VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-009-0147-y
Math. Slovaca **59** (2009), No. 5, 545–564

DATA DRIVEN RANK STATISTICS IN CHANGE POINT ANALYSIS

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Dedicated to Professor Andrej Pázman on the occasion of his 70th birthday

(*Communicated by Gejza Wimmer*)

ABSTRACT. The paper deals a class of rank based procedures for detection of changes with adaptively chosen scores. This is a certain continuation of the paper Antoch et al (2008). The limit behavior of the test procedures is studied both under the null as well as under a general class of alternatives. Accompanying simulation study focuses on various alternatives which are often met in practice.

1. Introduction

Let $V_1, \dots, V_N$ be independent observations obtained at some time moments $t_1 < t_2 < \dots < t_N$. Assume that V_i has continuous distribution function F_i. We are interested in the testing problem:

$$H_0: \ F_1 = \dots = F_N \tag{1.1}$$

against

$$A: \ \exists\, \eta \in (0,1) \text{ such that } F_1 = \dots = F_{\lfloor N\eta \rfloor} \neq F_{\lfloor N\eta \rfloor + 1} = \dots = F_N. \tag{1.2}$$

2000 Mathematics Subject Classification: Primary 62G10, 62E20.
Keywords: data driven procedures, rank statistics, change point.
The work was supported by grants GAČR 201/09/0755 and MSM 0021620839.

This is one of the classical versions of the change point problem. The number $m^\star = \lfloor N\eta \rfloor$ is called the change point. Notice that $A \equiv \bigcup_{m=1}^{N-1} A_m$, where

$$A_m: \ F_1 = \cdots = F_m \neq F_{m+1} = \cdots = F_N, \tag{1.3}$$

that clearly points out the relation to the classical two-sample problem.

The present paper contains further results on data driven (adaptive) rank based statistics for detection a change in distribution in a series of independent observations developed by Antoch et al. (2008). The paper combines ideas of procedures based on functionals of simple linear rank statistics for the above formulated problem and two-sample rank test with estimated scores, developed by Janic-Wróblewska and Ledwina (1997), (2000). As a result a rank based test procedure with effectively chosen scores has been obtained. While the paper [1] focuses on the restrictive max-type tests, the present paper considers also nonrestrictive max-type test procedures.

For m known we have a two-sample problem with F_1 and F_N being the distributions of the first and seconds sample, respectively. Janic-Wróblewska and Ledwina (2000) developed data driven rank statistics for H_0 against A_m. Let us recall shortly the idea. Denoting

$$H(x) = \frac{m}{N} F_1(x) + \frac{N-m}{N} F_N(x), \quad x \in R^1,$$

the random variables $H(V_1)$ and $H(V_N)$ have densities (w.r.t. the Lebesgue measure)

$$h_1(u) = 1 + \frac{N-m}{N}\, \overline{a}(u), \quad \text{and} \quad h_2(u) = 1 - \frac{m}{N}\, \overline{a}(u), \quad u \in (0,1), \tag{1.4}$$

respectively, where $\overline{a}$ is a measurable function on $(0,1)$ such that $\overline{a} \equiv 0 \iff F_1 \equiv F_N$, i.e., $\overline{a}$ contains full information concerning validity of H_0 or A. The densities $h_j(\cdot)$, $j = 1, 2$, are approximated by

$$h_{1N}(u; \boldsymbol{\theta}) = d_{1k}(\boldsymbol{\theta}) \exp\Big\{ \frac{N-m}{N} \sum_{j=1}^{k} \theta_j b_j(u) \Big\}, \quad u \in (0,1), \tag{1.5}$$

and

$$h_{2N}(u; \boldsymbol{\theta}) = d_{2k}(\boldsymbol{\theta}) \exp\Big\{ -\frac{m}{N} \sum_{j=1}^{k} \theta_j b_j(u) \Big\}, \quad u \in (0,1), \tag{1.6}$$

where $\theta_1, \ldots, \theta_k$ are unknown parameters, $b_j(\cdot)$ denotes the jth normalized orthonormal Legendre polynomial, $d_{jk}(\boldsymbol{\theta})$, $j = 1, 2$, are related constants. Within these families the validity of H_0 is equivalent to $\theta_1 = \cdots = \theta_k = 0$. This leads to a quadratic test statistic

$$\mathscr{T}_N(k;m) = \sum_{j=1}^{k} \mathscr{L}^2(m;b_j) \tag{1.7}$$

of the vector of the rank statistics

$$\mathscr{L}(m;b_j) = \sum_{i=1}^{N} c_{mi} b_j\Big(\frac{R_i - 1/2}{N}\Big), \quad j = 1,\ldots,k, \tag{1.8}$$

where $R_1,\ldots,R_N$ are the ranks of $V_1,\ldots,V_N$ and

$$c_{mi} = \begin{cases} \sqrt{\frac{m(N-m)}{N}}\, m^{-1}, & i = 1,\ldots,m, \\ -\sqrt{\frac{m(N-m)}{N}}\, \big(N-m\big)^{-1}, & i = m+1,\ldots,N. \end{cases}$$

The question is a suitable choice of k.

Janic-Wróblewska and Ledwina (2000) suggested simplified Schwarz's selection rule which leads to the test statistics $\mathscr{T}_N\big(\mathscr{S}(m,p_N);m\big)$, i.e., in (1.7) one puts $k = \mathscr{S}\big(m,p_N\big)$, where

$$\mathscr{S}(m,p_N) = \min\Big\{j:\, \mathscr{T}_N(j;m) - jp_N \ge \mathscr{T}_N(\ell;m) - \ell p_N, \\ \ell = 1,\ldots,d(N);\ 1 \le j \le d(N)\Big\}, \tag{1.9}$$

with $\big\{d(N)\big\}$ being a nondecreasing sequence of natural numbers representing a maximal dimension of the model (1.5) and (1.6) and $\{p_N\}$ being a sequence of positive numbers representing a penalty. In the following we denote $\mathscr{S}(p_N) = \big\{\mathscr{S}(m,p_N):\ m = 1,\ldots,N-1\big\}$.

In our setup m is unknown and for testing H_0 against A we use functionals of $\mathscr{T}_N\big(\mathscr{S}(m,p_N);m\big)$, $m = 1,\ldots,N-1$. As a motivation we can use functionals studied in the past, e.g., [11, pp. 49–50], [7], [10], [2] and [3], [6] or [12]. They are mostly sum- or max-type ones. We focus on max-type that are obtained by the so called union-intersection principle from the test procedures for the two-sample problem. Particularly, we consider

$$\mathscr{M}_N(\varepsilon;p_N) = \max_{\lfloor \varepsilon N\rfloor \le m < \lfloor (1-\varepsilon)N\rfloor} \mathscr{T}_N\big(\mathscr{S}(m,p_N);m\big), \tag{1.10}$$

where $\varepsilon \in [0,1/2)$ is prechosen and can be considered depending on N. Large values of $\mathscr{M}_N(\varepsilon,p_N)$ indicate that the null hypothesis is not true. The proposed test is based on the combination of $\mathscr{M}_N\big(\varepsilon;p_N\big)$, $0 < \varepsilon < 1/2$, and $\mathscr{M}_N\big(0;p_N\big)$.

The rest of the paper is organized as follows. Section 2 contains theoretical results and the description of the test procedure. Section 3 presents simulation study. The proofs of the theoretical results formulated in Section 2 are in Section 4.

2. Main results

At first we formulate assertions on asymptotic distribution of $\mathscr{M}_N\big(\varepsilon; p_N\big)$ and $\mathscr{M}_N\big(0; p_N\big)$ under H_0 and A that give motivation for the construction of the test.

Theorem 2.1. *Let $V_1, \dots, V_N$ be independent identically distributed (i.i.d.) random variables with common continuous parent distribution function F. Let $\{p_N\}$ be a sequence of positive numbers and let $\{d_N\}$ be a sequence of positive integers such that*

$$\lim_{N\to\infty} d_N = \infty \quad \text{and} \quad \lim_{N\to\infty} d_N\big(d_N + \log N\big)/p_N^2 = 0. \tag{2.1}$$

Then

$$\lim_{N\to\infty} P\left(\max_{1\le m<N} \mathscr{S}(m, p_N) = 1\right) = 1, \tag{2.2}$$

$$\lim_{N\to\infty} P\big(q_1(\log N)\mathscr{M}_N(0; p_N) \le x + q_2(\log N)\big) = \exp\big\{-2\exp\{-x\}\big\}, \quad x \in R^1, \tag{2.3}$$

where $q_1(t) = \sqrt{2\log t}$, $q_2(t) = 2\log t + \frac{1}{2}\log\log t - \frac{1}{2}\log \pi$, $t > 1$, and

$$\lim_{N\to\infty} P\big(\mathscr{M}_N(\varepsilon; p_N) \le x\big) = P\Big(\sup_{\varepsilon\le t\le 1-\varepsilon} \frac{B^2(t)}{t(1-t)} \le x\Big), \quad x > 0, \tag{2.4}$$

for any $\varepsilon \in (0, 1/2)$, where $\big\{B(t) : t \in (0,1)\big\}$ is a Brownian bridge. Moreover, $\mathscr{M}_N\big(\varepsilon; p_N\big)$, $0 < \varepsilon < 1/2$, and $\mathscr{M}_N\big(0; p_N\big)$ are asymptotically independent.

Theorem 2.2. *Let $V_1, \dots, V_{m^*}$ and $V_{m^*+1}, \dots, V_N$ be independent random samples with respective parent continuous distributions F and G, respectively, $F \ne G$. Let function $\overline{a}$ defined in (1.4) have the property:*

$$0 < \int_0^1 \overline{a}^2(u)\,\mathrm{d}u. \tag{2.5}$$

Assume that the change point fulfill $m^* = \lfloor \eta N \rfloor$ *for some* $\eta \in (0,1)$. *Let* $\{d_N\}$ *and* $\{p_N\}$ *be sequences of positive integers satisfying the assumptions of* Theorem 2.1. *Then, as* $N \to \infty$,

$$N^{-1}\,\mathscr{T}_N\big(k; \lfloor Nt \rfloor\big) \to g(t,\eta) \sum_{j=1}^{k} \bigg(\int_0^1 b_j(u)\overline{a}(u)\,\mathrm{d}u \bigg)^2, \quad k = 1, 2, \ldots \quad a.s., \tag{2.6}$$

$$N^{-1}\,\mathscr{T}_N\big(\mathscr{S}\big(\lfloor Nt \rfloor, p_N\big); \lfloor Nt \rfloor\big) \to g(t,\eta) \int_0^1 \overline{a}^2(u)\,\mathrm{d}u, \quad t \in (0,1) \quad a.s. \tag{2.7}$$

and

$$N^{-1}\,\mathscr{M}_N\big(\varepsilon; p_N\big) \to \max_{\varepsilon \le t \le 1-\varepsilon} g(t,\eta) \bigg(\int_0^1 \overline{a}^2(u)\,\mathrm{d}u \bigg)^2, \quad t \in (0,1), \quad a.s., \tag{2.8}$$

for $0 \le \varepsilon < 1/2$, *where*

$$g(t,\eta) = I\{0 < t \le \eta\} t(1-\eta)^2/(1-t) + I\{\eta < t < 1\}(1-t)\eta^2/t.$$

Remark 2.3. Theorems 2.1 and 2.2 imply that the test statistic $\mathscr{M}_N\big(\varepsilon; p_N\big)$ provides a consistent test for the considered class of alternatives for any $0 \le \varepsilon < 1/2$. The assertion of Theorem 2.2 moreover indicates that if $\eta \le \varepsilon$ the test procedure based on $\mathscr{M}_N\big(\varepsilon; p_N\big)$ is asymptotically best one in a sense that it asymptotically maximize the power within a certain class of max-type tests.

Remark 2.4. Notice that the choice of penalty term p_N does not influence the asymptotic behavior of the test procedure (of course, it has to satisfy (2.1)). Notice also that under H_0 the selection rule $\mathscr{S}\big(p_N\big)$ leads to a decision using only Wilcoxon type test with probability tending to 1. By (2.6) under alternatives (2.5) as $N \to \infty$

$$N^{-1}\,\mathscr{T}_N\big(d_N^*; \lfloor Nt \rfloor\big) \to g(t,\eta) \sum_{j=1}^{\infty} \bigg(\int_0^1 b_j(u)\overline{a}(u)\,\mathrm{d}u \bigg)^2 = g(t,\eta) \int_0^1 \overline{a}^2(u)\,\mathrm{d}u, \ \ a.s.$$

for any $d_N^* \to \infty$. In practice one usually chooses $p_N = c \log N$, $c > 0$, where suitable choice of c is discussed in [1].

If the change point m^* lies in the interval $(\varepsilon N, (1-\varepsilon)N)$, then the test based on $\mathscr{M}_N(\varepsilon; p_N)$, $\varepsilon \in (0, 1/2)$ has higher power than that based on $\mathscr{M}_N(0; p_N)$. Otherwise $\mathscr{M}_N(0; p_N)$ provides better test then $\mathscr{M}_N(\varepsilon; p_N)$. Since $\mathscr{M}_N(0; p_N)$

and $\mathscr{M}_N(\varepsilon; p_N)$ are asymptotically independent under H_0 for any $\varepsilon \in (0, 1/2)$ fixed, we propose the following test procedure:

Description of the test procedure

- Choose $d_N,\ p_N,\ \alpha_j \in (0,1),\ j = 1,2,\ \varepsilon \in (0, 1/2)$.
- Calculate $\mathscr{M}_N(\varepsilon; p_N)$ and an approximation $m_{N,\alpha_1}(\varepsilon)$ to its $(1-\alpha_1)100\%$ -quantile. If it leads to the rejection of the null hypothesis, i.e.,

$$\mathscr{M}_N(\varepsilon; p_N) \geq m_{N,\alpha_1}(\varepsilon),$$

  then our inference is that the data indicate that the null hypothesis is not true.
- Otherwise we proceed further and calculate $\mathscr{M}_N\big(0; p_N\big)$ and the respective approximation to the critical value m_{N,α_2}. If $\mathscr{M}_N\big(0; p_N\big) \geq m_{N,\alpha_2}$ then the null hypothesis is rejected.

In other words, the null hypothesis is rejected either if $\mathscr{M}_N(\varepsilon; p_N) \geq m_{N,\alpha_1}(\varepsilon)$ or $\mathscr{M}_N(0; p_N) \geq m_{N,\alpha_2}$. The asymptotic level of the test is $\alpha = \alpha_1 + (1-\alpha_1)\alpha_2$.

Since the considered test statistics are distribution free under H_0, approximations to the quantiles m_{N,α_j}, $j = 1, 2$, can be obtained through simulations. Alternatively, the limit distributions (see Theorem 2.1) under H_0 can be also used. However, simulations show that the convergence to the limit null distribution of $\mathscr{M}_N(0; p_N)$ is quite slow and therefore the asymptotic critical value provides asymptotically correct approximation for moderate number of observations then this approximation need not work reasonably well. Concerning asymptotic approximation to $m_{N,\alpha_1}(\varepsilon)$, the explicit form of the limit distribution of the respective statistic is unknown and the limit distribution has to be simulated.

3. Simulation study

Simulations mainly concern the procedure based on statistic $\mathscr{M}_N(\varepsilon; p_N),\ \varepsilon \in (0, 1/2)$. The usefulness of the new procedure based on $\mathscr{M}_N(0; p_N)$ is discussed at the end of Section 3. The main aims were:

- To see the power of the procedure based on statistic $\mathscr{M}_N(\varepsilon; p_N)$.

- To see the influence of the location of the change point. To this purpose we considered the change either in the middle, at first quarter or at the very beginning of the data, typically close to $\lfloor \varepsilon N \rfloor$.
- To see the "concentrations" of terms $\widetilde{S}(\log N) = \max_{1 \le m < N} S(m, \log N)$ forming the statistic $\mathscr{T}_N(\cdot;\cdot)$, see (1.7) and (1.9). By the term "concentration" we understand the distribution of $\widetilde{S}(\log N)$. Evidently, this distribution is discrete and concentrated on $\{1, \ldots, d(N)\}$. Therefore, in Figures 4–10 the histograms of $\widetilde{S}(\log N)$ are presented. To save the space, multiple histograms are plotted in one figure.

Following distributions has been considered:

(1) Laplace distribution, change of the location parameter μ, i.e.
$f(x) = (1/2) \cdot \exp\{-|x|\}$
$g(x) = (1/2) \cdot \exp\{-|x-\mu|\},\ \mu \in \{0.25,\ 0.5,\ 0.75,\ 1,\ 1.5,\ 2\}$

(2) Laplace distribution, change of the scale parameter σ, i.e.
$f(x) = (1/2) \cdot \exp\{-|x|\}$
$g(x) = (1/2) \cdot \exp\{-|x|/\sigma\},\ \sigma \in \{1.25,\ 1.5,\ 1.75,\ 2,\ 2.5,\ 3\}$

(3) Logistic distribution, change of the location parameter μ, i.e.
$F(x) = \exp\{x\}/(1 + \exp\{x\})$
$G(x) = \exp\{x-\mu\}/(1 + \exp\{x-\mu\}),\ \mu \in \{0.25,\ 0.5,\ 0.75,\ 1,\ 1.5,\ 2\}$

(4) Lognormal distribution, change of the parameter μ, i.e.
$F(x) = \Phi(\log x)$, where $\Phi(x)$ stands for the standard normal cdf
$G(x) = \Phi(\log(x-\mu)),\ \mu \in \{0.25,\ 0.5,\ 0.75,\ 1,\ 1.5,\ 2\}$

(5) Lognormal distribution, change of the parameter σ, i.e.
$F(x) = \Phi(\log x)$
$G(x) = \Phi\big(\sigma^{-1} \log x\big),\ \sigma \in \{1.25,\ 1.5,\ 1.75,\ 2,\ 2.5,\ 3\}$

(6) Pareto distribution, change of the shape parameter a, i.e.
$F(x) = 1 - 1/x,\ x > 1$
$G(x) = 1 - 1/x^a,\ x > 1,\ a \in \{1.25,\ 1.5,\ 1.75,\ 2,\ 2.5,\ 3,\ 5,\ 10\}$

(7) Weibull distribution, change of the shape parameter b, i.e.
$F(x) = 1 - \exp\{-x\},\ x > 0$
$G(x) = 1 - \exp\{-x^b\},\ x > 0,\ b \in \{\ 1.25,\ 1.5,\ 1.75,\ 2,\ 2.5,\ 3\}$

On the contrary to the paper of Antoch et al. (2008), where many distributions interesting rather from the theoretical point of view were considered, we have concentrated in this study mainly distributions often used in practice.

Concerning the other parameters of the simulations, we set:

- trimming proportion $\varepsilon = 0.1$;
- sample size $N = 100, 200$;
- maximal number of terms $d(N) = N/10$.

For each combination of the parameters we run 10 000 simulations and:

- Estimated distribution of the test statistic $\mathscr{M}_N\big(\varepsilon; \log\, N\big)$, defined in (1.10) from the simulations both under the hypothesis and alternatives using the cumulative empirical distribution functions. Typical example for the case when changing the location parameter of the Laplace distribution can be found in Figure 1. Notice that the vertical bar denotes 95% empirical quantile, i.e. the 5 % sample critical value calculated under H_0.
- Calculated the power relating to the 95% critical value estimated under the null hypothesis of no change, i.e. $F(x)$.
- Calculated the "concentrations" of terms $\widetilde{S}(\log\, N)$ forming the test statistic $\mathscr{T}_N(\cdot;\cdot)$.

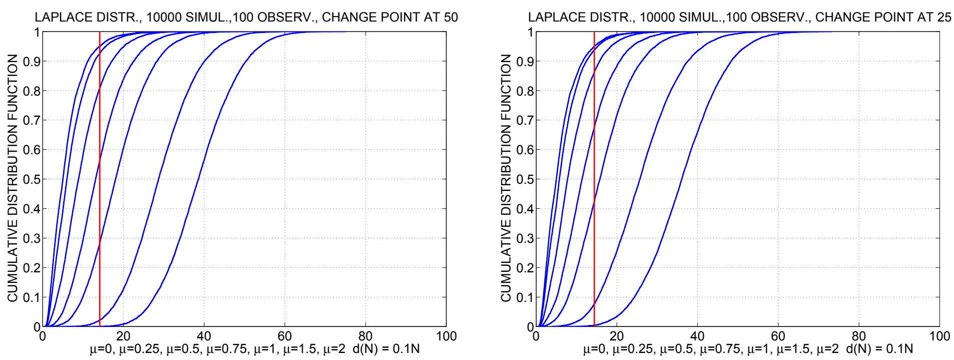

FIGURE 1. Empirical cumulative distribution functions of statistics $\mathscr{M}_N\big(0.1; \log\, N\big)$ when changing the location parameter μ of the Laplace distribution, the case $N = 100,\ m = 50$ on the left and $N = 100,\ m = 25$ on the right.

Our simulations show that there is only a small difference in the results when we move the change point from the middle of the sample to the quarter of the sample. As expected, the power is highest when the change point is in the middle of the sample. However, the situation is much worse when we move the change point either close to the beginning or to the end of the data set. It is seen from Figures 4–10 that decrease in the power is quite large.

To see the decrease of our test procedure related to $\mathscr{M}_N\big(0.1; \log\, N\big)$ with other rank procedures, we used the procedure based on $\max\limits_{1\le m<N} \mathscr{L}(m; \text{Savage})$, where $\mathscr{L}(m; \text{Savage})$ is the Savage statistic for sample sizes m and $N-m$. $\mathscr{L}(m; \text{Savage})$ is asymptotically optimal for the two-sample situation for equality of scales when the the underlying distribution is exponential, for details see [5]. Notice the relation between the exponential and Laplace distributions.

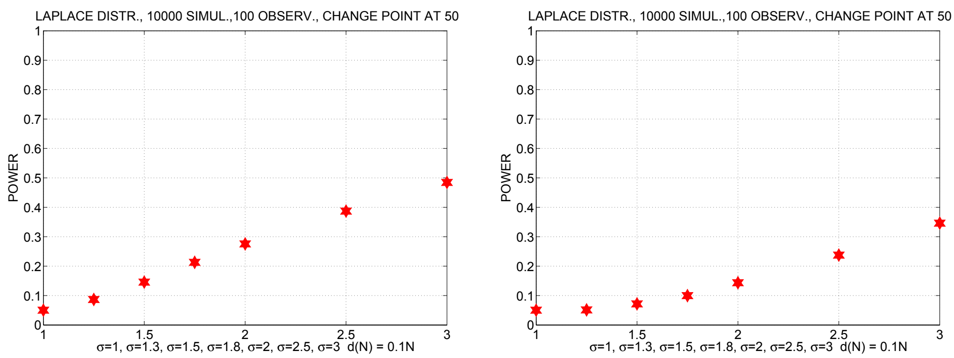

FIGURE 2. Power of Savage statistic (left) and statistic $\mathscr{M}_N\big(0.1; \log\, N\big)$ (right) when changing scale parameter σ of the Laplace distribution.

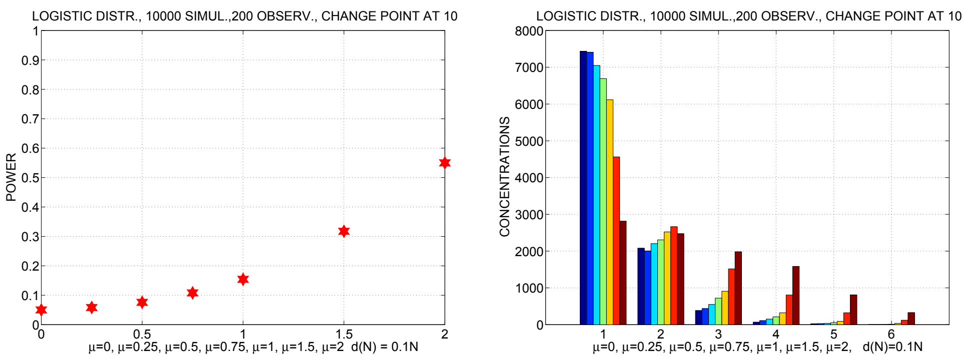

FIGURE 3. Power and concentrations $\widetilde{S}(\log N)$, when changing the location parameter μ of the logistic distribution, of statistic $\mathscr{M}_N(\varepsilon; p_N)$ for change point $\lfloor \eta N \rfloor$ below the trimming proportion $\lfloor \varepsilon N \rfloor$.

We were also interested what will happen if the change point $\lfloor \eta N \rfloor$ will be smaller than the trimming proportion $\lfloor \varepsilon N \rfloor$. Results for such a situation when

$N = 200$, $\varepsilon = 0.1$, $d(N) = 0.1N$ and $\eta = 0.05$ are shown in Figure 3. Comparing it with Figure 6 ($N = 100$, $\varepsilon = 0.1$, $d_N = 0.1N$ and $\eta = 0.1$ see Figure 6) notice two points, i.e.

- smaller power in Figure 3 in spite of larger number of observations;
- high negative skewness of the concentrations of statistics $\widetilde{S}(\log N)$.

As seen from Figures 4–10, a general rule is that closer is τ to zero more negatively skewed are the concentrations.

In Figures 4–10 can be found complete results for the distributions mentioned above when changing either the location or scale/shape parameter. The lhs figures represent the powers. The rhs figures represent empirical distributions of $\widetilde{S}(\log N)$, each vertical bar corresponds to the underlying distribution. We omit empirical distribution functions because the most important information concerning our simulations is described by the simulated power and concentrations of respective test procedures.

We were also interested in the difference between the behavior of the statistic $\mathscr{M}_N(\varepsilon; p_N)$ and newly proposed test statistic in the case when the change point $\lfloor \eta N \rfloor$ is smaller than the trimming proportion. Because the conclusions were similar in all considered cases, we present here only the situation analogous to that described in Figure 3. This means that we concentrated on the logistic distribution with $N = 200$, $\varepsilon = 0.1$, $d(N) = 0.1N$, $\eta = 0.05$. Further, we fixed $\alpha_1 = 0.04$ and $\alpha_2 = 1/96$, $\alpha_1 = 0.03$ and $\alpha_2 = 2/97$, $\alpha_1 = 0.025$ and $\alpha_2 = 25/975$, $\alpha_1 = 0.02$ and $\alpha_2 = 3/98$, and $\alpha_1 = 0.01$ and $\alpha_2 = 4/99$, respectively. This choice ensures the asymptotic level of the test $\alpha = 0.05$. For all these settings we calculated the power of $\mathscr{M}_N(0; p_N)$, $\mathscr{M}_N(0.1; p_N)$ and newly suggested test statistic. The results are summarized in Table 1, where:

- μ denotes the size of shift of the location parameter of the logistic distribution (cf distribution (3) in Section 3);
- $pow_{0.1}$ denotes the (sample) power of the statistic $\mathscr{M}_N(0.1; p_N)$;
- pow_0 denotes the (sample) power of the statistic $\mathscr{M}_N(0; p_N))$;
- pow_{AH} denotes the (sample) power of newly suggested test statistic.

From Table 1. we can see that the newly proposed test procedure slightly improves the situation if the desired level of the test would be α_1. On the other hand, if we wish to keep the composite level α, it appears that for the sample size $n = 200$ and selected parameters of the simulation the original procedure from [1] is still slightly better. The reason seems to be the fact that asymptotic results do not apply for such small sample size as is, unfortunately, often the case in the change point detection.

TABLE 1. Powers of $\mathscr{M}(0.1; p_N)$, $\mathscr{M}(0; p_N)$ and newly suggested test statistic.

	$\alpha_1 = 0.01$			$\alpha_1 = 0.02$			$\alpha_1 = 0.025$		
μ	$pow_{0.1}$	pow_0	pow_{AH}	$pow_{0.1}$	pow_0	pow_{AH}	$pow_{0.1}$	pow_0	pow_{AH}
0.00	0.010	0.040	0.049	0.020	0.030	0.049	0.025	0.025	0.049
0.25	0.011	0.040	0.050	0.023	0.031	0.053	0.028	0.026	0.053
0.50	0.017	0.041	0.057	0.031	0.032	0.062	0.037	0.027	0.062
0.75	0.026	0.045	0.069	0.048	0.034	0.080	0.056	0.028	0.082
1.00	0.050	0.050	0.096	0.077	0.036	0.110	0.088	0.030	0.115
1.50	0.149	0.072	0.200	0.198	0.049	0.233	0.216	0.039	0.244
2.00	0.333	0.126	0.383	0.405	0.077	0.436	0.427	0.061	0.451
	$\alpha_1 = 0.03$			$\alpha_1 = 0.04$			$\alpha_1 = 0.05$		
μ	$pow_{0.1}$	pow_0	pow_{AH}	$pow_{0.1}$	pow_0	pow_{AH}	$pow_{0.1}$	pow_0	pow_{AH}
0.00	0.030	0.021	0.047	0.040	0.010	0.050	0.050	0.000	0.050
0.25	0.034	0.020	0.051	0.044	0.010	0.054	0.058	0.000	0.058
0.50	0.043	0.021	0.061	0.054	0.010	0.064	0.075	0.000	0.075
0.75	0.066	0.022	0.083	0.082	0.010	0.091	0.108	0.000	0.108
1.00	0.100	0.024	0.118	0.122	0.010	0.131	0.154	0.000	0.154
1.50	0.238	0.028	0.254	0.272	0.010	0.280	0.316	0.001	0.317
2.00	0.448	0.035	0.460	0.488	0.010	0.493	0.549	0.001	0.550

4. Proofs

Proof of Theorem 2.1. The assertion (2.2) is a consequence of [1, Theorem 2] with $N_1 = 1$ and $N_2 = N - 1$. The assertion (2.4) coincides with [1, (5)]. Concerning (2.3), by (2.2) we have, as $n \to \infty$, that

$$\sup_{x \in R^1} \left| P\Big(\sqrt{\mathscr{M}_N(0; p_N)} \le x\Big) - P\Big(\max_{1 \le m < N} \big|\mathscr{L}(m; b_1)\big| \le x\Big) \right| \to 0.$$

By [6, Theorem 2] we have

$$P\Big(q_1\big(\log N\big) \max_{1 \le m < N} \big|\mathscr{L}(m; b_1)\big| \le x + q_2\big(\log N\big)\Big) \to \exp\big\{-2\exp\{-x\}\big\},$$

$$x \in R^1,$$

hence the assertion (2.3) is proven.

In order to show the asymptotic independence of $\mathscr{M}_N\big(0;p_N\big)$ and $\mathscr{M}_N\big(\varepsilon;p_N\big)$, $\varepsilon>0$, it suffices to prove the asymptotic independence of $\max\limits_{1\le m<N}\big|\mathscr{L}(m;b_1)\big|$ and $\max\limits_{N\varepsilon\le m<N(1-\varepsilon)}\big|\mathscr{L}(m;b_1)\big|$, $\varepsilon>0$. Toward this we have to go into the proof of [6, Theorem 2], where we realize that the asymptotic distribution of $\max\limits_{1\le m<N}\big|\mathscr{L}(m;b_1)\big|$ is determined by $V_1,\dots,V_{\lfloor N\varepsilon_N\rfloor},V_{N-\lfloor N\varepsilon_N\rfloor},\dots,V_N$ for some $\varepsilon_N\to 0$ while the asymptotic distribution of $\mathscr{M}_N(\varepsilon;p_N)$ is determined by $V_{\lfloor N\varepsilon_N\rfloor},\dots,V_{N-\lfloor N\varepsilon_N\rfloor}$. □

P r o o f o f T h e o r e m 2.2. The assertion (2.6) is an immediate consequence of the results of H á j e k (1974). Since $\mathscr{T}_N\big(k;\lfloor Nt\rfloor\big)$ is nondecreasing in k for each $t\in(0,1)$ and since (2.6) holds, we have for each $t\in(0,1)$

$$N^{-1}\mathscr{T}_N\big(\mathscr{S}(\lfloor Nt\rfloor,p_N);\lfloor Nt\rfloor\big)\to g(t,\eta)\sum_{k=1}^{\infty}\bigg(\int_0^1 b_k(u)\overline{a}(u)\,\mathrm{d}u\bigg)^2,\ \ t\in(0,1),\ \ a.s.$$

The assertion (2.7) can be concluded from the above relation, the assumption (2.5) and properties of Legendre's polynomials, i.e.,

$$\sum_{k=1}^{\infty}\bigg(\int_0^1 b_k(u)\overline{a}(u)\,\mathrm{d}u\bigg)^2=\int_0^1\overline{a}^2(u)\,\mathrm{d}u.$$

Finally, (2.8) is a direct consequence of (2.7). □

Acknowledgement. The authors wish to thank to three referees for very careful reading the manuscript and valuable comments.

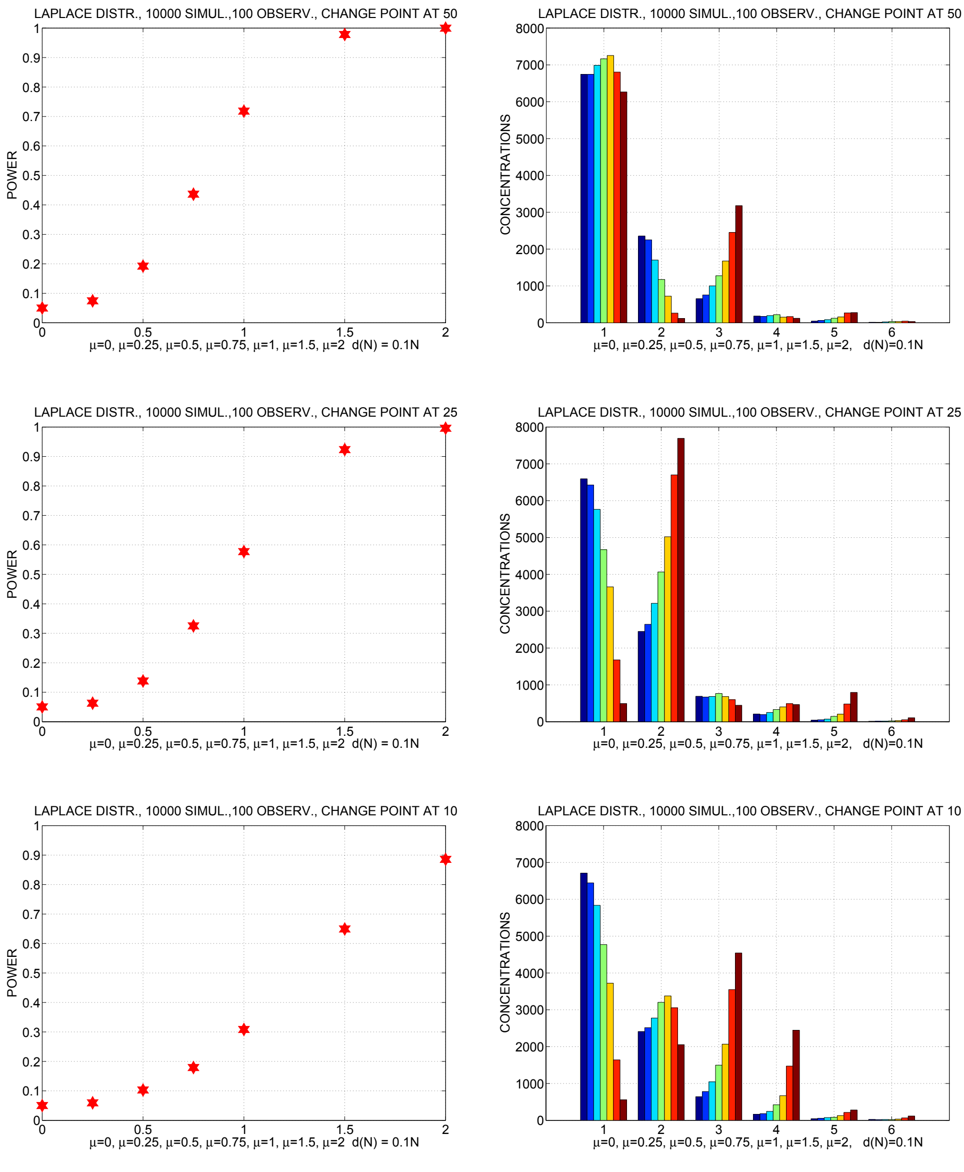

FIGURE 4. Powers of $\mathscr{M}_N(0.1; \log N)$ and concentrations of $\widetilde{S}(\log N)$ for the change in the shift parameter of the Laplace distribution.

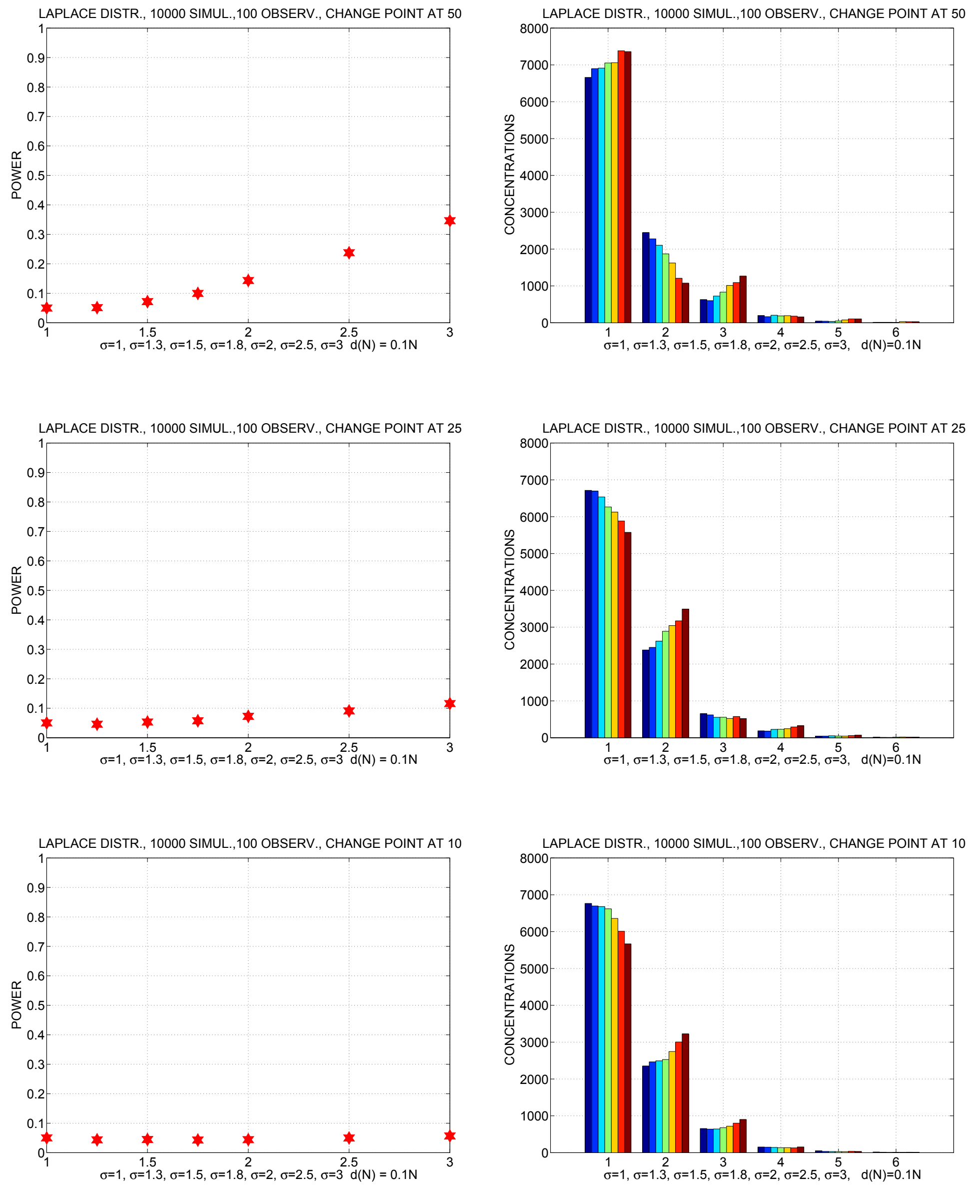

FIGURE 5. Powers of $\mathscr{M}_N(0.1; \log N)$ and concentrations of $\widetilde{S}(\log N)$ when changing scale parameter σ of the Laplace distribution.

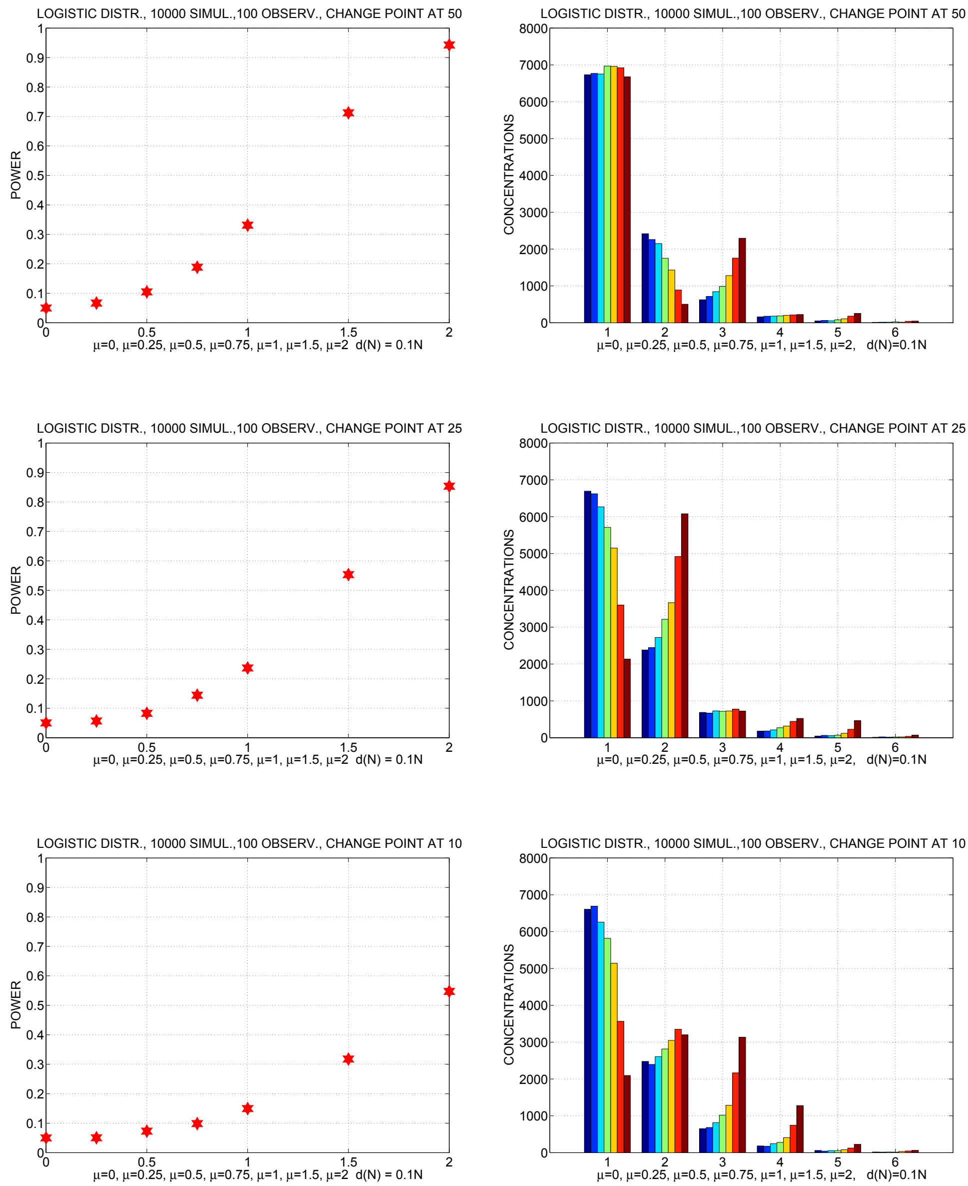

FIGURE 6. Powers of $\mathscr{M}_N(0.1; \log N)$ and concentrations of $\widetilde{S}(\log N)$ when changing the shift parameter μ of the logistic distribution.

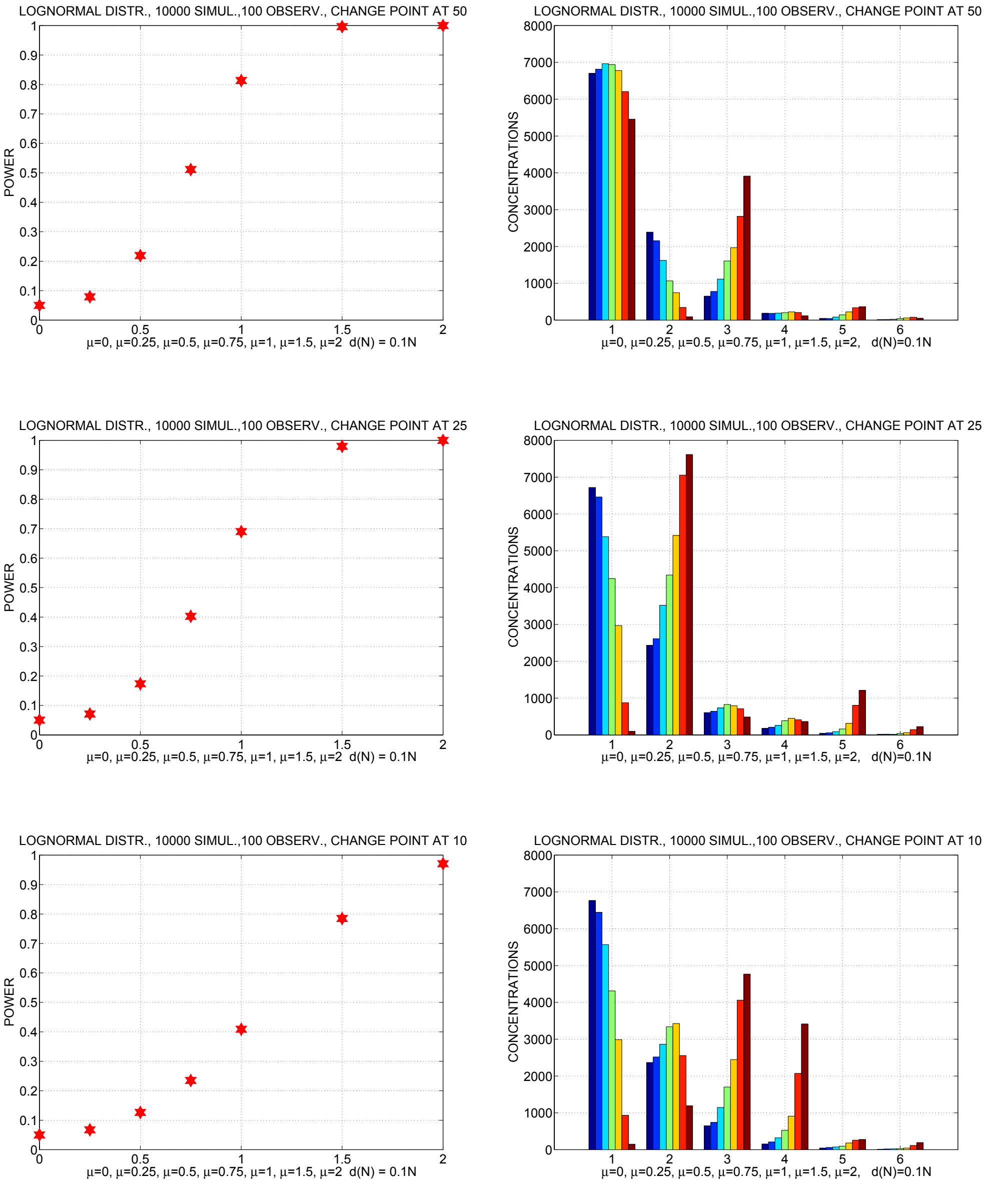

FIGURE 7. Powers of $\mathscr{M}_N(0.1; \log N)$ and concentrations of $\widetilde{S}(\log N)$ when changing the parameter μ of the lognormal distribution.

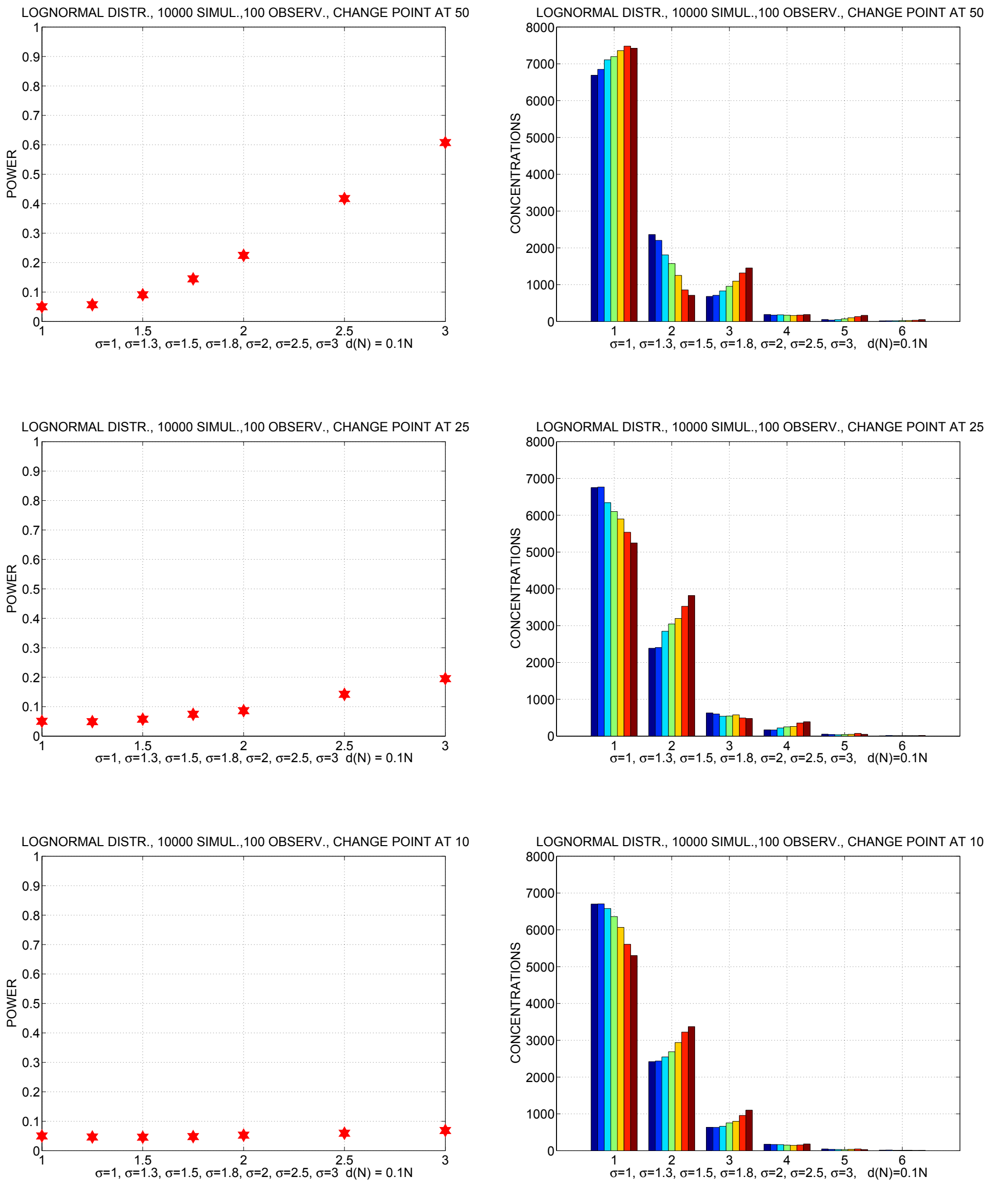

FIGURE 8. Powers of $\mathscr{M}_N(0.1; \log N)$ and concentrations of $\widetilde{S}(\log N)$ when changing parameter σ of the lognormal distribution.

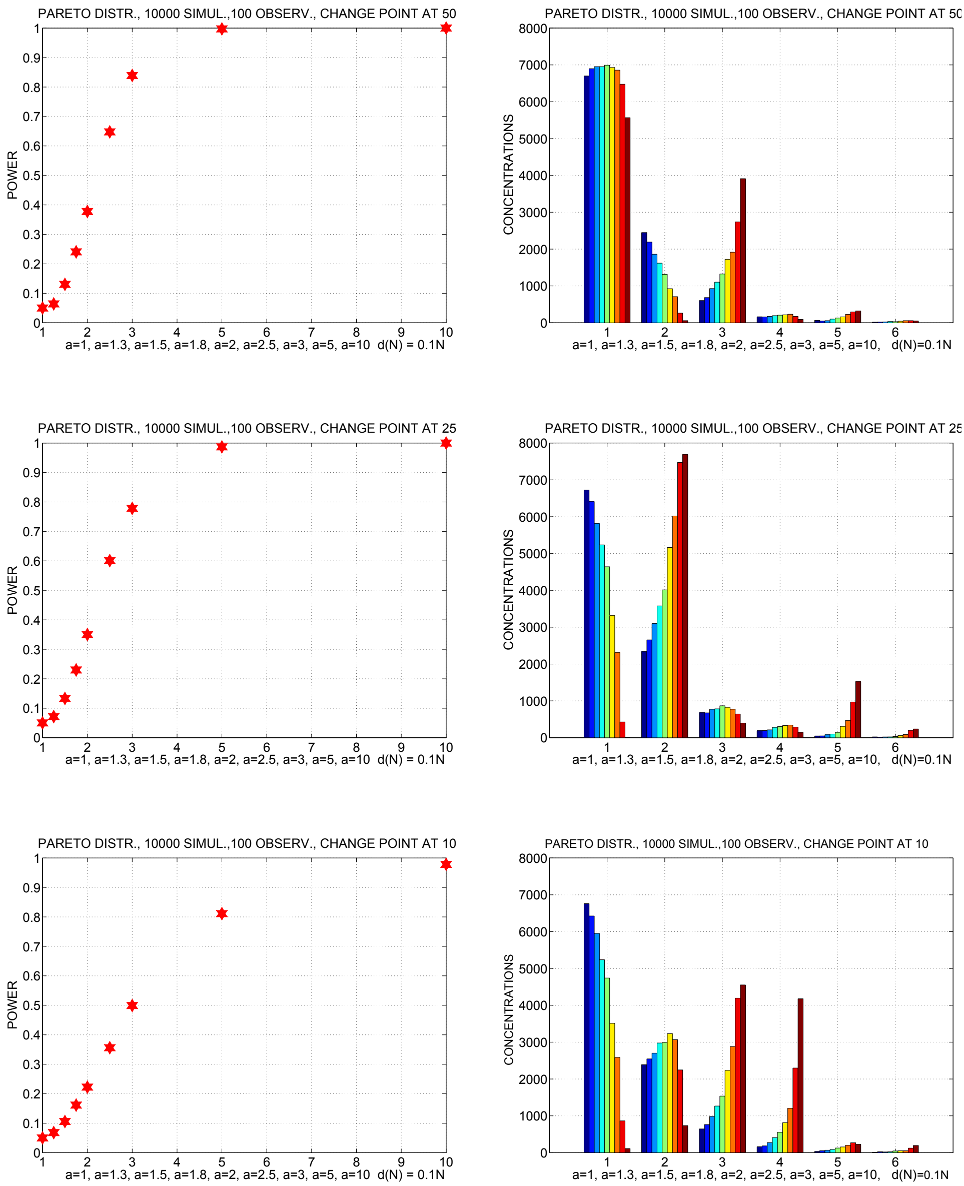

FIGURE 9. Powers of $\mathscr{M}_N(0.1; \log N)$ and concentrations of $\widetilde{S}(\log N)$ when changing shape parameter a of the Pareto distribution.

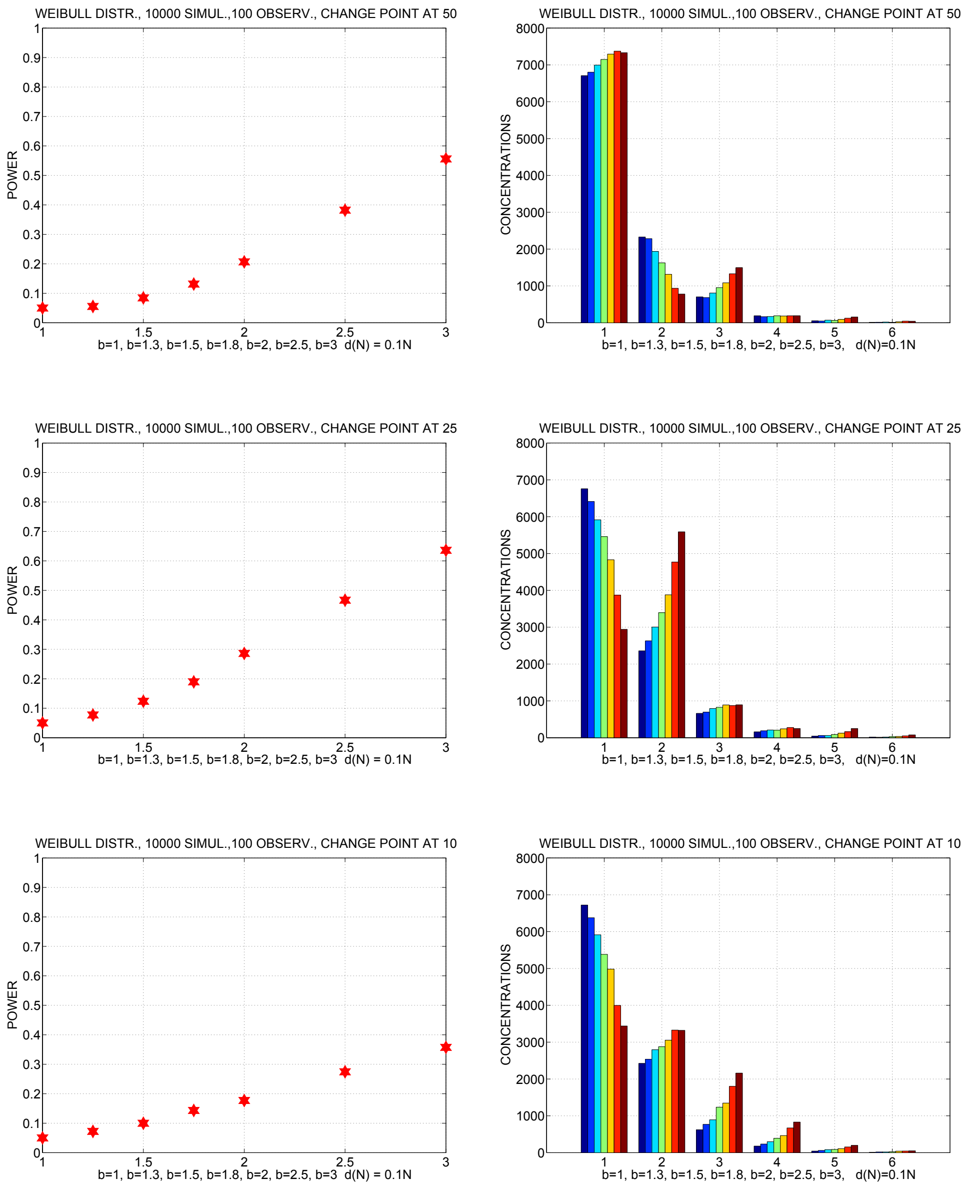

Figure 10. Powers of $\mathscr{M}_N(0.1; \log N)$ and concentrations of $\widetilde{S}(\log N)$ when changing shape parameter b of the Weibull distribution.

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Received 24. 4. 2008

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