1+\varepsilon} y Q_1^{1/2} Q_2^{1/2}}{q Z_1 Z_2} \ll \frac{X^{3+3\varepsilon} y}{q Z_1^2 Z_2^2}, \]

\[ (5.5) \]

where \( \mu(\mathcal{A}') \) is the Lebesgue measure of \( \mathcal{A}' \). Thus we have

\[ \mathcal{B}(\mathcal{A}') \ll Z_1 Z_2 X^{1/3+1/4+1/5} \mu(\mathcal{A}') \min(r^2, y^{-2}). \]
\[ \tau \frac{X^{27/60+3\varepsilon}}{q Z_1 Z_2^3} \ll \tau X^{77/60-1/16+\varepsilon}. \]

Summing over all possible values of \( Z_1, Z_2, y, Q_1, Q_2 \), we conclude that
\[
\mathcal{B}(m_2) \ll \tau X^{77/60-1/32+\varepsilon}. \tag{5.5}
\]

**Lemma 5.2.** We have
\[
\mathcal{B}(m_3) \ll \tau X^{77/60-1/32+\varepsilon}. \tag{5.6}
\]

**Proof.** The proof is similar to that of Lemma 5.1, we only give a brief proof. We divide \( m_3 \) into disjoint sets such that for \( a \in A(Z_1, Z_2, y) \), we have
\[
Z_1 \leq |S_1(\lambda_1 a)| < 2Z_1 \text{ or } Z_2 \leq |S_4(\lambda_4 a)| < 2Z_2 \text{ or } y \leq |a| < 2y,
\]
where \( Z_1 = X^{1/6+\varepsilon} t_1, Z_2 = X^{1/6-1/96+\varepsilon} t_2, \) \( y = 2^s \) for some positive integers \( t_1, t_2, s \). Thus, by Corollaries 3.2 and 3.4, there exist two pairs of coprime integers \((a_1, q_1), (a_2, q_2)\) with \( a_1 a_2 \neq 0 \) and
\[
1 \leq q_1 \ll (X/Z_1)^2 X^\varepsilon, \quad |q_1 \lambda_1 a - a_1| \ll (X/Z_1)^2 X^{\varepsilon-1};
\]
\[
1 \leq q_2 \ll (X^{1/4}/Z_2)^2 X^\varepsilon, \quad |q_2 \lambda_4 a - a_2| \ll (X^{1/4}/Z_2)^2 X^{\varepsilon-1}.
\]

Let \( A' = A(Z_1, Z_2, y, Q_1, Q_2) \) be the subset of \( A(Z_1, Z_2, y) \) for which \( q \sim Q_j \). Then,
\[
\left| a_2 q_1 \frac{\lambda_1}{\lambda_4} - a_1 q_2 \right| \ll \frac{X^{1/2+2\varepsilon}}{Z_1^2 Z_2^2} \ll X^{-31/48-2\varepsilon}.
\]

Also
\[
|a_2 q_1| \ll y Q_1 Q_2.
\]

Since \( q' \approx q = X^{5/12} \), we have
\[
\left| a_2 q_1 \frac{\lambda_1}{\lambda_4} \right| \leq \frac{1}{4 q'}, \quad q_1 \sim Q_1, \quad a_2 \sim y Q_2.
\tag{5.7}
\]

Hence, we conclude that
\[
\mu(A') \ll X^e y Q_1 Q_2 \frac{X^{1/4+\varepsilon}}{q' Z_1 Z_2 Q_1^{1/2} Q_2^{1/2}} \ll \frac{X^{1/4+2\varepsilon} y Q_1^{1/2} Q_2^{1/2}}{q' Z_1 Z_2} \ll X^{3/2+3\varepsilon} y \frac{X^{3/2+3\varepsilon} y}{q' Z_1 Z_2}. \tag{5.8}
\]

Thus by Lemma 3.7, we have
\[
\mathcal{B}(A') \ll \left( \int_{A'} |S_1(\lambda_1 a)S_4(\lambda_4 a)| K_1(a) da \right)^{1/2} \left( \int_{A'} |S_2(\lambda_2 a)S_3(\lambda_3 a)S_5(\lambda_5 a)|^2 K_1(a) da \right)^{1/2}
\ll \left( \tau X^{16/15+\varepsilon} \right)^{1/2} \left( \min(\tau^2, y^{-2}) Z_1^2 Z_2^2 \frac{X^{3/2+3\varepsilon} y}{q' Z_1 Z_2} \right)^{1/2}
\ll \tau \frac{X^{77/60+2\varepsilon}}{(q')^{1/2}} \ll \tau X^{77/60-5/24+2\varepsilon}.
\]

Summing over all possible values of \( Z_1, Z_2, y, Q_1, Q_2 \), we conclude that
\[
\mathcal{B}(m_3) \ll \tau X^{77/60-1/32+\varepsilon}.
\]
Lemma 5.3. We have
\[ \mathcal{J}(m_\alpha) \ll \tau X^{7/60-1/32+\epsilon}. \] (5.9)

Proof. We use the method of the first author and Zhao [4]. First, by Cauchy’s inequality, we get
\[ \mathcal{J}(m_\alpha) \ll \left( \int_{\mathbb{R}} |S_1(\lambda_1 \alpha)|^2 K_\alpha(\alpha) \, d\alpha \right)^{1/2} \mathcal{J}(2)^{1/2} \ll (\tau X^{1+\epsilon})^{1/2} \mathcal{J}(2)^{1/2}, \] (5.10)
where
\[ \mathcal{J}(t) = \int_{m_\alpha} |S_2(\lambda_2 \alpha)^2 S_3(\lambda_3 \alpha)^3 S_4(\lambda_4 \alpha)^2 S_5(\lambda_5 \alpha)^2| K_\alpha(\alpha) \, d\alpha. \] (5.11)

Then we have
\[ \mathcal{J}(2) = \sum_{p \in \mathcal{P}_3} (\log p) \int_{m_\alpha} \left| e(\alpha \lambda_3 p^3) S_3(-\lambda_3 \alpha) S_2(\lambda_2 \alpha)^2 S_4(\lambda_4 \alpha)^2 S_5(\lambda_5 \alpha)^2| K_\alpha(\alpha) \, d\alpha \right|. \]

Then, by Cauchy’s inequality, we get
\[ \mathcal{J}(2) \ll P_3^{1/2} (\log X) \mathcal{L}^{1/2}, \] (5.12)
where
\[ \mathcal{L} = \sum_{n \in \mathcal{P}_3} \left| \int_{m_\alpha} e(\alpha \lambda_3 n^3) S_3(-\lambda_3 \alpha) S_2(\lambda_2 \alpha)^2 S_4(\lambda_4 \alpha)^2 S_5(\lambda_5 \alpha)^2| K_\alpha(\alpha) \, d\alpha \right|^2. \]

For the sum \( \mathcal{L} \), we have
\[ \mathcal{L} = \sum_{n \in \mathcal{P}_3} \left| \int_{m_\alpha} \left( S_2(\lambda_2 \alpha)^2 S_4(\lambda_4 \alpha)^2 S_5(\lambda_5 \alpha)^2 S_2(\lambda_2 \alpha)^2 S_4(\lambda_4 \alpha)^2 S_5(\lambda_5 \alpha)^2 \right| - \right. S_3(-\lambda_3 \alpha) S_3(\lambda_3 \beta) e(\lambda_3 n^3 (\alpha - \beta)) K_\alpha(\alpha) K_\beta(\beta) \, d\alpha \, d\beta \leq \int_{m_\alpha} \left| S_2(\lambda_2 \alpha)^2 S_4(\lambda_4 \alpha)^2 S_5(\lambda_5 \alpha)^2 S_3(-\lambda_3 \alpha) T(\lambda_3 (\alpha - \beta)) K_\alpha(\alpha) \, d\alpha \right|, \] (5.13)

where
\[ F(\beta) = \int_{m_\alpha} \left| S_2(\lambda_2 \alpha)^2 S_4(\lambda_4 \alpha)^2 S_5(\lambda_5 \alpha)^2 S_3(-\lambda_3 \alpha) T(\lambda_3 (\alpha - \beta)) \right| K_\alpha(\alpha) \, d\alpha \] (5.14)
and
\[ T(x) = \sum_{n \in \mathcal{P}_3} e(xn^3). \]

Let \( M_\beta(r, b) = \{ \alpha \in m_\alpha : |r\lambda_3 (\alpha - \beta) - b| \leq P_3^{-9/4} \}. \) Then the set \( M_\beta(r, b) \neq \emptyset \) forces that
\[ |b + r\lambda_3 \beta| \leq |r\lambda_3 (\alpha - \beta) - b| + |r\lambda_3 \alpha| \leq P_3^{-9/4} + r|\lambda_3| |\xi|. \]

Let \( \mathcal{B} = \{ b \in \mathbb{Z} : |b + r\lambda_3 \beta| \leq P_3^{-9/4} + r|\lambda_3| |\xi| \}. \) We divide the set \( \mathcal{B} \) into two sets \( \mathcal{B}_1 = \{ b \in \mathbb{Z} : |b + r\lambda_3 \beta| \leq r|\lambda_3| |\xi|^{-1} \} \) and \( \mathcal{B}_2 = \mathcal{B} \setminus \mathcal{B}_1. \) Let
\[ M_\beta = \bigcup_{1 \leq r \leq P_3^{-9/4}} \bigcup_{|\lambda_3| > 1} M_\beta(r, b). \]
Then by Lemma 3.8, we have

\[
F(\beta) \ll P_3 \int_{M_\beta} \frac{|S_2(\lambda_3^2)S_4(\lambda_3^2)S_5(\lambda_3^2)S_3(-\lambda_3^2)|w_3(r)K_r(\alpha)}{1 + P_3^3|\lambda_3(\alpha - \beta) - b/r|} \, da + P_3^{3/4 + \epsilon} \gamma(1),
\] (5.15)

where \(w_3(r)\) is defined as in (3.3). Note that \(|S_2(\lambda_3^2)| \leq P_2X^{-1/16+\epsilon}\) and \(|S_4(\lambda_3^2)| \leq P_4X^{-1/96+\epsilon}\) for \(\alpha \in m_4\).

Then, by Cauchy’s inequality, we get

\[
\leq \left( \int_{m_0} \frac{|S_2(\lambda_3^2)S_3(\lambda_3^2)S_4(\lambda_3^2)S_5(\lambda_3^2)|w_3(r)K_r(\alpha)}{1 + P_3^3|\lambda_3(\alpha - \beta) - b/r|} \, da \right)^{1/2}
\]

\[
\ll P_2P_4X^{-7/96+\epsilon} \gamma(2)^{1/2} \gamma(\beta)^{1/2},
\] (5.16)

where

\[
\gamma(\beta) = \int_{M_\beta} |S_5(\lambda_5^2)|w_3(r)K_r(\alpha) \frac{\, da}{(1 + P_3^3|\lambda_3(\alpha - \beta) - b/r|)^2}.
\] (5.17)

Now we begin to estimate the integral \(\gamma(\beta)\). First, we divide it into two parts.

\[
\gamma(\beta) = \sum_{1 \leq \lambda \leq P_3^{1/4}} \sum_{b \in \mathbb{Z}, \alpha \in M_\beta(r,b)} \int_{M_\beta} \frac{|S_2(\lambda_3^2)S_3(\lambda_3^2)S_4(\lambda_3^2)S_5(\lambda_3^2)|w_3(r)K_r(\alpha)}{1 + P_3^3|\lambda_3(\alpha - \beta) - b/r|} \, da
\]

\[
= \gamma_1(\beta) + \gamma_2(\beta),
\] (5.18)

where

\[
\gamma_1(\beta) = \sum_{1 \leq \lambda \leq P_3^{1/4}} \sum_{b \in \mathbb{Z}, \alpha \in M_\beta(r,b)} \int_{M_\beta} \frac{|S_2(\lambda_3^2)S_3(\lambda_3^2)S_4(\lambda_3^2)S_5(\lambda_3^2)|w_3(r)K_r(\alpha)}{1 + P_3^3|\lambda_3(\alpha - \beta) - b/r|} \, da.
\] (5.19)

For the first part, we have

\[
\gamma_1(\beta) \ll r^2 \sum_{1 \leq \lambda \leq P_3^{1/4}} w_3(r)^2 \sum_{b \in \mathbb{Z}, \alpha \in M_\beta(r,b)} \int_{M_\beta} \frac{|S_5(\lambda_5(\beta + \gamma) + b\lambda_5/(r\lambda_3))|^2}{(1 + P_3^3|\lambda_3\gamma|)^2} \, d\gamma
\]

\[
\ll r^2 \sum_{1 \leq \lambda \leq P_3^{1/4}} w_3(r)^2 \int_{|\lambda_3\gamma| \leq P_3^{1/4}} \frac{U(B_1^*)}{(1 + P_3^3|\lambda_3\gamma|)^2} \, d\gamma,
\]

where

\[
U(B_1^*) = \sum_{b \in B_1^*} |S_5(\lambda_5(\beta + \gamma) + b\lambda_5/(r\lambda_3))|^2,
\]

and

\[
B_1^* = \{ b \in \mathbb{Z} : -r((|\lambda_3|^2r^{-1} + 1) < b + r\lambda_3\beta \leq r((|\lambda_3|^2r^{-1} + 1)\}
\]

Since \(\lambda_3/\lambda_3\) is rational, we take \(\lambda_3/\lambda_3 = u/v\) with \(u, v \in \mathbb{Z}\) and \((u, v) = 1\). We take \(r_1 = \frac{r}{(u,v)}\). Then we have

\[
U(B_1^*) = \sum_{p_1, p_2 \leq \lambda_3} \sum_{b \in B_1^*} e((\lambda_5(\beta + \gamma) + b\lambda_5/(r\lambda_3))(p_1^2 - p_2^2))
\]
Thus, by Lemma 3.9, we have
\[
\left| \sum_{p_1, p_2 \in \mathcal{P}_3} \sum_{b \in \mathcal{B}_q^*} e \left( \frac{bu}{r^2} (p_1^2 - p_2^2) \right) \right| 
\leq 2 rv (|\lambda_3| \nu^{-1} r^{-1} + 1) \sum_{p_1, p_2 \in \mathcal{P}_3} \sum_{b \in \mathcal{B}_q^*} 1 
\ll r^{-1} \sum_{p_1, p_2 \in \mathcal{P}_3} 1 
\ll r^{-1} P_3^2 (r_1 \nu)^{-2} \sum_{1 \leq b \leq r_1 \nu, \nu b \equiv \lambda_3 (\nu \mod \nu)} 1 
\ll r^{-1} P_3^2 (r_1 \nu)^{-1} \sum_{1 \leq b \leq \nu} 1 
\ll r^{-1} P_3^2 d(r)^c.
\]

Thus, by Lemma 3.9, we have
\[
\mathcal{J}_1(\beta) \ll \tau P_3^2 \sum_{1 \leq r \leq P_3^{1/4}} w_3(r)^2 d(r)^c \int \frac{1}{(1 + P_3^3 |\lambda_3 \gamma|)^2} d\gamma 
\ll \tau P_3^2 X^{-1} \sum_{1 \leq r \leq P_3^{1/4}} w_3(r)^2 d(r)^c \ll \tau P_3^2 X^{-1+\epsilon}. \tag{5.20}
\]

Now, we begin to estimate \( \mathcal{J}_2(\beta) \). First, without loss of generality we need only consider the set
\[
\mathcal{B}_2 = \{ b \in \mathbb{Z} : r|\lambda_3| \nu^{-1} < b + r\lambda_3 \beta \leq P_3^{9/4} + r|\lambda_3| \nu \}
\]
which falls in the set
\[
\mathcal{B}_2^* = \{ b \in \mathbb{Z} : rv \kappa_1 < b + r\lambda_3 \beta \leq rv \kappa_2 \},
\]
where \( \kappa_1 = |\lambda_3| \nu^{-1} \) and \( \kappa_2 = |\lambda_3| \nu^{-1} \xi + 2 \). Then we have
\[
\mathcal{J}_2(\beta) \ll \sum_{1 \leq r \leq P_3^{1/4}} \int \frac{|S_2(\lambda_3 \alpha)|^2 w_3(r)^2 K_\tau(a)}{(1 + P_3^3 |\lambda_3 \alpha - b/r|)^2} da 
\ll \sum_{1 \leq r \leq P_3^{1/4}} w_3(r)^2 \int \frac{|S_2(\lambda_3 \alpha)|^2 |a|^{-2}}{(1 + P_3^3 |\lambda_3 \alpha - b/r|)^2} da 
\ll \sum_{1 \leq r \leq P_3^{1/4}} w_3(r)^2 \sum_{k_1 \leq k_2} \frac{1}{(k-1)^2} \sum_{rv \kappa_1 < b + r\lambda_3 \beta \leq rv \kappa_2} \int \frac{|S_2(\lambda_3 \alpha)|^2}{(1 + P_3^3 |\lambda_3 \alpha - b/r|)^2} da 
\ll \sum_{1 \leq r \leq P_3^{1/4}} w_3(r)^2 \sum_{k_1 \leq k_2} \frac{1}{(k-1)^2} \int \frac{U(c_k)}{(1 + P_3^3 |\lambda_3 \gamma|)^2} d\gamma,
\]
where \( c_k = \{ b \in \mathbb{Z} : rv \kappa_1 < b + r\lambda_3 \beta \leq rv (k + 1) \} \). On the other hand, similar to the above estimate of \( U(\mathcal{B}_2^*) \), we have \( U(c_k) \ll P_3^2 d(r)^c \). Thus we have
\[
\mathcal{J}_2(\beta) \ll P_3^2 X^{-1} \sum_{1 \leq r \leq P_3^{1/4}} w_3(r)^2 d(r)^c \sum_{k_1 \leq k_2} \frac{1}{(k-1)^2} \ll \tau P_3^2 X^{-1+\epsilon}. \tag{5.21}
\]
Combining (5.15)-(5.21), we have
\[
F(\beta) \ll \tau^{1/2} P_2 P_3 P_4 P_5 X^{-1/2-7/9\epsilon} \gamma(2)^{1/2} + P_3^{1/6+\epsilon} \gamma(1) \tag{5.22}
\]
uniformly for $\beta \in \mathbb{R}$.

Hence, by (5.12), (5.13) and (5.22), we have

$$\mathcal{J}(2) \ll P_2^{7/8} \mathcal{J}(1) + \tau^{1/4} (P_2 P_3^2 P_4 P_5) 1/2 X^{-1/4-7/192+\epsilon} \mathcal{J}(1)^{1/2} \mathcal{J}(2)^{1/4}. \quad (5.23)$$

By Hölder’s inequality and Lemma 3.7, we have

$$\mathcal{J}(1) \leq \mathcal{J}(2)^{1/3} \left( \int_{\mathbb{R}} |S_2(\lambda_2 a)^2 S_4(\lambda_4 a)^6 |K_T(a) da \right)^{1/3} \left( \int_{\mathbb{R}} |S_2(\lambda_2 a)^2 S_3(\lambda_3 a)^2 |K_T(a) da \right)^{1/6} \left( \int_{\mathbb{R}} |S_2(\lambda_2 a)^2 S_5(\lambda_5 a)^2 |K_T(a) da \right)^{1/6}$$

$$\ll \mathcal{J}(2)^{1/3} (\tau P_2^2 P_3^6 X^{-1+\epsilon})^{1/3} (\tau P_2^2 P_3^2 P_4^2 X^{-1+\epsilon})^{1/6} (\tau P_2^2 P_5^6 X^{-1+\epsilon})^{1/6} \ll \mathcal{J}(2)^{1/3} (\tau P_2^2 P_3^{1/2} P_4^2 P_5^2 X^{-1+\epsilon})^{2/3}.$$

Thus we have

$$\mathcal{J}(2) \ll P_2^{7/8} \mathcal{J}(2)^{1/3} (\tau P_2^2 P_3^{1/2} P_4^2 P_5^2 X^{-1+\epsilon})^{2/3} + \mathcal{J}(2)^{5/12} \tau^{7/12} (P_2 P_3 P_4 P_5)^{7/6} X^{-7/12-7/192+\epsilon}.$$

Then this implies

$$\mathcal{J}(2) \ll \tau P_2^2 P_3^{1/2} P_4^2 P_5^2 X^{-1+\epsilon} + \tau (P_2 P_3 P_4 P_5)^2 X^{-1-1/16+\epsilon} \ll \tau X^{47/30-1/16+\epsilon}. \quad (5.24)$$

Then (5.9) follows from (5.10) and (5.24) immediately. \hfill \square

### 6 Completion of the proof of Theorem 1.1

We take $\tau = X^{1/32+2\epsilon}$. Combining (2.4), (5.1) and Lemmas 4.1, 4.2, 5.1, 5.2, 5.3, we deduce that $\beta(\mathbb{R}) \gg \tau^2 X^{77/60}$. Thus by (2.3), we have

$$\mathcal{N}(\eta, X) \gg \tau X^{77/60} (\log X)^{-5}.$$

Note that $\max(p_j) = X$, so $\tau \asymp \max(p_j)^{-1/32+2\epsilon}$. Then we see that the following inequality

$$|\lambda_1 p_1 + \cdots + \lambda_5 p_5^2 + \eta| \ll \max(p_j)^{-1/32+2\epsilon}$$

has $\tau X^{77/60} (\log X)^{-5}$ solutions in primes $p_j$. Since $X = q^{12/5}$ and $\lambda_1/\lambda_2$ is irrational, there are infinitely many pairs of integers $q, a$. This implies that the last inequality has infinitely many solutions in primes $p_j$.

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