Non-Fourier heat removal from hot nanosystems through graphene layer

Abstract
Nonlocal effects on heat transport beyond a simple Fourier description are analyzed in a thermodynamical model. In the particular case of hot nanosystems cooled through a graphene layer, it is shown that these effects may increase in a ten percent the amount of removed heat, as compared with classical predictions based on the Fourier law.

Keywords
Nonlocal effects • nanosystems cooling • non-Fourier theory

1. Introduction
The raise of nanotechnology requires even increasing efforts to better understand the thermal-transport properties of nanodevices, whose performance and reliability are much influenced by memory, nonlocal and nonlinear effects [14, 24, 25, 41, 45]. One of the aspects which are being studied is the heat removal and consequent cooling of very small areas heated by some external source, either static (as hot spot due to a very miniaturized working device), or a fast laser pulse, for instance. The analysis of heat transfer at very small distances in these circumstances provides a tool for the exploration of the behavior of phonon transport. Since the phonon collision time depends on the phonon frequency, one of the topics of research is the measurement of the frequency dependence of phonon mean-free path (mfp). Low-frequency phonons may have long mfps and behave ballistically, whereas high-frequency phonons have short mfps and behave diffusively.

Here we will study heat removal from a small hot spot by using a surrounding cooling layer. Our analysis is based on the following generalized evolution equation for the heat flux $\mathbf{q}$ [15, 24, 42]:

$$\tau R \dot{\mathbf{q}} + \mathbf{q} = -\lambda_0 \nabla T + \ell^2 \left( \nabla^2 \mathbf{q} + 2 \nabla \nabla \cdot \mathbf{q} \right),$$

(1)

$\tau R$ being a relaxation time, $\lambda_0$ the thermal conductivity, $T$ the temperature, and $\ell$ the mfp of the heat carriers. Equation (1), which is well-known in literature as the Guyer-Krumhansl equation [1, 21, 22] and follows from the linearized Boltzmann equation for phonons, reduces to the also well-known Maxwell-Cattaneo equation [12] whenever the last terms (that is, the product of $\ell^2$ and the second-order spatial derivatives of $\mathbf{q}$) are neglected in it. Indeed, these nonlocal terms arise in the kinetic theory, in the so-called Burnett approximation [10, 13], and play a crucial role whenever the inhomogeneities in $\mathbf{q}$ are relevant enough in $\ell$-size regions [24, 39]. These terms are well-known in molecular hydrodynamics [23, 24], but they have not received sufficient attention in heat transport at nanoscale. Thermodynamic derivations of Eq. (1) and of other hydrodynamic-like equations for phonon heat transport have been discussed in many occasions, as they imply modifications to the entropy and entropy flux [11, 15–17, 19, 20, 23–25, 34, 43, 44]. The hypothesis of a single average

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Fig 1. Cylindrical nanodevice connected to a graphene layer. The internal device (red in figure) is characterized by a radius $r_0$ and a thickness $h_0$. The surrounding graphene layer (grey in figure) is characterized, instead, by an outer radius $r_g$ and a thickness $h_g$. Indeed, for graphene one could practically speak about a sheet as its thickness $h_g$ is approximately equal to $3.35 \times 10^{-10}$ m, namely, it is of the order of the atomic diameter of carbon. The graphene layer removes heat from the core device, which acts so as a steady heat source.

The mfp in Eq. (1) may have some limitations, as compared with the possibility of two populations characterized one by a long mfp, and another one by a short mfp [26–28]. However, here we will make the simplest assumption, in order to explore the main qualitative consequences of nonlocal contributions to the heat flux.

In nanoscale devices, a mfp larger than the typical size of the device reduces the thermal conductivity to a size-dependent value [33, 38], because the heat carriers scatter at the boundaries before they should do in the bulk. However, outside of the device, and in absence of boundaries, a long mfp of heat carriers will increase heat removal, because the heat flow will proceed without many collisions. This would be the case of a hot spot connected to a graphene layer around it. The heat removed from the hot spot per unit time through the graphene layer will be higher than that in more usual materials, because the mfp in graphene is very long, of the order hundreds nanometers at room temperature [7, 8, 18, 31, 32]. But, in these systems, a long mfp may also have additional effects in heat transport, beyond those appearing in the classical Fourier law [30, 33]. This may be of special interest for heat removal from nanosystems, for example.

Some physical consequences of Eq. (1) have been discussed for nanowires and thin layers [2, 3, 35, 36], but the application to plane situation with axial symmetry have been only discussed in a few occasions [39]. Here we study the consequences of that equation on heat removal from hot nanosystems through graphene layers. Such removal becomes more and more necessary as electronic devices are increasingly miniaturized. Indeed, such miniaturization, as well as an increase in the computational rate, much enhances the rate of heat per unit volume which has to be dissipated, thus being a limiting factor in current computer miniaturization [6]. The faster heat is removed, the more efficient the devices run and the longer they last. The heat is usually removed by diffusing it along thin plates, from which it is given away to the environment. However, as we said above, the thermal conductivity gets smaller for thinner plates. For these purposes, the use of graphene is especially promising because of its high thermal conductivity [6, 7, 18, 31, 32]. Graphene could become so a part of computer chips alongside silicon, as well as transferring heat from solar panels.

The plan of the paper is the following. In Sec. 2 we compare the transversal rate of heat removal from a cylindrical miniaturized device connected to a concentric circular layer of graphene, when a Fourier like and a non-Fourier like heat-transport equation are considered. In Sec. 3 a more realistic situation involving the heat exchange between the graphene layer and the surrounding environment is considered. Section 4 is devoted to summarize and discuss the obtained results.

2. Radial heat removal from a cylindrical nanodevice

Higher circuit densities and faster clock speeds yield chips run so hot that heat is becoming one of the major problem in the design of integrated circuits. Incorporation of graphene into chip design could yield devices that are faster, less noisy, and run cooler [6]. Indeed, the usefulness of graphene depends not only on its physical properties, but also on its compatibility with the physics at low dimensions. Here we study the influence of nonlocal effects in removing heat from a hot device through flat a system. In more specific details, we suppose a cylindrical nanodevice (whose radius is $r_0$, and the thickness is $h_0$) acting as a steady heat source (the temperature of which is held at the constant value $T_0$) and connected to a graphene circular layer (whose outer radius is $r_g$, and the thickness is $h_g$), which removes the heat from that nanodevice (see Fig. 1 for a qualitative sketch). We assume the steady-state situation, in which the heat removed is equal to the heat dissipated, thus allowing the temperature to remain constant.

As a first step of the analysis, here we compare the steady-state results of Eq. (1) with those obtained from the
classical Fourier law. The motivation for this analysis is that the phonon mfp $\ell$ in graphene at room temperature is very long, i.e., it is of the order of $775 \times 10^{-9}\text{m}$ \cite{7, 8, 18, 31, 32}. Thus, if the radius of the core nanodevice is of the order of $\ell$, and the width of the graphene layer is a few times $\ell$, the nonlocal effects related to the second-order spatial derivatives of $q$ in Eq. (1) are expected to be very relevant. Before making a full analysis, in the present section we will estimate the relevance of such nonlocal effects in a much simplified situation, in which heat is carried away radially along the layer, without being transversally transferred to the environment. This analysis allows to carry out a first set of illustrative results.

Due to the axial symmetry, in steady-state situations the heat flux in the graphene circular layer has only one component. To obtain the radial dependence of $q$, one may observe that in the absence of transversal exchanges, the total heat flux flowing across each concentric circular area is always the same. In more details, once two different concentric circular areas of radial distance from the source equal to $r$ and $r + dr$ have been chosen, then the radial profile of the local heat flux is such that

$$2\pi q (r) = 2\pi (r + dr) q (r + dr) \approx 2\pi (r + dr) \left[ q (r) + \frac{dq}{dr} dr \right] \Rightarrow \frac{dq}{dr} + q (r) = 0, \quad (2)$$

when a first-order approximation in $dr$ is used. Equation (2) may be easily integrated in order to have

$$q (r) = \frac{\Gamma}{r}, \quad (3)$$

with $\Gamma = Q / (2\pi h \ell)$ as a constant value, being $Q$ the heat removed from the hot device per unit time. The use of this radial heat-flux profile implicitly yields that we are assuming the layer behaves isotropically, namely, we suppose that the heat propagates in the same way along all directions in the surrounding layer.

The introduction of Eq. (3) in the steady-state version of Eq. (1), for $r > r_0$, at first gets

$$\frac{\lambda_0}{\ell} \frac{dT}{dr} = \Gamma \left( \frac{\ell^2}{r^2} - \frac{1}{r} \right), \quad (4)$$

and then, by a simple integration of this ordinary differential equation, we are led to the following temperature profile:

$$T_{df} (r) = T_0 + \frac{\Gamma}{\lambda_0} \left[ \frac{\ell^2}{2r_0^2} \left( 1 - \frac{r_0^2}{r^2} \right) + \ln \left( \frac{r_0}{r} \right) \right], \quad (5)$$

where the subscript $f$ means the use of a non-Fourier law for the heat flux.

In obtaining the temperature profile (5) we assumed that the temperature of the hot device is $T_0$, i.e., $T (r) \equiv T_0$ for $r < r_0$. This assumption seems to be logical in systems working at regime wherein the hot device has reached a limit value of tolerable temperature, due to the incessant heat produced by the high computation rate, for example. The consequent use of a cooling system is needed to avoid the temperature overcomes the fusion threshold. In other words, we are assuming the worst case in which the layer removes the running produced heat from the hot source which, therefore, remains at a fixed and constant (but acceptable) temperature. Here we have assumed that the temperature $T_0$ of the graphene layer in contact with the device is the same as the device temperature. However, in the presence of a thermal resistance between the device and the graphene layer, there would be a discontinuity in temperature in the device-graphene contact. Note, however, that we are interested in the heat flux across the graphene layer, so that what appears in our analysis is the graphene temperature rather than the device temperature.

In the absence of the nonlocal terms (i.e., if $\ell = 0$ in Eq. (1)), instead, one would simply have

$$T_i (r) = T_0 + \frac{\Gamma}{\lambda_0} \ln \left( \frac{T_0}{r} \right), \quad (6)$$

and the subscript $f$ indicates the use of the classical Fourier law for $q$. The differences between the profiles given by Eqs. (3) and (6) have been analyzed from the physical point of view in Sellitto et al. \cite{39}. However, if we aim to cool
the hot cylindrical device (that is, if we aim to keep it at the constant temperature \(T_0\)) by the circular graphene layer (the outer edge of which is at a smaller temperature \(T_g\)), we have to focus the attention on \(\Gamma\). In this situation, setting \(r = r_g\) from Eqs. (5) and (6) we have

\[
\Gamma_{ad} = \lambda_0 \left[ \frac{T_g - T_0}{\frac{\ell^2}{2r_g^2} \left( 1 - \frac{r^2}{r_g^2} \right) + \ln \left( \frac{r_0}{r_g} \right)} \right],
\]

\[
\Gamma_i = \lambda_0 \left[ \frac{T_0 - T_g}{\ln \left( \frac{r_0}{r_g} \right)} \right],
\]

wherein it is understood that \(\lambda_0\) and \(\ell\) refer to the graphene layer. For the sake of simplicity, we suppose them as constant, since we only aim to illustrate the role of the nonlocal effects, rather than to do a detailed practical calculation. In a more detailed analysis, one could ask for the effects of possible backward reflection of the phonons when arriving at the external edge of the graphene layer, which could reduce the heat flow with respect to the value computed here.

In Fig. 2 we compare the predictions arising from Eqs. (7) (the circle marker stands for the results of the non-Fourier law, and the triangle one stands for the results of the Fourier law) as a function of the ratio \(\Phi = r_g/\ell\) for a hot component characterized by \(r_0 = 7.75 \times 10^{-3} m\) (i.e., equal to the mfp of graphene at the room temperature, in order to consider a sufficiently miniaturized device, comparable to the graphene mfp) and \(T_0 = 500 K\), being cooled through a graphene circular layer (with an outer edge kept at \(T_g = 300 K\)). For a given value of the external radius of the graphene circular layer \(r_g\) (such that \(r_g \leq 10\ell\), for the sake of illustration), it is seen that the nonlocal effects appreciably influence the heat removal. In particular, Eqs. (7) predict that a greater amount of heat has to be removed when the non-Fourier theory is valid, than that predicted by the Fourier theory. Of course, the bigger the external radius of the graphene circular layer, the less important the role played by nonlocal effects.

Indeed, in practical applications one should fix the value \(Q^*\) which has to be removed from the hot core, and consequently should size the cooling system. In this light, from Fig. 2 it follows that the external radius of the graphene circular layer has to be sensibly bigger when nonlocal effects are taken into account (i.e., if Eq. (7a) holds), in contrast with the predictions of the classical Fourier law (i.e., if Eq. (7b) holds). These results are a natural consequence of the role played by nonlocal effects in steady-state radial heat transport. In fact, as pointed out in Ref. [39], in such a situation the nonlocal effects strongly modify the temperature behavior in a range of radial distances which are several times greater than the mfp. In particular, in those regions, the temperature predicted by Eq. (1) is higher than that predicted by the Fourier law. Thus, to find the same temperature difference between two distinct points (turning out the ability of the layer to remove heat in our case) one has to consider radial distances much bigger when nonlocal effects are taken into account, with respect to the case when they are neglected.

Let us observe that these results are only a direct consequence of the assumption that the outer edge’s temperature \(T_g\) is taken at a constant value, which is indeed logical in practical applications. This way, in fact, the cooling layer is always submitted to the same temperature gradient which is so a well-known working condition. Moreover, this way the average temperature of the layer is constant and its physical properties are not modified.

### 3. Effect of lateral heat transfer from graphene layer to the environment

In Sec. 2 we have neglected the lateral heat transfer from the heated graphene layer to the neighboring environment. If this assumption is relaxed in view of a more realistic approach (in practical applications the heat transversally transferred to the environment may be removed through cooling fans, for example), then the steady-state heat-flux profile is no longer given by Eq. (3), since the differential equation (2) changes in

\[
\frac{dq}{dr} + \frac{q}{r} = -\frac{2\sigma}{r_g} \left[ T(r) - T_g \right],
\]

still following from the conservation of the total heat flux. In Eq. (8) \(\sigma\) is the heat-exchange coefficient between graphene and environment, and \(T(r)\) is the temperature profile in the graphene layer. Note that whenever \(\sigma = 0\) (or when the
Fig 2. Heat removed from a cylindrical nanodevice with a radius \( r_0 = 7.75 \times 10^{-2} \) m through a graphene layer, whose outer radius is \( r_g \): comparison between the prediction of the non-Fourier law (i.e., Eq. (7a)) and the Fourier law (i.e., Eq. (7b)) as a function of the ratio \( \Phi = r_g/\ell \). For a given value of \( r_g \), Eq. (7a) predicts that a bigger amount of heat is removed with respect to the prediction of Eq. (7b). At the room temperature for graphene one has \( \ell = 775 \times 10^{-9} \) m, and \( \lambda_0 = 5150 \text{ W m}^{-1} \text{ K}^{-1} \) [7, 18, 31, 32]. The thickness of the graphene layer has been taken as \( h_g = 3.33 \times 10^{-10} \) m.

ratio \( \sigma/h_g \) gets very small values), Eq. (8) reduces to Eq. (3), and the results of Sec. 2 are recovered. The solution of Eq. (8) is not so obvious. However, in order to derive the behavior of the temperature profile in this new situation, we may couple Eq. (8) with the steady-state version of Eq. (1). To do this, we express the radial heat flux as

\[
q(r) = \frac{\Gamma(r)}{r},
\]

being \( \Gamma(r) \) a regular function of the radial distance \( r \) from the hot device. From the combination of Eqs. (8) and (9) one gets

\[
\frac{d\Gamma}{dr} = -\frac{2\sigma r}{h_g} [T(r) - T_g],
\]

in the hypothesis that \( \sigma \) does not depend on the radial distance. In our computations the heat-exchange coefficient between graphene and environment will be estimated as \( \sigma = 4\sigma_{SB} T_g^3 \), being \( \sigma_{SB} = 5.67 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-2} \) the Stefan-Boltzmann constant. Indeed, since \( \sigma \) may be temperature dependent (and the latter varies along the graphene layer), it would be a function of \( r \), in principle. Although more refined models (accounting for the dependencies of \( \lambda_0, \ell \) and \( \sigma \) on the temperature) may be introduced in technologically realistic situations, here we use these simplistic assumptions just for the sake of illustration of the physical problem.

Inserting Eq. (9) into Eq. (1) one straightforwardly has

\[
\lambda_0 \frac{dT}{dr} = -\frac{\Gamma}{r} + \ell^2 \frac{1}{r} \frac{d}{dr} \left( \frac{1}{r} \frac{d}{dr} \left( \frac{\Gamma}{r} \right) \right).
\]
the combination of which with Eq. (10) yields

$$\lambda_0^* \frac{dT}{dr} = -\frac{\Gamma}{r} \left( 1 - \frac{\ell^2}{r^2} \right),$$

(12)

with $\lambda_0^* = \lambda_0 + 2\ell^2 a/h_0$. By differentiating Eq. (12) with respect to $r$ and combining this differential form once again with Eqs. (10) and (12), one has the following dimensionless equation

$$\frac{d^2\theta}{d\xi^2} + \frac{1}{\xi} \left( 1 - \frac{2}{\xi^2 - 1} \right) \frac{d\theta}{d\xi} = \frac{2\sigma \ell^2}{\lambda_0^* h_0} \left( 1 - \frac{1}{\xi^2} \right) \theta,$$

(13)

where $\theta(r) = [T(r) - T_0]/(T_0 - T_0)$, and $\xi = r/\ell$. Its analytical solution is quite complicated. However, that boundary value problem can be solved by numerical methods. In particular, by using a variable-step fourth-order accurate Runge-Kutta method (10^-3 relative error, and 10^-6 absolute error), the results are shown in Fig. 3. In that figure the behavior of $\theta$, as a function of $\xi$, has been obtained by assuming $\theta(\xi_0) = 1$ and for different values of $d\theta/d\xi|_{\xi_0} = 0.25$, $d\theta/d\xi|_{\xi_0} = 0.25$, $d\theta/d\xi|_{\xi_0} = 0.3$ and $d\theta/d\xi|_{\xi_0} = 0.35$, being $\xi_0 = r_0/\ell$. This dimensionless gradient is firstly related to the temperature gradient holding at the interface between the cylindrical hot device and the surrounding graphene layer, and, consequently, to the amount of heat (per unit time) $Q^*$ which has to be removed from the hot device to keep its temperature constant. In fact, from Eqs. (9) and (12) one has

$$Q^* = 2\pi h_0 r_0 q(r_0) = 2\pi h_0 \theta^* \left( \frac{r_0^2}{r_0^2 - \ell^2} \right) \left( \frac{r_0^2}{r_0^2 - \ell^2} \right).$$

(14)

Since $\theta$ vanishes when $T(r) \equiv T_0$ (i.e., when $r \equiv r_0$), the temperature behavior in Fig. 3 points out that, for a given value of $d\theta/d\xi|_{\xi_0}$, there is a minimum radius $r_m = \xi^*$ ($\xi^*$ being the value of the independent variable $\xi$ nullifying $\theta$ in Eq. (13), as it can be seen in that figure) which the graphene layer must have in order to be a heat-removing tool. These values of $r_m$, which may be useful in practical applications to check the right dimensions of the systems, are quoted in Tab. 1.

As it can be seen, the bigger the temperature gradient $d\theta/d\xi|_{\xi_0}$, the smaller $\xi^*$. Indeed, this result is logical since for a given temperature difference between two different points (in our case $T_0 - T_0$), the smaller the radial distance, the bigger the temperature gradient. It is worth noticing that one can not arbitrary reduce the size of the cooling layer, since the outer edge has to be taken at a constant temperature, lower than that of the inner edge. The smaller the distance between the two edges, the more difficult to keep a constant temperature gradient between them.

When nonlocal terms are not taken into account (i.e., if the Fourier law holds and the second-order spatial derivatives of $q$ are neglected in Eq. (1)) one simply has

$$\frac{d^2\theta}{d\xi^2} + \frac{1}{\xi} \frac{d\theta}{d\xi} = \frac{2\sigma \ell^2}{\lambda_0^* h_0} \theta,$$

(15)

the presence of $\ell$ in the right-hand side being only due to the use of the dimensionless variable $\xi$. Equation (15) has the following analytical solution:

$$\theta(\xi) = C_1 j_0 \left( i\xi \sqrt{\frac{2\sigma}{\lambda_0^* h_0}} \right) + C_2 Y_0 \left( i\xi \sqrt{\frac{2\sigma}{\lambda_0^* h_0}} \right),$$

(16)

$$\begin{array}{|c|c|c|c|}
| \hline
r_m [m] & 1.14 \times 10^{-5} & 7.97 \times 10^{-6} & 6.10 \times 10^{-6} \\
|\hline
\end{array}$$

Note: The table above provides minimum values of the outer radius for a heat-removing graphene layer in order to keep the temperature constant in the inner component, for different values of temperature gradient at the interface between the hot device and the graphene layer. The radius of the device is $r_0 = 15.5 \times 10^{-7}$ m (i.e., two times bigger than the mtp of graphene at the room temperature), and its temperature is $T_0 = 500$ K. For the outer edge of the graphene layer we assume $T_0 = 300$ K.
Temperature behavior in the graphene layer at 300 K

\[ \frac{d\theta}{d\xi} |_{\xi=0} = 0.2 \]
\[ \frac{d\theta}{d\xi} |_{\xi=0} = 0.25 \]
\[ \frac{d\theta}{d\xi} |_{\xi=0} = 0.3 \]
\[ \frac{d\theta}{d\xi} |_{\xi=0} = 0.35 \]

\[ \xi^* \]

Fig 3. Temperature profile arising from Eq. (13): numerical solution for \( \theta |_{\xi=0} = 1 \) and for different values of the temperature gradient \( \frac{d\theta}{d\xi} |_{\xi=0} \) (i.e., \( \frac{d\theta}{d\xi} |_{\xi=0} = 0.2, \frac{d\theta}{d\xi} |_{\xi=0} = 0.25, \frac{d\theta}{d\xi} |_{\xi=0} = 0.3 \) and \( \frac{d\theta}{d\xi} |_{\xi=0} = 0.35 \)), with \( \xi = r/\ell \). For a given value of \( \frac{d\theta}{d\xi} |_{\xi=0} \), related to the amount of heat which has to be removed from the hot device, the figure shows the value \( \xi^* \) (or in other terms, the value of the radius \( r_m = \xi^* \)) for which the temperature of the layer becomes equal to the environment temperature.

where \( J_0 \) means the first-kind zero-order Bessel function of the indicated argument, that is,

\[ J_0 (z) = 1 - \frac{z^2}{2^2} + \frac{z^4}{2^2 \cdot 4^2} - \frac{z^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots \]

and \( Y_0 \) is the second-kind zero-order Bessel function of the indicated argument, namely,

\[ Y_0 (z) = \frac{2}{\pi} \left[ \ln \left( \frac{z}{2} \right) + \gamma \right] J_0 (z) + \frac{2}{\pi} \left[ \frac{z^2}{2^2} - \frac{z^4}{2^2 \cdot 4^2} \left( 1 + \frac{1}{2} \right) + \frac{z^6}{2^2 \cdot 4^2 \cdot 6^2} \left( 1 + \frac{1}{2} + \frac{1}{3} \right) + \cdots \right] \]

\( \gamma = 0.58 \) being the Euler-Mascheroni constant. In Eq. (16) the two constants \( C_1 \) and \( C_2 \) may be obtained through the boundary conditions \( \theta |_{\xi=0} = 1 \), and \( \theta |_{\xi_0} = 0 \), being \( \xi_0 = r_0/\ell \).

However, since here we are especially interested in pointing out the importance of nonlocal effects, instead of obtaining the exact temperature profile, we solve numerically Eq. (15) again, as a function of the temperature gradient at the interface cylindrical hot device–graphene layer. To do this, we apply again the same variable-step fourth-order accurate Runge–Kutta method used to solve Eq. (13). Note that, we can not take into account the same values for \( d\theta/d\xi |_{\xi=0} \) used to compute the results in Fig. (3), in this case. To make a consistent comparison between the non-Fourier theory and the Fourier one, for example, here we assume a fixed value for the outer radius of the the graphene layer, and evaluate the consequent amount of heat \( Q^* \) (per unit time) which has to be removed both for the non-Fourier (nf) theory, and for the Fourier (f) one. The results are quoted in Tab. 2. As it can be seen, again we find that the predicted values of removed \( Q^* \) are always bigger in the case of a non-Fourier theory, than in the Fourier theory.
Table 2. Heat flux removed from a hot device for different external radii of the graphene layer: comparison between Eq. (1) and the Fourier law.

<table>
<thead>
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<th>$r_0$ [m]</th>
<th>$Q_0^+$ [W]</th>
<th>$Q_0^−$ [W]</th>
<th>$Q_1^+$ [W]</th>
<th>$Q_1^−$ [W]</th>
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<td>1.33 × 10⁻¹</td>
<td>1.61 × 10⁻⁴</td>
<td>1.83 × 10⁻⁴</td>
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</table>

4. Conclusions

In the present paper we have studied the consequences of nonlocal terms related to long mfp in heat removal from nanosystems. Usually, the Fourier law is assumed for the radial heat flux across the heat-removing layer, and the only effect of a long mfp results in an enhancement of the thermal conductivity. However, when second-order nonlocal effects are considered, as in Eq. (1) for example, a long value of the mfp has still some additional effects, which may be pointed out in radial heat transport. In the special case that heat is radially removed through a cooling layer (which in the present case has been supposed to be made of graphene), the role played by nonlocal effects have been shown in Secs. 2 and 3. In particular, therein we showed that for a chosen value of the outer radius of the graphene layer, the amount of heat (per unit time) which is removed from the hot device is predicted to be bigger (in some ten percent, approximately) in the non-Fourier, than in the classical Fourier law. Substantial differences in the transport properties in graphene, compared to the basal planes in graphite or three-dimensional bulk crystals, have been also shown both theoretically and experimentally in Ref. [30].

Although the aim of the present paper was to show the role played by nonlocal effects, estimating it in a very simple situation, as a proof-of-concept exploration, rather than in a fully detailed technological situation, at the very end let us observe that it would be also useful to point out the role of nonlocal effects in deriving the minimum radius $r_m$ for the cooling layer. In fact, along with what observed in Sec. 2, one may also ask in practical applications what will be $r_m$ when a given amount of removed heat per unit time has been supposed to cool the core device, both in the non-Fourier theory, and in the Fourier one. This analysis could be still made by solving numerically Eqs. (13) and (15). To this end we observe that quantum confinement effects, arising when a material is confined into a nanostructure, strongly modify the thermal transport properties. Thus, in nanosystems the surface scattering is expected to play a very important role [4, 5, 9, 40], in such a way that thermal perturbations across the transversal section of a layer are well described by an effective thermal conductivity which decreases for decreasing thickness $h$ of the layer [2, 3, 35, 36, 39]. That effective thermal conductivity should also take into account the phenomenon of phonon backscattering, which may cause a strong reduction and even a suppression of the thermal conductivity [29, 35]. Therefore, in analyzing the thermal transport in nanosystems, one should still use the evolution equation (1) for the heat flux, but with the value of the thermal conductivity $\lambda_0$ replaced by an effective value of the thermal conductivity $\lambda_{\text{eff}}$, depending on the ratio between $\ell$ and $h$, that is, the so-called Knudsen number $\text{Kn} = \ell/h$. This effective thermal conductivity, which may be explored from mesoscopic perspectives by using a phonon-hydrodynamic approach [37, 38], has been omitted here just for the sake of illustration of the importance of nonlocal effects. In fact, the phenomenological analyses we made in Secs. 2 and 3 do not aim to be a complete and conclusive study, but only as a first step forward to a more refined model. In this light it will be also interesting to analyze what is the influence of nonlocal effects when the characteristic dimension of the whole system (i.e., hot device and cooling layer) is smaller than (or comparable to) the mfp of heat carriers. In these situations, in fact, anomalous behavior in the temperature profile may occur when heat propagates radially away from a point source [39], showing that the thermodynamics at the scale of the mfp deserves a deeper analysis. Since the miniaturization of modern devices continually progresses, we feel that checking both the right amount of removed heat, and the right sizes of a nanodevice may be a very important playground of theoretical researches.

Acknowledgements

A. S. acknowledges the University of Basilicata for funding the research project Modeling heat and electric transport in nanosystems in the presence of memory, nonlocal and nonlinear effects, the financial support from the Dirección General de Investigación of the Spanish Ministry of Science and Innovation under grant FIS No. 2009-13370-C02-01, and that of Gruppo Nazionale per la Fisica Matematica - GNFM under grant Progetto Giovani 2012 for his stay at the Autonomous University of Barcelona.
F. X. A acknowledges the financial support from the Dirección General de Investigación of the Spanish Ministry of Science and Innovation under grant FIS No. 2009-13370-C02-01, the Consolider Project NanoTherm (grant CSD-2010-00044), and the Direcció General de Recerca of the Generalitat of Catalonia under grant No. 2009-SGR-00164.

The authors cordially thank Prof. D. Jou from Autonomous University of Barcelona for his useful suggestions and comments.

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