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BOUNDED PSEUDO-HOOPS WITH INTERNAL STATES

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ABSTRACT. State MV-algebras were introduced by Flaminio and Montagna as MV-algebras with internal states. Di Nola and Dvurečenskij presented the notion of state-morphism MV-algebra which is a stronger variation of a state MV-algebra. Rachůnek and Šalounová introduced state GMV-algebras (pseudo-MV algebras) and state-morphism GMV-algebras, while the state BL-algebras and state-morphism BL-algebras were defined by Ciungu, Dvurečenskij and Hyčko. Recently, Dvurečenskij, Rachůnek and Šalounová presented state Rℓ-monoids and state-morphism Rℓ-monoids. In this paper we study these concepts for more general fuzzy structures, namely pseudo-hoops and we present state pseudo-hoops and state-morphism pseudo-hoops.

1. Introduction

There is a strong motivation to revise the classical probability theory and to introduce more general probability models based on non-classical logics. In analogy to probability measure, the states on multiple-valued logic algebras proved to be the most suitable models for averaging the truth-value in their corresponding logics. The notion of a state is a basic notion in quantum structures (for a survey on quantum structures, mathematical foundations of quantum mechanics, see e.g. [25]). The state on MV-algebras was introduced by Mundici ([45]) and the state on BL-algebras was introduced by Riečan ([47]) as functions defined on these algebras with values in $[0, 1]$. Bosbach and Riečan states, introduced in [34], have as domain a pseudo-BL-algebra A and as codomain the real interval $[0, 1]$ and the notions were generalized in [26], [27] for bounded non-commutative Rℓ-monoids. For the case of residuated lattices the states were investigated in

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[5], for semi-divisible residuated lattices in [48], [49], for pseudo-hoops in [8] and for pseudo-BCK algebras in [9], [6]. Nowadays, states on other algebraic structures are also intensively studied ([3], [29]–[32], [39]).

Recently, for the case of residuated lattices the notion of a state was generalized as a function with values in a residuated lattice ([12], [13]) and this concept was extended to the case of pseudo-BCK algebras and pseudo-hoops ([14]).

Flaminio and Montagna ([33]) endowed the MV-algebras with a unary operation called an internal state or a state operator satisfying some basic properties of states and the new structures are called state MV-algebras. In fact, they developed a unified approach of the states and probabilistic many-valued logic in a logic and algebraic setting. A more powerful type of logic can be given by algebraic structures with internal states, and they are also very interesting varieties of universal algebras. Di Nola and Dvurečenskij introduced the notion of state-morphism MV-algebra which is a stronger variation of state MV-algebra ([15], [16]). Subdirectly irreducible state-morphism MV-algebras were characterized in [18] and some classes of state-morphism MV-algebras were given in [17]. The notion of a state operator was extended by Rachůnek and Šalounová in [46] for the case of GMV-algebras (pseudo-MV algebras). State operators and state-morphism operators on BL-algebras were introduced and investigated in [10] and subdirectly irreducible state-morphism BL-algebras were studied in [22]. Recently, Dvurečenskij, Rachůnek and Šalounová introduced state $R\ell$-monoids and state-morphism $R\ell$-monoids([28]).

In this paper we study these concepts for the more general fuzzy structures, namely pseudo-hoops and we discuss the state pseudo-hoops and state-morphism pseudo-hoops. We define the notions of state operator, strong state operator, state-morphism operator, weak state-morphism operator and we study their properties. We prove that every strong state pseudo-hoop is a state pseudo-hoop and any state operator on an idempotent pseudo-hoop is a weak state-morphism operator. Glivenko and (mN) properties are defined and it is proved that for an idempotent pseudo-hoop A having these properties a state operator on $\mathrm{Reg}(A)$ can be extended to a state operator on A. One of the main results of the paper consists of proving that every perfect pseudo-hoop admits a nontrivial state operator. Other results refer to the connection between the state operators and the states and generalized states on a pseudo-hoop. Some conditions are given for a state operator to be a generalized state and for a generalized state to be a state operator.

2. Basic definitions and results

Pseudo-hoops were introduced in [38] as a generalization of hoops which were originally defined and studied by Bosbach in [1] and [2] under the name of complementary semigroups.

It was proved that a pseudo-hoop has the pseudo-divisibility condition and it is a meet-semilattice, so a bounded Rℓ-monoid can be viewed as a bounded pseudo-hoop together with the join-semilattice property. In other words, a pseudo-hoop is a meet-semilattice ordered residuated, integral and divisible monoid.

In what follows we recall some basic notions and results regarding the pseudo-hoops.

Definition 2.1. ([38]) A *pseudo-hoop* is an algebra $(A, \odot, \to, \rightsquigarrow, 1)$ of the type $(2, 2, 2, 0)$ such that, for all $x, y, z \in A$:

(A_1) $x \odot 1 = 1 \odot x = x$;

(A_2) $x \to x = x \rightsquigarrow x = 1$;

(A_3) $(x \odot y) \to z = x \to (y \to z)$;

(A_4) $(x \odot y) \rightsquigarrow z = y \rightsquigarrow (x \rightsquigarrow z)$;

(A_5) $(x \to y) \odot x = (y \to x) \odot y = x \odot (x \rightsquigarrow y) = y \odot (y \rightsquigarrow x)$.

In the sequel, we will agree that $\odot$ has higher priority than the operations $\to$, $\rightsquigarrow$.

If the operation $\odot$ is commutative, or equivalently $\to = \rightsquigarrow$, then the pseudo-hoop is said to be *hoop*. Properties of hoops were studied in [1] and [2].

On the pseudo-hoop A we define $x \le y$ iff $x \to y = 1$ (equivalent to $x \rightsquigarrow y = 1$) and $\le$ is a partial order on A. A pseudo-hoop A is *bounded* if there is an element $0 \in A$ such that $0 \le x$ for all $x \in A$.

In the sequel we will also refer to the pseudo-hoop $(A, \odot, \to, \rightsquigarrow, 1)$ by its universe A.

Let $(A, \odot, \to, \rightsquigarrow, 0, 1)$ be a bounded pseudo-hoop. We define two negations $^-$ and $^\sim$: for all $x \in A$, $x^- = x \to 0$, $x^\sim = x \rightsquigarrow 0$.

A bounded pseudo-hoop A is called *good* if $x^{-\sim} = x^{\sim-}$ for all $x \in A$.

If $x^{-\sim} = x^{\sim-} = x$ for all $x \in A$, then the bounded pseudo-hoop A is said to have the *pseudo-double negation* property, (pDN) for short.

The elements $x \in A$ with the property $x^{-\sim} = x^{\sim-} = x$ are sometimes called also *regular elements*.

We recall that every pseudo-MV algebra is good ([35], [36]), every linearly ordered pseudo-BL algebra is good ([20]) and every linearly ordered pseudo-hoop is good ([21]).

Recently, it was proved that there exist pseudo-BL algebras that are not good ([24]) solving an open problem from [19].

A *pseudo-BCK algebra* (more precisely, *reversed left-pseudo-BCK algebra*) is a structure $\mathcal{A} = (A, \le, \to, \rightsquigarrow, 1)$ where $\le$ is a binary relation on A, $\to$ and $\rightsquigarrow$ are binary operations on A and 1 is an element of A satisfying, for all $x, y, z \in A$, the axioms:

(bck$_1$) $x \to y \leq (y \to z) \rightsquigarrow (x \to z)$, $x \rightsquigarrow y \leq (y \rightsquigarrow z) \to (x \rightsquigarrow z)$;

(bck$_2$) $x \leq (x \to y) \rightsquigarrow y$, $x \leq (x \rightsquigarrow y) \to y$;

(bck$_3$) $x \leq x$;

(bck$_4$) $x \leq 1$;

(bck$_5$) if $x \leq y$ and $y \leq x$, then $x = y$;

(bck$_6$) $x \leq y$ iff $x \to y = 1$ iff $x \rightsquigarrow y = 1$.

A pseudo-BCK algebra with *(pP) condition* (i.e. with *pseudo-product* condition) or a *pseudo-BCK(pP) algebra* for short, is a pseudo-BCK algebra $\mathcal{A} = (A, \leq, \to, \rightsquigarrow, 1)$ satisfying (pP) condition:

(pP) there exists, for all $x, y \in A$, $x \odot y = \min\{z \mid x \leq y \to z\}$.
$$= \min\{z \mid y \leq x \rightsquigarrow z\}$$

For more details about the properties of a pseudo-BCK algebra we refer te reader to [36] and [42].

One can easily prove that any pseudo-hoop is a pseudo-BCK algebra with pseudo-product ([11]). It follows that all the properties of a pseudo-BCK algebra with pseudo-product proved in [40] and [41] are also valid in a pseudo-hoop.

Proposition 2.2. ([38], [41]) *In every pseudo-hoop $(A, \odot, \to, \rightsquigarrow, 1)$ the following hold:*

(1) $(A, \odot, 1)$ *is a monoid;*

(2) $(A, \leq)$ *is a meet-semillatice with* $x \wedge y = (x \to y) \odot x = x \odot (x \rightsquigarrow y)$;

(3) $x \odot y \leq z$ *iff* $x \leq y \to z$ *iff* $y \leq x \rightsquigarrow z$;

(4) $x \odot y \leq x \wedge y$, $x \leq y \to x$ *and* $x \leq y \rightsquigarrow x$;

(5) $x \to y \leq (y \to z) \rightsquigarrow (x \to z)$ *and* $x \rightsquigarrow y \leq (y \rightsquigarrow z) \to (x \rightsquigarrow z)$;

(6) $x \leq y$ *implies* $z \to x \leq z \to y$ *and* $z \rightsquigarrow x \leq z \rightsquigarrow y$;

(7) $x \leq y$ *implies* $y \to z \leq x \to z$ *and* $y \rightsquigarrow z \leq x \rightsquigarrow z$;

(8) $x \to y \wedge z = (x \to y) \wedge (x \to z)$ *and* $x \rightsquigarrow y \wedge z = (x \rightsquigarrow y) \wedge (x \rightsquigarrow z)$;

(9) $y \leq x \to y \odot x$ *and* $y \leq x \rightsquigarrow x \odot y$.

If A is bounded, then:

(10) $x \leq x^{-\sim}$, $x \leq x^{\sim -}$;

(11) $x \to y^{\sim} = y \rightsquigarrow x^{-}$ *and* $x \rightsquigarrow y^{-} = y \to x^{\sim}$;

(12) $x^{-\sim -} = x^{-}$, $x^{\sim - \sim} = x^{\sim}$;

(13) $x \to y^{-\sim} = y^{-} \rightsquigarrow x^{-} = x^{-\sim} \to y^{-\sim}$ *and*
$x \rightsquigarrow y^{\sim -} = y^{\sim} \to x^{\sim} = x^{\sim -} \rightsquigarrow y^{\sim -}$;

(14) $x \to y^{-} = (x \odot y)^{-}$ *and* $x \rightsquigarrow y^{\sim} = (y \odot x)^{\sim}$.

Proposition 2.3. ([8]) *Let $(A, \odot, \to, \rightsquigarrow, 0, 1)$ be a good bounded pseudo-hoop. We define a binary operation $\oplus$ on A by $x \oplus y := y^{\sim} \to x^{\sim -}$. Then for all $x, y \in A$ the following hold:*

(1) $x \oplus y = x^{-} \rightsquigarrow y^{\sim -}$;

(2) $x, y \leq x \oplus y$;

(3) $x \oplus 0 = 0 \oplus x = x^{\sim -}$;

(4) $x \oplus 1 = 1 \oplus x = 1$;

(5) $(x \oplus y)^{-\sim} = x \oplus y = x^{-\sim} \oplus y^{-\sim}$;

(6) $x \oplus y = (y^{-} \odot x^{-})^{\sim} = (y^{\sim} \odot x^{\sim})^{-}$;

(7) $\oplus$ *is associative.*

For any $n \in \mathbb{N}$, $x \in A$ we put $x^0 = 1$ and $x^{n+1} = x^n \odot x = x \odot x^n$.

If A is a bounded pseudo-hoop, then the *order* of $x \in A$, denoted $\mathrm{ord}(x)$ is the smallest $n \in \mathbb{N}$ such that $x^n = 0$. If there is no such n, then $\mathrm{ord}(x) = \infty$.

We say that A is *locally finite* if for any $x \in A$, $x \neq 1$ implies $\mathrm{ord}(x) < \infty$.

Let $(A, \odot, \to, \rightsquigarrow, 1)$ be a pseudo-hoop. A non-empty subset F of A is a *filter* of A if for all $x, y \in A$ the following conditions are satisfied:

(F_1) $x, y \in F$ implies $x \odot y \in F$;

(F_2) $x \in F$ and $x \leq y$ implies $y \in F$.

A filter F of A is *proper* if $F \neq A$.

If $X \subseteq A$, we denote by $\langle X \rangle$ the filter *generated* by X in A, that is the intersection of all filters F of A such that $X \subseteq F$. If $X = \{x\}$, then the filter generated by X will be denoted $\langle x \rangle$ instead of $\langle \{x\} \rangle$ and it is called the *principal filter* generated by the element $x \in A$.

A filter H of A is called *normal* if for every $x, y \in A$, $x \to y \in A$ iff $x \rightsquigarrow y \in A$.

A *maximal* filter or *ultrafilter* is a proper filter F of A that is not included in any other proper filter of A.

A pseudo-hoop A is called *simple* if $\{1\}$ is the unique proper normal filter of A.

A pseudo-hoop A is called *strongly simple* if $\{1\}$ is the unique proper filter of A.

Obviously, any strongly simple pseudo-hoop is simple.

When A is a hoop, since filters and normal filters coincide, the notions of simple and strongly simple hoop coincide.

Proposition 2.4. ([38]) *For any pseudo-hoop A the following are equivalent:*

(a) *A is strongly simple;*

(b) *for all $x \in A$, if $x \neq 1$ then $\langle x \rangle = A$.*

A pseudo-hoop $(A, \odot, \to, \rightsquigarrow, 1)$ is said to be *cancellative* if the monoid $(A, \odot, 1)$ is cancellative, that is $x \odot a = y \odot a$ implies $x = y$ and $a \odot x = a \odot y$ implies

$x = y$ for all $x, y, a \in A$. A pseudo-hoop A is cancellative iff $y \to x \odot y = x$ and $y \rightsquigarrow y \odot x = x$ for all $x, y \in A$.

Proposition 2.5. ([38]) *Let A be a cancellative pseudo-hoop. Then for all $x, y, z \in A$ the following hold:*

(cc$_1$) $x \to y = x \odot z \to y \odot z$ *and* $x \rightsquigarrow y = z \odot x \rightsquigarrow z \odot y$;

(cc$_2$) $x \odot z \leq y \odot z$ *iff* $x \leq y$ *and* $z \odot x \leq z \odot y$ *iff* $x \leq y$.

In the next sections we will also use the notations:

$$x \vee_1 y = (x \to y) \rightsquigarrow y \quad \text{and} \quad x \vee_2 y = (x \rightsquigarrow y) \to y.$$

We mention that the above notations differ from the ones introduced in [38], but we use them to be in line with other works ([37], [9], [8]). Note that in [8], $x \vee_1 y$ and $x \vee_2 y$ defined in [38] were replaced for the same reason with the notations:

$$x \cup_1 y = ((x \to y) \rightsquigarrow y) \wedge ((y \to x) \rightsquigarrow x) = (x \vee_1 y) \wedge (y \vee_1 x),$$
$$x \cup_2 y = ((x \rightsquigarrow y) \to y) \wedge ((y \rightsquigarrow x) \to x) = (x \vee_2 y) \wedge (y \vee_2 x).$$

Proposition 2.6. ([9]) *In any pseudo-hoop A the following hold for all $x, y \in A$:*

(1) $1 \vee_1 x = x \vee_1 1 = 1 = 1 \vee_2 x = x \vee_2 1$.

(2) $x \leq y$ *implies* $x \vee_1 y = y$ *and* $x \vee_2 y = y$.

(3) $x \vee_1 x = x \vee_2 x = x$.

(4) *If* $x_1 \leq x_2$ *and* $y_1 \leq y_2$, *then* $x_1 \vee_1 y_1 \leq x_2 \vee_1 y_2$ *and* $x_1 \vee_2 y_1 \leq x_2 \vee_2 y_2$.

Proposition 2.7. ([8]) *Let A be a pseudo-hoop. Then for all $x, y \in A$ the following hold:*

(1) $x \vee_1 y \to y = x \to y$ *and* $x \vee_2 y \rightsquigarrow y = x \rightsquigarrow y$;

(2) $x \vee_1 y \to x = y \to x$ *and* $x \vee_2 y \rightsquigarrow x = y \rightsquigarrow x$.

A pseudo-hoop $(A, \odot, \to, \rightsquigarrow, 1)$ is said to be *Wajsberg* if it satisfies the following conditions:

(W$_1$) $(x \to y) \rightsquigarrow y = (y \to x) \rightsquigarrow x$;

(W$_2$) $(x \rightsquigarrow y) \to y = (y \rightsquigarrow x) \to x$.

A pseudo-hoop $(A, \odot, \to, \rightsquigarrow, 1)$ is said to be *basic* if it satisfies the following conditions:

(BH$_1$) $(x \to y) \to z \leq ((y \to x) \to z) \to z$;

(BH$_2$) $(x \rightsquigarrow y) \rightsquigarrow z \leq ((y \rightsquigarrow x) \rightsquigarrow z) \rightsquigarrow z$.

Taking $y = 0$ in (W$_1$) and (W$_2$), it follows that a bounded Wajsberg pseudo-hoop is with (pDN). As a consequence, every bounded Wajsberg pseudo-hoop is good.

We also recall that every strongly simple basic pseudo-hoop is a linear Wajsberg pseudo-hoop ([38: Cor. 4.15]).

A *bounded Rℓ-monoid* is an algebra $(A, \odot, \vee, \wedge, \rightarrow, \rightsquigarrow, 0, 1)$ of the type $(2, 2, 2, 2, 2, 0, 0)$ satisfying the following conditions:

($R\ell_1$) $(A, \odot, 1)$ is a monoid;

($R\ell_2$) $(A, \vee, \wedge, 0, 1)$ is a bounded lattice with bounds 0 and 1 (bottom and top);

($R\ell_3$) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ iff $y \leq x \rightsquigarrow z$ for all $x, y, z \in A$;

($R\ell_4$) $(x \rightarrow y) \odot x = y \odot (y \rightsquigarrow x) = x \wedge y$ for all $x, y \in A$.

For more details about the properties of a bounded Rℓ-monoid we refer the reader to [26] and [27].

Proposition 2.8. ([8]) *Every bounded Wajsberg pseudo-hoop is a bounded Rℓ-monoid.*

Example 2.9. ([38]) Let $\mathbf{G} = (G, +, -, 0, \vee, \wedge)$ be an arbitrary ℓ-group and $N(G)$ the negative cone of $\mathbf{G}$, that is $N(G) = \{x \in G \mid x \leq 0\}$. On $N(G)$ we define the following operations:

$$x \odot y = x + y,$$
$$x \rightarrow y = (y - x) \wedge 0,$$
$$x \rightsquigarrow y = (-x + y) \wedge 0.$$

Then $\mathbf{N(G)} = (N(G), \odot, \rightarrow, \rightsquigarrow, 0)$ is a cancellative pseudo-hoop.

Definition 2.10. ([7]) Let A be a good pseudo-hoop and $x, y \in A$. We say that x is *orthogonal* to y and write $x \perp y$, if $x^{-\sim} \leq y^{\sim}$.

Proposition 2.11. ([7]) *If A is a good pseudo-hoop, then the following are equivalent:*

(a) $x \perp y$; (b) $y^{-\sim} \leq x^{-}$; (c) $y^{-\sim} \odot x^{-\sim} = 0$.

Proposition 2.12. ([7]) *Let A be a good pseudo-hoop. For all $x, y \in A$ we have:*

(1) $x \perp 0$ *and* $0 \perp x$;

(2) *if* $x \leq y$, *then* $x \perp y^{-}$ *and* $y^{\sim} \perp x$;

(3) $x \perp x^{-}$ *and* $x^{\sim} \perp x$;

(4) *if* $x \perp y$, *then* $x^{n} \perp y^{m}$ *for all* $n, m \in \mathbb{N}$;

(5) *if* $x \perp y$, *then* $y \odot x = 0$;

(6) *if* $x \perp y$, *then* $x^{-\sim} \perp y^{-\sim}$;

(7) $x^{-} \perp y^{-}$ *iff* $x^{\sim} \perp y^{\sim}$.

Definition 2.13. Let A be a good pseudo-hoop.

(1) We say that the elements x and y are *N-orthogonal*, denoted $x \perp_{no} y$, if $x^{-} \leq y^{-\sim}$.

(2) A has the *strong orthogonality* property (SO for short), if $x \perp y$ implies $x \perp_{no} y$ for all $x, y \in A$ such that $x \neq 0$ and $y \neq 0$.

Remark 2.14. If A is a good pseudo-hoop, then:

(1) $x \perp_{no} y$ iff $y^{\sim} \leq x^{-\sim}$.

(2) $x \perp_{no} y$ iff $x^{-} \perp y^{-}$.

According to Proposition 2.12(7) we also have $x \perp_{no} y$ iff $x^{\sim} \perp y^{\sim}$.

(3) $x \perp_{no} 1$ and $1 \perp_{no} x$ for all $x \in A$.

(4) $x \perp_{no} 0$ iff $x^{-} = x^{\sim} = 0$.

(5) $0 \perp_{no} x$ iff $x^{-} = x^{\sim} = 0$.

Remark 2.15. If A is a good pseudo-hoop. Then A has (SO) property iff $x^{-} = y^{-\sim}$ iff $y^{\sim} = x^{-\sim}$ for all $x \neq 0$ and $y \neq 0$.

In a Wajsberg pseudo-hoop we can define two *distance functions*:

$$d_1(x, y) = (x \to y) \wedge (y \to x) = (x \vee y) \to (x \wedge y)$$

and

$$d_2(x, y) = (x \rightsquigarrow y) \wedge (y \rightsquigarrow x) = (x \vee y) \rightsquigarrow (x \wedge y).$$

3. State pseudo-hoops

Flaminio and Montagna ([33]) have endowed the MV-algebras with a unary operation called an internal state or a state operator satisfying some basic properties of states and the new structures are called state MV-algebras. In fact, they developed a unified treatment of states and probabilistic many-valued logic in a logic and algebraic setting. The notion of a state operator has been extended for the case of GMV-algebras (pseudo-MV algebras), [46], BL-algebras, [10], and Rℓ-monoids, [28]. With algebraic structures with internal states more powerful logic can be interpreted, but they are also very interesting varieties of universal algebras.

In this paper we study these concepts for the more general fuzzy structures, namely pseudo-hoops and we present state pseudo-hoops and state-morphism pseudo-hoops. We define the notions of state operator, strong state operator, state-morphism operator, weak state-morphism operator and we study their properties. We prove that every strong state pseudo-hoop is a state pseudo-hoop and any state operator on an idempotent pseudo-hoop is a weak state-morphism operator. Glivenko and (mN) properties are defined and it is proved that for an idempotent pseudo-hoop A having these properties a state operator on $\mathrm{Reg}(A)$ can be extended to a state operator on A. One of the main results of the paper consists of proving that every perfect pseudo-hoop admits a nontrivial state operator. Other results refer to the connection between the state operators and the states and generalized states on a pseudo-hoop. Some conditions are given for a state operator to be a generalized state and for a generalized state to be a state operator.

In what follows A will be a bounded pseudo-hoop.

Definition 3.1. A *state pseudo-hoop* is a pair (A, σ) where A is a bounded pseudo-hoop and $\sigma\colon A \to A$ is a mapping, called *state operator*, such that for any $x, y \in A$ the following conditions are satisfied:

(S_1) $\sigma(0) = 0$;

(S_2) $\sigma(x \to y) = \sigma(x) \to \sigma(x \wedge y)$ and $\sigma(x \rightsquigarrow y) = \sigma(x) \rightsquigarrow \sigma(x \wedge y)$;

(S_3) $\sigma(x \odot y) = \sigma(x) \odot \sigma(x \rightsquigarrow x \odot y) = \sigma(y \to x \odot y) \odot \sigma(y)$;

(S_4) $\sigma(\sigma(x) \odot \sigma(y)) = \sigma(x) \odot \sigma(y)$;

(S_5) $\sigma(\sigma(x) \to \sigma(y)) = \sigma(x) \to \sigma(y)$ and $\sigma(\sigma(x) \rightsquigarrow \sigma(y)) = \sigma(x) \rightsquigarrow \sigma(y)$.

Denote $\mathrm{Ker}(\sigma) = \{x \in A \mid \sigma(x) = 1\}$ called the *kernel* of σ.

A state operator is called *faithful* if $\mathrm{Ker}(\sigma) = 1$.

Proposition 3.2. *If (A, σ) is a state-pseudo-hoop, then for all $x, y \in A$ the following hold:*

(1) $\sigma(1) = 1$;

(2) $\sigma(x^-) = \sigma(x)^-$ *and* $\sigma(x^\sim) = \sigma(x)^\sim$;

(3) $x \le y$ *implies* $\sigma(x) \le \sigma(y)$;

(4) $\sigma(x \odot y) \ge \sigma(x) \odot \sigma(y)$;

If $x \odot y = 0$, *then* $\sigma(x \odot y) = \sigma(x) \odot \sigma(y)$;
If A is good and $y \perp x$, *then* $\sigma(x \odot y) = \sigma(x) \odot \sigma(y)$;

(5) $\sigma(x \odot y^\sim) \ge \sigma(x) \odot \sigma(y)^\sim$ *and* $\sigma(y^- \odot x) \ge \sigma(y)^- \odot \sigma(x)$;

If $x \le y$, *then* $\sigma(x \odot y^\sim) = \sigma(x) \odot \sigma(y)^\sim$ *and* $\sigma(y^- \odot x) = \sigma(y)^- \odot \sigma(x)$;

(6) $\sigma(x \wedge y) = \sigma(x) \odot \sigma(x \rightsquigarrow y) = \sigma(y \to x) \odot \sigma(y)$;

(7) $\sigma(x \to y) \le \sigma(x) \to \sigma(y)$ *and* $\sigma(x \rightsquigarrow y) \le \sigma(x) \rightsquigarrow \sigma(y)$.

If x and y are comparable, then $\sigma(x \to y) = \sigma(x) \to \sigma(y)$ *and* $\sigma(x \rightsquigarrow y) = \sigma(x) \rightsquigarrow \sigma(y)$;

(8) $\sigma(x \to y) \odot \sigma(y \to x) \le d_1(\sigma(x), \sigma(y))$ *and* $\sigma(x \rightsquigarrow y) \odot \sigma(y \rightsquigarrow x) \le d_2(\sigma(x), \sigma(y))$;

(9) $\sigma^2(x) = \sigma(x)$;

(10) *If A is good, then:*
$\sigma(x \oplus y) \leq \sigma(x) \oplus \sigma(y)$;
$\sigma(\sigma(x) \oplus \sigma(y)) = \sigma(x) \oplus \sigma(y)$;
If $x \perp_{no} y$, then $\sigma(x \oplus y) = \sigma(x) \oplus \sigma(y)$;
$\sigma(x \oplus x^-) = \sigma(x^\sim \oplus x) = 1$;

(11) $\sigma(A) = \{x \in A \mid \sigma(x) = x\}$;

(12) $\sigma(x \to y) = \sigma(x) \to \sigma(y)$ *iff* $\sigma(y \to x) = \sigma(y) \to \sigma(x)$ *iff*
$\sigma(x \rightsquigarrow y) = \sigma(x) \rightsquigarrow \sigma(y)$ *iff* $\sigma(y \rightsquigarrow x) = \sigma(y) \rightsquigarrow \sigma(x)$;

(13) $\sigma(\sigma(x) \wedge \sigma(y)) = \sigma(x) \wedge \sigma(y)$;

(14) $\sigma(\sigma(x) \vee_1 \sigma(y)) = \sigma(x) \vee_1 \sigma(y)$ *and* $\sigma(\sigma(x) \vee_2 \sigma(y)) = \sigma(x) \vee_2 \sigma(y)$;
$\sigma(x \vee_1 y) \leq \sigma(x) \vee_1 \sigma(y)$ *and* $\sigma(x \vee_2 y) \leq \sigma(x) \vee_2 \sigma(y)$;

If x and y are comparable, then $\sigma(x \vee_1 y) = \sigma(x) \vee_1 \sigma(y)$ and $\sigma(x \vee_2 y) = \sigma(x) \vee_2 \sigma(y)$;

(15) *If σ is faithful, then $x < y$ implies $\sigma(x) < \sigma(y)$;*

(16) *If σ is faithful, then either $\sigma(x) = x$ or $\sigma(x)$ and x are not comparable;*

(17) *If A is linearly ordered and σ is faithful, then $\sigma(x) = x$ for all $x \in A$.*

Proof. (1) $\sigma(1) = \sigma(0 \to 0) = \sigma(0) \to \sigma(0 \wedge 0) = 1$.

(2) $\sigma(x^-) = \sigma(x \to 0) = \sigma(x) \to \sigma(x \wedge 0) = \sigma(x) \to \sigma(0) = \sigma(x) \to 0 = \sigma(x)^-$. Similarly for $\sigma(x^\sim) = \sigma(x)^\sim$.

(3) By Proposition 2.2(2) we get $x = y \odot (y \rightsquigarrow x)$, so:
$\sigma(x) = \sigma(y \odot (y \rightsquigarrow x)) = \sigma(y) \odot \sigma(y \rightsquigarrow y \odot (y \rightsquigarrow x)) \leq \sigma(y)$.

(4) From $x \odot y \leq x \odot y$ we get $y \leq x \rightsquigarrow x \odot y$, so by (3) we have $\sigma(y) \leq \sigma(x \rightsquigarrow x \odot y)$.

Applying (S_3) we get: $\sigma(x \odot y) = \sigma(x) \odot \sigma(x \rightsquigarrow x \odot y) \geq \sigma(x) \odot \sigma(y)$. If $x \odot y = 0$, then $\sigma(x \odot y) = 0$, so that $\sigma(x \odot y) = \sigma(x) \odot \sigma(y) = 0$. If A is good and $y \perp x$, by Proposition 2.12 we have $x \odot y = 0$, hence $\sigma(x \odot y) = \sigma(x) \odot \sigma(y) = 0$.

(5) $\sigma(x \odot y^\sim) \geq \sigma(x) \odot \sigma(y)^\sim$ and $\sigma(y^- \odot x) \geq \sigma(y)^- \odot \sigma(x)$ follow from (4) and (2). If $x \leq y$ we have $y^\sim \leq x^\sim$, $y^- \leq x^-$, so $x \odot y^\sim \leq x \odot x^\sim = 0$ and $y^- \odot x \leq x^- \odot x = 0$.

It follows that $\sigma(x \odot y^\sim) = \sigma(y^- \odot x) = 0$, hence $\sigma(x \odot y^\sim) = \sigma(x) \odot \sigma(y)^\sim = 0$ and $\sigma(y^- \odot x) = \sigma(y)^- \odot \sigma(x) = 0$.

(6) $\sigma(x \wedge y) = \sigma(x \odot (x \rightsquigarrow y)) = \sigma(x) \odot \sigma(x \rightsquigarrow (x \odot (x \rightsquigarrow y))) =$
$\sigma(x) \odot \sigma(x \rightsquigarrow x \wedge y)) = \sigma(x) \odot \sigma(x \rightsquigarrow y)$ and
$\sigma(x \wedge y) = \sigma((y \to x) \odot y) = \sigma(y \to ((y \to x) \odot y)) \odot \sigma(y) =$
$\sigma(y \to x \wedge y) \odot \sigma(y) = \sigma(y \to x) \odot \sigma(y)$.

(7) By (S_2) and Proposition 2.2(6) we have:
$\sigma(x \to y) = \sigma(x) \to \sigma(x \wedge y) \leq \sigma(x) \to \sigma(y)$ and
$\sigma(x \rightsquigarrow y) = \sigma(x) \rightsquigarrow \sigma(x \wedge y) \leq \sigma(x) \rightsquigarrow \sigma(y)$.

If $x \leq y$, then $\sigma(x) \leq \sigma(y)$ and $\sigma(x \to y) = \sigma(x) \to \sigma(x \wedge y) = \sigma(x) \to \sigma(x) = 1$. We also have $\sigma(x) \to \sigma(y) = 1$, thus $\sigma(x \to y) = \sigma(x) \to \sigma(y)$. Similarly, $\sigma(x \rightsquigarrow y) = \sigma(x) \rightsquigarrow \sigma(y)$.

If $y \leq x$, then $x \wedge y = y$ and the equalities follow from (S_2).

(8) By (7) we have $\sigma(x \to y) \leq \sigma(x) \to \sigma(y)$ and $\sigma(y \to x) \leq \sigma(y) \to \sigma(x)$, hence $\sigma(x \to y) \odot \sigma(y \to x) \leq d_1(\sigma(x), \sigma(y))$. Similarly, $\sigma(x \rightsquigarrow y) \odot \sigma(y \rightsquigarrow x) \leq d_2(\sigma(x), \sigma(y))$.

(9) Applying (1) and (S_4) we have: $\sigma^2(x) = \sigma(\sigma(x)) = \sigma(\sigma(x) \odot \sigma(1)) = \sigma(x) \odot \sigma(1) = \sigma(x)$.

(10) From $\sigma(y^- \odot x^-) \geq \sigma(y^-) \odot \sigma(x^-)$ we get $(\sigma(y^-) \odot \sigma(x^-))^\sim \geq (\sigma(y^- \odot x^-))^\sim$. Applying (2) it follows that $(\sigma(y)^- \odot \sigma(x)^-)^\sim \geq \sigma((y^- \odot x^-)^\sim)$. Thus $\sigma(x \oplus y) \leq \sigma(x) \oplus \sigma(y)$.

By (2) and (9) we get:

$$\begin{aligned}\sigma(\sigma(x) \oplus \sigma(y)) &= \sigma((\sigma(y)^- \odot \sigma(x)^-)^\sim) = (\sigma(\sigma(y^-) \odot \sigma(x^-)))^\sim \\ &= (\sigma(y^-) \odot \sigma(x^-))^\sim = (\sigma(y)^- \odot \sigma(x)^-)^\sim = \sigma(x) \oplus \sigma(y).\end{aligned}$$

Obviously, $\sigma(x \oplus 0) = \sigma(x) \oplus \sigma(0)$ and $\sigma(0 \oplus x) = \sigma(0) \oplus \sigma(x)$. Since $x \perp_{no} y$, we have $x^- \perp y^-$, so by (4) and (2) we have $\sigma(y^- \odot x^-) = \sigma(y^-) \odot \sigma(x^-) = \sigma(y)^- \odot \sigma(x)^-$.

Hence $\sigma(x \oplus y) = \sigma((y^- \odot x^-)^\sim) = (\sigma(y^- \odot x^-))^\sim = (\sigma(y)^- \odot \sigma(x)^-)^\sim = \sigma(x) \oplus \sigma(y)$. For the last assertion we have:
$\sigma(x \oplus x^-) = (\sigma(x^{-\sim} \odot x^\sim)^-) = (\sigma(x^{\sim -} \odot x^\sim))^- = (\sigma(0))^- = 1$ and
$\sigma(x^\sim \oplus x) = (\sigma(x^- \odot x^{\sim -})^\sim) = (\sigma(x^- \odot x^{-\sim}))^\sim = (\sigma(0))^\sim = 1$.

(11) Consider $y \in \sigma(A)$, so there exists $x \in A$ such that $y = \sigma(x)$. Hence $\sigma(y) = \sigma^2(x) = \sigma(x) = y$. It follows that $y \in \{x \in A \mid \sigma(x) = x\}$. Conversely, if $y \in \{x \in A \mid \sigma(x) = x\}$ it follows that $y \in \sigma(A)$.

(12) Suppose $\sigma(x \to y) = \sigma(x) \to \sigma(y)$. Applying ($S_2$), (6) and Proposition 2.2(2) we get:

$$\begin{aligned}\sigma(y \to x) &= \sigma(y) \to \sigma(y \wedge x) = \sigma(y) \to \sigma(x \to y) \odot \sigma(x) \\ &= \sigma(y) \to (\sigma(x) \to \sigma(y)) \odot \sigma(x) = \sigma(y) \to \sigma(x) \wedge \sigma(y) = \sigma(y) \to \sigma(x).\end{aligned}$$

Similarly, if $\sigma(y \to x) = \sigma(y) \to \sigma(x)$, then $\sigma(x \to y) = \sigma(x) \to \sigma(y)$. Suppose again that $\sigma(x \to y) = \sigma(x) \to \sigma(y)$, so that $\sigma(y \to x) = \sigma(y) \to \sigma(x)$. Then we have:

$$\begin{aligned}\sigma(x \rightsquigarrow y) &= \sigma(x) \rightsquigarrow \sigma(x \wedge y) = \sigma(x) \rightsquigarrow (\sigma(y \to x) \odot \sigma(y)) \\ &= \sigma(x) \rightsquigarrow ((\sigma(y) \to \sigma(x)) \odot \sigma(y)) = \sigma(x) \rightsquigarrow \sigma(x) \wedge \sigma(y) = \sigma(x) \rightsquigarrow \sigma(y).\end{aligned}$$

Similarly, if $\sigma(x \rightsquigarrow y) = \sigma(x) \rightsquigarrow \sigma(y)$, then $\sigma(x \to y) = \sigma(x) \to \sigma(y)$.

Finally, we can prove in the same manner that
$\sigma(x \rightsquigarrow y) = \sigma(x) \rightsquigarrow \sigma(y)$ implies $\sigma(y \rightsquigarrow x) = \sigma(y) \rightsquigarrow \sigma(x)$.

(13) Applying (6), (9), (S_5) and Proposition 2.2(2) we get:

$$\begin{aligned}\sigma(\sigma(x) \wedge \sigma(y)) &= \sigma^2(x) \odot \sigma(\sigma(x) \rightsquigarrow \sigma(y)) = \sigma(x) \odot (\sigma(x) \rightsquigarrow \sigma(y)) \\ &= \sigma(x) \wedge \sigma(y).\end{aligned}$$

(14) Applying (S_5) and (9) we get:

$$\begin{aligned}\sigma(\sigma(x) \vee_1 \sigma(y)) &= \sigma((\sigma(x) \to \sigma(y)) \rightsquigarrow \sigma(y)) = \sigma(\sigma(x) \to \sigma(y)) \rightsquigarrow \sigma(y) \\ &= (\sigma(x) \to \sigma(y)) \rightsquigarrow \sigma(y) = \sigma(x) \vee_1 \sigma(y).\end{aligned}$$

Similarly, $\sigma(\sigma(x) \vee_2 \sigma(y)) = \sigma(x) \vee_2 \sigma(y)$. The second part follows applying (7) twice.

(15) By (3) $x < y$ implies $\sigma(x) \le \sigma(y)$. Suppose $\sigma(x) = \sigma(y)$. From (S_2) it follows that $\sigma(y \to x) = \sigma(y) \to \sigma(x) = 1$, that is $y \to x \in \mathrm{Ker}(\sigma) = \{1\}$. Thus $y \to x = 1$, hence $y \le x$, a contradiction. It follows that $\sigma(x) < \sigma(y)$.

(16) Consider $x \in A$ such that $\sigma(x) \ne x$ and let x and $\sigma(x)$ be comparable. We have $x < \sigma(x)$ or $\sigma(x) < x$, so $\sigma(x) < \sigma(x)$, a contradiction. It follows that either $\sigma(x) = x$ or $\sigma(x)$ and x are not comparable.

(17) Since A is linearly ordered it follows that x and $\sigma(x)$ are comparable. Hence by (16), $\sigma(x) = x$. □

Corollary 3.3. *Let (A, σ) be a linearly ordered state pseudo-hoop. Then for all $x, y \in A$ the following hold:*

(1) $\sigma(x \to y) = \sigma(x) \to \sigma(y)$ *and* $\sigma(x \rightsquigarrow y) = \sigma(x) \rightsquigarrow \sigma(y)$;

(2) $\sigma(x \vee_1 y) = \sigma(x) \vee_1 \sigma(y)$ *and* $\sigma(x \vee_2 y) = \sigma(x) \vee_2 \sigma(y)$;

(3) *If A has (SO) property, then:*
$\sigma(x \oplus y^-) = \sigma(x) \oplus \sigma(y^-)$ *and* $\sigma(y^\sim \oplus x) = \sigma(y^\sim) \oplus \sigma(x)$ *or*
$\sigma(y \oplus x^-) = \sigma(y) \oplus \sigma(x^-)$ *and* $\sigma(x^\sim \oplus y) = \sigma(x^\sim) \oplus \sigma(y)$.

Proof.

(1) It follows from Proposition 3.2(7).

(2) It follows from Proposition 3.2(14).

(3) Consider the following cases:

(a) If $x = 0$, then applying Proposition 2.3(3) and Proposition 3.2(2) we have:

$$\sigma(0 \oplus y^-) = \sigma(y^{-\sim-}) = \sigma(y^-) = \sigma(y)^- \quad \text{and}$$
$$\sigma(0) \oplus \sigma(y^-) = 0 \oplus \sigma(y^-) = \sigma(y^-)^{\sim-} = \sigma(y)^-.$$

Thus $\sigma(x \oplus y^-) = \sigma(x) \oplus \sigma(y^-)$ and similarly $\sigma(y^\sim \oplus x) = \sigma(y^\sim) \oplus \sigma(x)$.

(b) If $y = 0$, then according to Proposition 2.3(4) we have:

$$\sigma(x \oplus 0^-) = \sigma(x \oplus 1) = \sigma(1) = 1 \quad \text{and} \quad \sigma(1) \oplus \sigma(y^-) = 1 \oplus \sigma(y^-) = 1.$$

Thus $\sigma(x \oplus y^-) = \sigma(x) \oplus \sigma(y^-)$ and similarly $\sigma(y^\sim \oplus x) = \sigma(y^\sim) \oplus \sigma(x)$.

(c) Assume $x \neq 0$, $y \neq 0$ and $x \leq y$. According to Proposition 2.12(2), $x \perp y^-$ and $y^\sim \perp x$. Applying Proposition 3.2(10) we have

$$\sigma(x \oplus y^-) = \sigma(x) \oplus \sigma(y^-) \quad \text{and} \quad \sigma(y^\sim \oplus x) = \sigma(y^\sim) \oplus \sigma(x).$$

Similarly, if $x \neq 0$, $y \neq 0$ and $y \leq x$ we get

$$\sigma(y \oplus x^-) = \sigma(y) \oplus \sigma(x^-) \quad \text{and} \quad \sigma(x^\sim \oplus y) = \sigma(x^\sim) \oplus \sigma(y).$$

□

Proposition 3.4. *Let (A, σ) be a state pseudo-hoop. Consider the properties:*

(a) $\sigma(x \to y) = \sigma(x) \to \sigma(y)$ *or* $\sigma(x \rightsquigarrow y) = \sigma(x) \rightsquigarrow \sigma(y)$ *for all* $x, y \in A$;

(b) $\sigma(x \wedge y) = \sigma(x) \wedge \sigma(y)$ *for all* $x, y \in A$;

(c) $\sigma(x \odot y) = \sigma(x) \odot \sigma(y)$ *for all* $x, y \in A$;

(d) $\sigma(x \vee_1 y) = \sigma(x) \vee_1 \sigma(y)$ *and* $\sigma(x \vee_2 y) = \sigma(x) \vee_2 \sigma(y)$ *for all* $x, y \in A$.

Then (a) *is equivalent with* (b) *and* (a) *implies* (c), (d).

Proof. According to Proposition 3.2(12), σ preserves $\to$ iff it preserves $\rightsquigarrow$.

(a) $\Longrightarrow$ (b) By Proposition 3.2(6) and Proposition 2.2(2) we have:

$$\sigma(x \wedge y) = \sigma(x) \odot \sigma(x \rightsquigarrow y) = \sigma(x) \odot (\sigma(x) \rightsquigarrow \sigma(y)) = \sigma(x) \wedge \sigma(y).$$

(b) $\Longrightarrow$ (a) Applying (S_2) we get:

$$\sigma(x \to y) = \sigma(x) \to \sigma(x \wedge y) = \sigma(x) \to (\sigma(x) \wedge \sigma(y)) = \sigma(x) \to \sigma(y).$$

Similarly, $\sigma(x \rightsquigarrow y) = \sigma(x) \rightsquigarrow \sigma(y)$.

(a) $\Longrightarrow$ (c) By (A_3) we have:

$$\begin{aligned}\sigma(x \odot y) \to \sigma(z) &= \sigma(x \odot y \to z) = \sigma(x \to (y \to z)) \\ &= \sigma(x) \to (\sigma(y) \to (\sigma(z)) = (\sigma(x) \odot \sigma(y)) \to \sigma(z).\end{aligned}$$

Taking $z = \sigma(x) \odot \sigma(y)$ we get:

$$\begin{aligned}\sigma(x \odot y) \to \sigma(\sigma(x) \odot \sigma(y)) &= (\sigma(x) \odot \sigma(y)) \to \sigma(\sigma(x) \odot \sigma(y)) \\ &= (\sigma(x) \odot \sigma(y)) \to (\sigma(x) \odot \sigma(y)) = 1.\end{aligned}$$

Thus $\sigma(x \odot y) \leq \sigma(\sigma(x) \odot \sigma(y)) = \sigma(x) \odot \sigma(y)$. Applying Proposition 3.2(4), we get $\sigma(x \odot y) = \sigma(x) \odot \sigma(y)$.

(a) $\Longrightarrow$ (d) It follows by the definitions of $\vee_1$ and $\vee_2$, applying (a). □

Let A be a bounded pseudo-hoop and $\sigma \colon A \to A$ be a mapping such that for all $x, y \in A$:

(S_3') $\sigma(x \odot y) = \sigma(y^- \vee_1 x) \odot \sigma(y) = \sigma(x) \odot \sigma(x^\sim \vee_2 y)$.

Definition 3.5. A mapping $\sigma\colon A \to A$ is called a *strong state operator* on A if σ satisfies conditions (S_1), (S_2), (S_3'), (S_4), (S_5).

A pair (A, σ) such that A is a bounded pseudo-hoop and σ is a strong state operator on A is called *strong state pseudo-hoop*.

A state operator σ is called *C-state operator* if it satisfies the following condition (C):

(C) $\sigma(x \vee_1 y) = \sigma(y \vee_1 x)$ and $\sigma(x \vee_2 y) = \sigma(y \vee_2 x)$.

A pair (A, σ) such that A is a bounded pseudo-hoop and σ is a C-state operator on A is called *C-state pseudo-hoop*. If a C-state operator is strong, then we call it *C-strong state operator*.

Remark 3.6. Every state Wajsberg pseudo-hoop is a C-state Wajsberg pseudo-hoop.

Proposition 3.7. *Let A be a bounded pseudo-hoop. If $\sigma\colon A \to A$ is an order-preserving mapping satisfying condition* (C)*, then $\sigma(x \vee_1 y) = \sigma(x \vee_2 y)$ for all $x, y \in A$.*

Proof. First we prove the equality for $y \leq x$. Applying Proposition 2.6(2) and condition (C) we get:

$$\sigma(x \vee_1 y) = \sigma(y \vee_1 x) = \sigma(x) \qquad \text{and} \qquad \sigma(x \vee_2 y) = \sigma(x),$$

i.e., $\sigma(x \vee_1 y) = \sigma(x \vee_2 y)$.

Assume now that x and y are arbitrary elements of A. Using again Proposition 2.6(2), condition (C) and the first part of the proof, we get:

$$\begin{aligned}
\sigma(x \vee_1 y) &= \sigma(x \vee_1 (x \vee_1 y)) = \sigma((x \vee_1 y) \vee_1 x) \\
&= \sigma((x \vee_1 y) \vee_2 x) \geq \sigma(y \vee_2 x) \\
&= \sigma(x \vee_2 (y \vee_2 x)) \geq \sigma(x \vee_2 y) \\
&= \sigma(y \vee_2 (x \vee_2 y)) = \sigma((x \vee_2 y) \vee_2 y) \\
&\geq \sigma(x \vee_1 y).
\end{aligned}$$

Thus $\sigma(x \vee_1 y) = \sigma(x \vee_2 y)$. $\square$

Corollary 3.8. *If σ is a C-state operator, then $\sigma(x \vee_1 y) = \sigma(x \vee_2 y)$.*

Theorem 3.9. *Every strong state pseudo-hoop is a state pseudo-hoop.*

Proof. Consider the strong state pseudo-hoop (A, σ) and $x, y \in A$. Taking into consideration that $y^- \leq y \to x$ and $x^\sim \leq x \rightsquigarrow y$ we get:

$$y^- \vee_1 (y \to x) = y \to x \qquad \text{and} \qquad x^\sim \vee_2 (x \rightsquigarrow y) = x \rightsquigarrow y.$$

Then we have:

$$\sigma(x \wedge y) = \sigma(x \odot (x \rightsquigarrow y)) = \sigma(x) \odot \sigma(x^\sim \vee_2 (x \rightsquigarrow y)) = \sigma(x) \odot \sigma(x \rightsquigarrow y).$$

It follows that

$$\sigma(x \odot y) = \sigma(x \wedge (x \odot y)) = \sigma(x) \odot \sigma(x \rightsquigarrow (x \odot y)).$$

Similarly,

$$\sigma(x \wedge y) = \sigma((y \to x) \odot y) = \sigma(y^- \vee_1 (y \to x)) \odot \sigma(y) = \sigma(y \to x) \odot \sigma(y),$$

so

$$\sigma(x \odot y) = \sigma((x \odot y) \wedge y) = \sigma(y \to (x \odot y)) \odot \sigma(y).$$

Thus condition (S_3') implies condition (S_3), hence σ is a state operator on A. □

Proposition 3.10. *If σ is a strong state operator on a bounded pseudo-hoop A such that $x^\sim \le y$ or $y^- \le x$ for some $x, y \in A$, then $\sigma(x \odot y) = \sigma(x) \odot \sigma(y)$.*

Proof. Since σ is a strong state operator, it satisfies the condition

$$\sigma(x \odot y) = \sigma(y^- \vee_1 x) \odot \sigma(y) = \sigma(x) \odot \sigma(x^\sim \vee_2 y).$$

According to Proposition 2.6, $y^- \le x$ implies $\sigma(y^- \vee_1 x) = \sigma(x)$ and $x^\sim \le y$ implies $\sigma(x^\sim \vee_2 y) = \sigma(y)$. Thus $\sigma(x \odot y) = \sigma(x) \odot \sigma(y)$. □

Proposition 3.11. *If σ is a state operator on a linearly ordered bounded pseudo-hoop A, then σ is a pseudo-hoop endomorphism such that $\sigma^2 = \sigma$.*

Proof. Since (A, σ) is a linearly ordered state pseudo-hoop, according to Corollary 3.3 σ preserves $\to$ and $\rightsquigarrow$. Applying Proposition 3.4, it follows that σ preserves $\odot$.

Taking into consideration that σ preserves also the constants 0 (by Definition 3.1(S_1)) and 1 (by Proposition 3.2(1)), we conclude that σ is an endomorphism. Condition $\sigma^2 = \sigma$ follows from Proposition 3.2(9). □

Proposition 3.12. *If (A, σ) is a state pseudo-hoop, then $\sigma(A)$ is a subalgebra of A.*

Proof. By (S_4), (S_5) and Proposition 3.2(1), $\sigma(A)$ is closed under the operations $\odot$, $\to$, $\rightsquigarrow$, 1. Thus $\sigma(A)$ is a subalgebra of A. □

Definition 3.13.

(1) A state operator σ on a bounded pseudo-hoop A is called a *weak state-morphism operator* on A if for all $x, y \in A$:
(S_6) $\sigma(x \odot y) = \sigma(x) \odot \sigma(y)$.
In this case (A, σ) is called a *weak state-morphism pseudo-hoop.*

(2) A bounded pseudo-hoop endomorphism $\sigma\colon A \to A$ is said to be a *state-morphism operator* if $\sigma^2 = \sigma$.

Obviously, a state-morphism operator is always a weak state-morphism operator.

Example 3.14. ([28])

(1) If A is a bounded pseudo-hoop, then the identity id_A is a state operator on A.

(2) Let A be a bounded pseudo-hoop and $B = A \times A$. Then the mappings $\sigma_1, \sigma_2 \colon B \to B$ such that $\sigma_1(x_1, x_2) = (x_1, x_1)$, $\sigma_2(x_1, x_2) = (x_2, x_2)$ are state-morphism operators on the bounded pseudo-hoop B.

Remark 3.15. From Propositions 3.4 and 3.10 it follows that:

(1) If σ is a state operator on A preserving $\to$ or preserving $\rightsquigarrow$, then σ is a weak state-morphism operator and a state-morphism operator.

(2) If σ is a strong state operator on a bounded pseudo-hoop A such that $x^{\sim} \leq y$ or $y^{-} \leq x$ for all $x, y \in A$, then σ is a weak state-morphism operator.

Proposition 3.16. *If A is a bounded cancellative pseudo-hoop, then any state operator σ on A is a weak state-morphism operator.*

Proof. According to (S_3) and taking into consideration that in a cancellative pseudo-hoop $y \to x \odot y = x$, we get:

$$\sigma(x \odot y) = \sigma(y \to x \odot y) \odot \sigma(y) = \sigma(x) \odot \sigma(y).$$

Thus σ is a weak state-morphism operator on A. (It can be proved similarly for the case $x \rightsquigarrow x \odot y = y$). □

An element a of a pseudo-hoop A is said to be an *idempotent* if $a^2 = a$. The set of all idempotents of A is denoted by $\mathrm{Id}(A)$.

A pseudo-hoop A is called *idempotent pseudo-hoop* if $\mathrm{Id}(A) = A$, that is all elements of A are idempotent.

It was proved in [23: Prop. 3.1] that, if $a \in \mathrm{Id}(A)$, then for all $x \in A$ we have:

(a) $a \odot x = a \wedge x = x \odot a$;

(b) $a \to x = a \rightsquigarrow x$.

According to [43], representable Brouwerian algebras are idempotent basic hoops and generalized Boolean algebras are idempotent Wajsberg hoops.

Theorem 3.17. *If A is a bounded idempotent pseudo-hoop, then any state operator σ on A is a weak state-morphism operator and a state-morphism operator.*

Proof. Consider $x, y \in A$. Applying the property of idempotent elements and Proposition 3.2(4) we get:

$$\sigma(x \wedge y) = \sigma(x \odot y) \geq \sigma(x) \odot \sigma(y) = \sigma(x) \wedge \sigma(y).$$

On the other hand, $\sigma(x \wedge y) \leq \sigma(x) \wedge \sigma(y) = \sigma(x) \odot \sigma(y)$. Thus $\sigma(x \wedge y) = \sigma(x \odot y) = \sigma(x) \odot \sigma(y) = \sigma(x) \wedge \sigma(y)$. Thus σ is a weak state-morphism operator on A.

Since $\wedge$ is preserved, according to Proposition 3.4($(a) \Leftrightarrow (b)$), one of $\rightarrow$, $\rightsquigarrow$ is preserved as well. The preservance of the second one follows from Proposition 3.2(12).

The constants 0 and 1 are preserved by Definition 3.1(S_1) and Proposition 3.2(1), respectively. Thus σ is an endomorphism on A.

Since from Proposition 3.2(9) we have $\sigma^2 = \sigma$, it follows that σ is also a state-morphism operator on A. ☐

Proposition 3.18. *If σ is a state operator on a bounded pseudo-hoop A, then* $\mathrm{Ker}(\sigma)$ *is a normal filter of A.*

Proof. Similarly as in [28: Prop. 5.6]. ☐

4. Glivenko and meet-negation properties

We introduce the Glivenko and meet-negation properties which will be used in the next sections.

For a bounded pseudo-hoop $(A, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ we define

$$\mathrm{Reg}(A) = \{a \in A \mid a^{-\sim} = a^{\sim -} = a\}.$$

Then $(\mathrm{Reg}(A), \odot, \rightarrow, \rightsquigarrow, 0, 1)$ is a subalgebra of A (see [44]). Obviously, A satisfies the (pDN) condition iff $A = \mathrm{Reg}(A)$. Moreover, if A is good, then $a^{-\sim} \in \mathrm{Reg}(A)$.

Based on the conditions introduced in [44] we introduce the notion of Glivenko property for a good pseudo-hoop.

Definition 4.1. A good pseudo-hoop A has the *Glivenko property* iff the following identities are satisfied for all $x, y \in A$:

$$(x \rightarrow y)^{-\sim} = x \rightarrow y^{-\sim}, \qquad (x \rightsquigarrow y)^{-\sim} = x \rightsquigarrow y^{-\sim}.$$

Remark 4.2. If A is a good bounded Rℓ-monoid, then according to Lemma 2.1 in [27], the following hold for all $x, y \in A$:

$$(x \rightarrow y)^{-\sim} = x^{-\sim} \rightarrow y^{-\sim}, \qquad (x \rightsquigarrow y)^{-\sim} = x^{-\sim} \rightsquigarrow y^{-\sim}.$$

Applying Proposition 2.2(13) it follows that

$$(x \rightarrow y)^{-\sim} = x \rightarrow y^{-\sim}, \qquad (x \rightsquigarrow y)^{-\sim} = x \rightsquigarrow y^{-\sim}.$$

Thus any good bounded Rℓ-monoid satisfies Glivenko property.

On the other hand by Proposition 2.8, every bounded Wajsberg pseudo-hoop is a bounded Rℓ-monoid. It follows that every bounded Wajsberg pseudo-hoop has Glivenko property.

Remark 4.3. By Proposition 2.2(13), in any good pseudo-hoop A satisfying Glivenko property the following hold:

$$(x \to y)^{-\sim} = x^{-\sim} \to y^{-\sim}, \qquad (x \rightsquigarrow y)^{-\sim} = x^{-\sim} \rightsquigarrow y^{-\sim}$$

for all $x, y \in A$.

Definition 4.4. A good pseudo-hoop A is said to be with *meet-negation* property (mN for short) if

(mN) $(x \wedge y)^{-\sim} = x^{-\sim} \wedge y^{-\sim}$ for all $x, y \in A$.

Remark 4.5. According to [27: Lemma 2.1], any good bounded Rℓ-monoid satisfies (mN) property.

Applying Proposition 2.8, it follows that every bounded Wajsberg pseudo-hoop has (mN) property.

Proposition 4.6. *Let (A, σ) by an idempotent state pseudo-hoop. Then:*

(1) $\sigma(x \wedge y) = \sigma(x \odot y) = \sigma(x) \odot \sigma(y) = \sigma(x) \wedge \sigma(y)$ *for all* $x, y \in A$;

(2) $\sigma(x \to y) = \sigma(x) \to \sigma(y)$ *and* $\sigma(x \rightsquigarrow y) = \sigma(x) \rightsquigarrow \sigma(y)$ *for all* $x, y \in A$;

(3) *If A has* (mN) *property, then*
$(x \odot y)^{-\sim} = (x \wedge y)^{-\sim} = x^{-\sim} \wedge y^{-\sim} = x^{-\sim} \odot y^{-\sim}$ *for all* $x, y \in A$.

Proof. We remark that an idempotent pseudo-hoop is commutative, so that it is good.

(1) It follows from the proof of Theorem 3.17.

(2) By Proposition 3.4 it follows that

$$\sigma(x \to y) = \sigma(x) \to \sigma(y) \quad \text{or} \quad \sigma(x \rightsquigarrow y) = \sigma(x) \rightsquigarrow \sigma(y) \text{ for all } x, y \in A$$

which are equivalent according to Proposition 3.2(12).

(3) Since A is idempotent, $x \odot y = x \wedge y$ for all $x, y \in A$ and we get:

$$(x \odot y)^{-\sim} = (x \wedge y)^{-\sim} = x^{-\sim} \wedge y^{-\sim} = x^{-\sim} \odot y^{-\sim}.$$

□

Proposition 4.7. *Let (A, σ) be a state pseudo-hoop and $x, y \in \mathrm{Reg}(A)$. Then:*

(1) $\sigma(x) \in \mathrm{Reg}(A)$;

(2) *If A is good, then* $x \oplus y \in \mathrm{Reg}(A)$;

(3) *If A has Glivenko property, then* $x \to y$, $x \rightsquigarrow y$, $x \vee_1 y$, $x \vee_2 y \in \mathrm{Reg}(A)$;

(4) *If A has* (mN) *property, then* $x \wedge y \in \mathrm{Reg}(A)$;

(5) *If A is idempotent with* (mN) *property, then* $x \odot y \in \mathrm{Reg}(A)$.

Proof.

(1) By Proposition 3.2(2) we have:

$$\sigma(x)^{-\sim} = \sigma(x^{-\sim}) = \sigma(x) \quad \text{and} \quad \sigma(x)^{\sim-} = \sigma(x^{\sim-}) = \sigma(x),$$

thus $\sigma(x) \in \mathrm{Reg}(A)$.

(2) Applying Proposition 2.3(6) we get:

$$(x \oplus y)^{-\sim} = ((y^- \odot x^-)^\sim)^{-\sim} = (y^- \odot x^-)^\sim = x \oplus y.$$

From goodness property we have that $(x \oplus y)^{\sim-} = (x \oplus y)^{-\sim} = x \oplus y$. Thus $x \oplus y \in \mathrm{Reg}(A)$.

(3) By Proposition 2.2(13) we have:

$$(x \to y)^{-\sim} = x \to y^{-\sim} = x^{-\sim} \to y^{-\sim} = x \to y \quad \text{and}$$
$$(x \rightsquigarrow y)^{-\sim} = x \rightsquigarrow y^{-\sim} = x^{-\sim} \rightsquigarrow y^{-\sim} = x \rightsquigarrow y,$$

since A is good. As a consequence, it follows that $x \vee_1 y,\ x \vee_2 y \in \mathrm{Reg}(A)$.

(4) By (mN) property we have $(x \wedge y)^{-\sim} = x^{-\sim} \wedge y^{-\sim} = x \wedge y$, thus $x \wedge y \in \mathrm{Reg}(A)$.

(5) From Proposition 4.6 we have $(x \odot y)^{-\sim} = x^{-\sim} \odot y^{-\sim} = x \odot y$, so $x \odot y \in \mathrm{Reg}(A)$. □

5. On the existence of state operators on pseudo-hoops

In this section we investigate the existence of the state operators proving that every perfect pseudo-hoop admits a nontrivial state operator on it. In what follows A will be a bounded pseudo-hoop.

Lemma 5.1. *Any state operator σ on a locally finite pseudo-hoop is faithful.*

Proof. Assume that there exists $0 < x < 1$ such that $\sigma(x) = 1$. Then there is an integer $n \geq 1$ such that $x^n = 0$, hence $0 = \sigma(0) = \sigma(x^n) \geq \sigma(x)^n = 1$, a contradiction. Thus σ is faithful. □

Proposition 5.2. *If A is a strongly simple locally finite basic pseudo-hoop, then the identity is the unique state operator on A.*

Proof. Let σ be a state operator on A. By Lemma 5.1 it follows that σ is faithful. Since every strongly simple basic pseudo-hoop is linearly ordered, applying Proposition 3.2(17), we get $\sigma(x) = x$ for all $x \in A$. □

We remark that any bounded idempotent pseudo-hoop A is good. Indeed, applying the identity $a \to x = a \rightsquigarrow x$ for $x = 0$, we get $a^- = a^\sim$, so that $a^{-\sim} = a^{--} = a^{\sim-}$ for all $a \in A$.

Theorem 5.3. *Let A be an idempotent pseudo-hoop satisfying Glivenko and* (mN) *properties and* $\sigma\colon \mathrm{Reg}(A) \to \mathrm{Reg}(A)$ *be a state operator on* $\mathrm{Reg}(A)$*. Then the mapping* $\tilde{\sigma}\colon A \to A$ *defined by* $\tilde{\sigma}(x) = \sigma(x^{-\sim})$ *is a state operator on A such that* $\tilde{\sigma}_{|\mathrm{Reg}(A)} = \sigma$.

Proof. Obviously, $\tilde{\sigma}(0) = \sigma(0) = 0$, so the condition (S_1) is verified. Applying Proposition 4.6 we get:

$$\tilde{\sigma}(x \to y) = \sigma((x \to y)^{-\sim}) = \sigma(x^{-\sim} \to y^{-\sim}) = \sigma(x^{-\sim}) \to \sigma(x^{-\sim} \wedge y^{-\sim})$$
$$= \sigma(x^{-\sim}) \to \sigma((x \wedge y)^{-\sim}) = \tilde{\sigma}(x) \to \tilde{\sigma}(x \wedge y).$$

Similarly, $\tilde{\sigma}(x \rightsquigarrow y) = \tilde{\sigma}(x) \rightsquigarrow \tilde{\sigma}(x \wedge y)$, so $\tilde{\sigma}$ satisfies (S_2). By Proposition 4.6 we also have:

$$\tilde{\sigma}(x \odot y) = \sigma((x \odot y)^{-\sim}) = \sigma(x^{-\sim} \odot y^{-\sim}) = \sigma(x^{-\sim}) \odot \sigma(x^{-\sim} \rightsquigarrow x^{-\sim} \odot y^{-\sim})$$
$$= \sigma(x^{-\sim}) \odot \sigma(x^{-\sim} \rightsquigarrow (x \odot y)^{-\sim}) = \sigma(x^{-\sim}) \odot \sigma((x \rightsquigarrow x \odot y)^{-\sim})$$
$$= \tilde{\sigma}(x) \odot \tilde{\sigma}(x \rightsquigarrow x \odot y).$$

Similarly, $\tilde{\sigma}(x \odot y) = \tilde{\sigma}(y \to x \odot y) \odot \tilde{\sigma}(y)$, hence $\tilde{\sigma}$ satisfies (S_3). For the condition (S_4) we have:

$$\tilde{\sigma}(\tilde{\sigma}(x) \odot \tilde{\sigma}(y)) = \sigma((\sigma(x^{-\sim}) \odot \sigma(y^{-\sim}))^{-\sim}) = \sigma(\sigma(x^{-\sim})^{-\sim} \odot \sigma(y^{-\sim})^{-\sim})$$
$$= \sigma(\sigma(x^{-\sim}) \odot \sigma(y^{-\sim})) = \sigma(x^{-\sim}) \odot \sigma(y^{-\sim}) = \tilde{\sigma}(x) \odot \tilde{\sigma}(y)$$

thus it is verified too. Finaly we have:

$$\tilde{\sigma}(\tilde{\sigma}(x) \to \tilde{\sigma}(y)) = \sigma((\sigma(x^{-\sim}) \to \sigma(y^{-\sim}))^{-\sim}) = \sigma(\sigma(x^{-\sim})^{-\sim} \to \sigma(y^{-\sim})^{-\sim})$$
$$= \sigma(\sigma(x^{-\sim}) \to \sigma(y^{-\sim})) = \sigma(x^{-\sim}) \to \sigma(y^{-\sim}) = \tilde{\sigma}(x) \to \tilde{\sigma}(y)$$

and similarly $\tilde{\sigma}(\tilde{\sigma}(x) \rightsquigarrow \tilde{\sigma}(y)) = \tilde{\sigma}(x) \rightsquigarrow \tilde{\sigma}(y)$, that is the condition (S_5) for $\tilde{\sigma}$. We conclude that $\tilde{\sigma}$ is a state operator on A. If $x \in \mathrm{Reg}(A)$, then $\tilde{\sigma}(x) = \sigma(x^{-\sim}) = \sigma(x)$, so that $\tilde{\sigma}_{|\mathrm{Reg}(A)} = \sigma$. □

Corollary 5.4. *If A is an idempotent $R\ell$-monoid, then any state operator on* $\mathrm{Reg}(A)$ *can be extended to a state operator on A.*

In what follows we recall some notions and results regarding the perfect pseudo-hoops. Since every pseudo-hoop is a pseudo-BCK(pP) algebra, the results proved in [4] and [11] for the pseudo-BCK(pP) algebras are also valid for pseudo-hoops.

A pseudo-hoop A is called *local* if and only if it has a unique maximal filter. We will denote by:

$$D(A) = \{x \in A \mid \mathrm{ord}(x) = \infty\} \quad \text{and} \quad D(A)^* = \{x \in A \mid \mathrm{ord}(x) < \infty\}.$$

Obviously, $D(A) \cap D(A)^* = \emptyset$ and $D(A) \cup D(A)^* = A$. We can also remark that $1 \in D(A)$ and $0 \in D(A)^*$.

The following are equivalent:

(a) $D(A)$ is a filter of A;

(b) $D(A)$ is a proper filter of A;

(c) A is local;

(d) $D(A)$ is the unique maximal filter of A;

(e) for all $x, y \in A$, $\mathrm{ord}(x \odot y) < \infty$ implies $(\mathrm{ord}(x) < \infty$ or $\mathrm{ord}(y) < \infty)$.

A pseudo-hoop A is called *perfect* if it satisfies the following conditions:

(i) A is a local good pseudo-hoop;

(ii) for any $x \in A$, $\mathrm{ord}(x) < \infty$ iff $[\mathrm{ord}(x^-) = \infty$ and $\mathrm{ord}(x^\sim) = \infty]$.

The intersection of all maximal filters of A is called the *radical* of A and it is denoted by $\mathrm{Rad}(A)$.

Let A be a perfect pseudo-hoop. Then:

(i) $\mathrm{Rad}(A) = D(A)$;

(ii) $A = \mathrm{Rad}(A) \cup \mathrm{Rad}(A)^*$;

(iii) $\mathrm{Rad}(A)$ is a normal filter of A.

Let A be a perfect pseudo-hoop and $x \in \mathrm{Rad}(A)^*$, $y \in A$. Then the following properties hold:

(i) If $y \leq x$, then $y \in \mathrm{Rad}(A)^*$;

(ii) $x \odot y \in \mathrm{Rad}(A)^*$.

Theorem 5.5. *Any perfect pseudo-hoop admits a nontrivial state operator on it.*

Proof. Let A be a perfect pseudo-hoop, so $A = \mathrm{Rad}(A) \cup \mathrm{Rad}(A)^*$. We will prove that the map $\sigma\colon A \to A$ defined by

$$\sigma(x) = \begin{cases} 1 & \text{if } x \in \mathrm{Rad}(A) \\ 0 & \text{if } x \in \mathrm{Rad}(A)^* \end{cases}$$

is a state operator on A. Obviously, $\sigma(0) = 0$, hence (S_1) is satisfied.

We consider the following cases:

(1) $a, b \in \mathrm{Rad}(A)$. Obviously, $\sigma(a) = \sigma(b) = 1$. Since $\mathrm{Rad}(A)$ is a filter of A and $b \leq a \to b$, it follows that $a \to b \in \mathrm{Rad}(A)$. Hence $\sigma(a \to b) = 1$. Similarly, $\sigma(a \rightsquigarrow b) = 1$. From the definition of a filter we have $a \odot b, a \wedge b \in \mathrm{Rad}(A)$. Thus $\sigma(a) = \sigma(b) = \sigma(a \to b) = \sigma(a \rightsquigarrow b) = \sigma(a \wedge b) = \sigma(a \odot b) = 1$.

Since $a \odot b \in \mathrm{Rad}(A)$ and $a \odot b \leq a \rightsquigarrow a \odot b$, $a \odot b \leq b \to a \odot b$ it follows that $a \rightsquigarrow a \odot b,\ b \to a \odot b \in \mathrm{Rad}(A)$, so $\sigma(a \rightsquigarrow a \odot b) = \sigma(b \to a \odot b) = 1$.

One can easily check that the conditions (S_2)–(S_5) are satisfied.

(2) $a, b \in \mathrm{Rad}(A)^*$. In this case, $\sigma(a) = \sigma(b) = 0$ and we will prove that $a \to b,\ a \rightsquigarrow b \in \mathrm{Rad}(A)$. Indeed, suppose that $a \to b \in \mathrm{Rad}(A)^*$. Since $a \leq a^{-\sim}$, it follows that $a^{-\sim} \to b \leq a \to b$, so $a^{-\sim} \to b \in \mathrm{Rad}(A)^*$. But $a^- \leq a^{-\sim} \to b$, hence $a^- \in \mathrm{Rad}(A)^*$, a contradiction with the condition (ii) in

the definition of perfect pseudo-hoop ($a \in \mathrm{Rad}(A)^* = D(A)^*$ iff $a^- \in \mathrm{Rad}(A) = D(A)$). It follows that $a \to b \in \mathrm{Rad}(A)$ and similarly $a \rightsquigarrow b \in \mathrm{Rad}(A)$. Hence $\sigma(a \to b) = \sigma(a \rightsquigarrow b) = 1$. From $a \wedge b \leq b$, $a \odot b \leq b$, we get $a \wedge b,\ a \odot b \in \mathrm{Rad}(A)^*$, thus $\sigma(a \wedge b) = \sigma(a \odot b) = 0$.

We can see that the conditions (S_2)–(S_5) are also verified.

(3) $a \in \mathrm{Rad}(A)$, $b \in \mathrm{Rad}(A)^*$. Obviously, $\sigma(a) = 1$ and $\sigma(b) = 0$.

We show that $a \to b \in \mathrm{Rad}(A)^*$. Indeed, suppose that $a \to b \in \mathrm{Rad}(A)$. Because $b \leq b^{-\sim}$, we have $a \to b \leq a \to b^{-\sim}$, so $a \to b^{-\sim} \in \mathrm{Rad}(A)$. It means that $(a \odot b^{\sim})^- \in \mathrm{Rad}(A)$, that is, $a \odot b^{\sim} \in \mathrm{Rad}(A)^*$. On the other hand, since $\mathrm{Rad}(A)$ is a filter of A and $a, b^{\sim} \in \mathrm{Rad}(A)$ we have $a \odot b^{\sim} \in \mathrm{Rad}(A)$, a contradiction. We conclude that $a \to b \in \mathrm{Rad}(A)^*$, so $\sigma(a \to b) = 0$. Similarly, $a \rightsquigarrow b \in \mathrm{Rad}(A)^*$, so $\sigma(a \rightsquigarrow b) = 0$.

Since $a \wedge b \leq b$, $a \odot b \leq b$, we have $a \wedge b,\ a \odot b \in \mathrm{Rad}(A)^*$, so $\sigma(a \wedge b) = \sigma(a \odot b) = 0$. Moreover, $a \in \mathrm{Rad}(A)$ and $a \odot b \in \mathrm{Rad}(A)^*$ implies $a \rightsquigarrow a \odot b \in \mathrm{Rad}(A)^*$, hence $\sigma(a \rightsquigarrow a \odot b) = 0$.

It easy to see that the conditions (S_2)–(S_5) are satisfied.

(4) $a \in \mathrm{Rad}(A)^*$, $b \in \mathrm{Rad}(A)$. Taking into consideration that $b \leq a \to b$, $b \leq a \rightsquigarrow b$ we have $a \to b,\ a \rightsquigarrow b \in \mathrm{Rad}(A)$.

From $a \wedge b$, $a \odot b \leq a$ we get $a \wedge b,\ a \odot b \in \mathrm{Rad}(A)^*$. Hence $\sigma(a) = 0$, $\sigma(b) = 1$, $\sigma(a \wedge b) = \sigma(a \odot b) = 0$, $\sigma(a \to b) = \sigma(a \rightsquigarrow b) = 1$. Applying the case (3), $b \in \mathrm{Rad}(A)$, $a \odot b \in \mathrm{Rad}(A)^*$ implies $b \to a \odot b \in \mathrm{Rad}(A)^*$, so $\sigma(b \to a \odot b) = 0$.

Thus the conditions (S_2)–(S_5) are also satisfied.

We conclude that σ is a state operator on A, that is (A, σ) is a state pseudo-hoop. □

Remark 5.6. The state operator σ defined in Theorem 5.5 is a C-state operator. Indeed, in the cases (1), (3), (4) from the proof of Theorem 5.5 we have $a \vee_1 b$, $b \vee_1 a$, $a \vee_2 b$, $b \vee_2 a \in \mathrm{Rad}(A)$, so $\sigma(a \vee_1 b) = \sigma(b \vee_1 a) = 1$ and $\sigma(a \vee_2 b) = \sigma(b \vee_2 a) = 1$. In the case (2), $a \vee_1 b$, $b \vee_1 a$, $a \vee_2 b$, $b \vee_2 a \in \mathrm{Rad}(A)^*$, hence $\sigma(a \vee_1 b) = \sigma(b \vee_1 a) = 0$ and $\sigma(a \vee_2 b) = \sigma(b \vee_2 a) = 0$. Thus σ is a C-state operator.

6. State operators and states on pseudo-hoops

The notions of states on bounded pseudo-hoops have been investigated in [8]. A *Bosbach state* on the bounded pseudo-hoop A is a function $s\colon A \to [0,1]$ such that the following conditions hold for any $x, y \in A$:

(B_1) $s(x) + s(x \to y) = s(y) + s(y \to x)$;

(B_2) $s(x) + s(x \rightsquigarrow y) = s(y) + s(y \rightsquigarrow x)$;

(B_3) $s(0) = 0$ and $s(1) = 1$.

Let A be a good bounded pseudo-hoop. A *Riečan state* on A is a function $s\colon A \to [0,1]$ such that the following conditions hold for all $x, y \in A$:

(R_1) if $x \perp y$, then $s(x \oplus y) = s(x) + s(y)$;

(R_2) $s(1) = 1$.

It was proved in [8] that any Bosbach state on a good pseudo-hoop is a Riečan state. Let s be a Riečan state on a good pseudo-hoop A such that $s(x \vee_1 y) = s(y \vee_1 x)$ and $s(x \vee_2 y) = s(y \vee_2 x)$ for all $x, y \in A$.
If A satisfies the conditions:

$$(x \to y)^{-\sim} = x \vee_1 y \to y^{-\sim}$$
$$(x \rightsquigarrow y)^{-\sim} = x \vee_2 y \rightsquigarrow y^{-\sim},$$

then s is a Bosbach state on A.

It was also proved in [8] that, if s is a Riečan state on a bounded pseudo-hoop A with (pDN) satisfying the properties $s(x \vee_1 y) = s(y \vee_1 x)$ and $s(x \vee_2 y) = s(y \vee_2 x)$ for all $x, y \in A$, then s is a Bosbach state on A. As a consequence, every Riečan state on a bounded Wajsberg pseudo-hoop is a Bosbach state.

Theorem 6.1. *Let σ be a state operator on a bounded pseudo-hoop A preserving $\to$ or $\rightsquigarrow$. If s is a Bosbach state on A, then the mapping $s_\sigma\colon A \to [0,1]$ defined by $s_\sigma(x) = s(\sigma(x))$ is a Bosbach state on A.*

Proof. Obviously, $s_\sigma(0) = 0$ and $s_\sigma(1) = 1$, so (B_3) is verified.

It is sufficient to assume that just one of the arrows $\to$, $\rightsquigarrow$ is preserved, the preservance of the second one is implied by Proposition 3.2(12). It follows that:

$$s_\sigma(x) + s_\sigma(x \to y) = s(\sigma(x)) + s(\sigma(x \to y)) = s(\sigma(x)) + s(\sigma(x) \to \sigma(y))$$
$$= s(\sigma(y)) + s(\sigma(y) \to \sigma(x)) = s(\sigma(y)) + s(\sigma(y \to x)) = s_\sigma(y) + s_\sigma(y \to x).$$

Thus s_σ satisfies (B_1) and similarly s_σ satisfies (B_2). It follows that s_σ is a Bosbach state on A. □

Corollary 6.2. *Let (A, σ) be a linearly ordered state pseudo-hoop and s be a Bosbach state on A. Then the mapping $s_\sigma\colon A \to [0,1]$ defined by $s_\sigma(x) = s(\sigma(x))$ is a Bosbach state on A.*

Proof. According to Corollary 3.3, σ preserves $\to$ and $\rightsquigarrow$, hence by Theorem 6.1 s_σ is a Bosbach state on A. □

Corollary 6.3. *Let (A, σ) be an idempotent state pseudo-hoop and s be a Bosbach state on A. Then the mapping $s_\sigma\colon A \to [0,1]$ defined by $s_\sigma(x) = s(\sigma(x))$ is a Bosbach state on A.*

Proof. By Proposition 4.6 we have $\sigma(x \wedge y) = \sigma(x) \wedge \sigma(y)$. According to Proposition 3.4, σ preserves $\to$ and $\rightsquigarrow$, hence by Theorem 6.1 s_σ is a Bosbach state on A. □

Theorem 6.4. *Let A be a bounded pseudo-hoop with Glivenko property and σ be a state operator on A preserving $\to$ or $\rightsquigarrow$. If s is a Bosbach state on* $\mathrm{Reg}(A)$*, then the mapping $\tilde{s}_\sigma\colon A \to [0,1]$ defined by $\tilde{s}_\sigma(x) = s(\sigma(x^{-\sim}))$ is a Bosbach state on A.*

P r o o f. Obviously, $s_\sigma(0) = 0$ and $s_\sigma(1) = 1$, so (B_3) is verified.

If σ preserves one of the arrows $\to$, $\rightsquigarrow$, then by Proposition 3.2(12) the second one is also preserved. Applying Remark 4.3, we have:

$$\begin{aligned}
&\tilde{s}_\sigma(x) + \tilde{s}_\sigma(x \to y) = s(\sigma(x^{-\sim})) + s(\sigma((x \to y)^{-\sim}))\\
&= s(\sigma(x^{-\sim})) + s(\sigma(x^{-\sim} \to y^{-\sim})) = s(\sigma(x^{-\sim})) + s(\sigma(x^{-\sim}) \to \sigma(y^{-\sim}))\\
&= s(\sigma(y^{-\sim})) + s(\sigma(y^{-\sim}) \to \sigma(x^{-\sim})) = s(\sigma(y^{-\sim})) + s(\sigma(y^{-\sim} \to x^{-\sim}))\\
&= s(\sigma(y^{-\sim})) + s(\sigma((y \to x)^{-\sim})) = \tilde{s}_\sigma(y) + \tilde{s}_\sigma(y \to x).
\end{aligned}$$

Thus $\tilde{s}_\sigma$ satisfies the condition (B_1).

Similarly, $\tilde{s}_\sigma(x) + \tilde{s}_\sigma(x \rightsquigarrow y) = \tilde{s}_\sigma(y) + \tilde{s}_\sigma(y \rightsquigarrow x)$, so the condition (B_2) is also satisfied. It follows that $\tilde{s}_\sigma$ is a Bosbach state on A. $\square$

Corollary 6.5. *Let A be a bounded $R\ell$-monoid and σ be a state operator on A preserving $\to$ or $\rightsquigarrow$. If s is a Bosbach state on* $\mathrm{Reg}(A)$*, then the mapping $\tilde{s}_\sigma : A \to [0,1]$ defined by $\tilde{s}_\sigma(x) = s(\sigma(x^{-\sim}))$ is a Bosbach state on A.*

Theorem 6.6. *Let A be a good pseudo-hoop satisfying* (SO) *property, σ be a state operator and s be a Riečan state on A. Then the mapping $s_\sigma\colon A \to [0,1]$ defined by $s_\sigma(x) = s(\sigma(x))$ is a Riečan state on A.*

P r o o f. Obviously, $s_\sigma(1) = 1$. It is easy to check that $s_\sigma(x \oplus 0) = s_\sigma(x) + s_\sigma(0)$ and $s_\sigma(0 \oplus x) = s_\sigma(0) + s_\sigma(x)$. Consider $x, y \in A$ such that $x \neq 0$, $y \neq 0$ and $x \perp y$. It follows that $\sigma(x) \perp \sigma(y)$.

By (SO) property we have $x \perp_{no} y$ and applying Proposition 3.2(10) we get $\sigma(x \oplus y) = \sigma(x) \oplus \sigma(y)$. Hence:

$$s_\sigma(x \oplus y) = s(\sigma(x \oplus y)) = s(\sigma(x) \oplus \sigma(y)) = s(\sigma(x)) + s(\sigma(y)) = s_\sigma(x) + s_\sigma(y).$$

Thus s_σ is a Riečan state on A. $\square$

Theorem 6.7. *Let A be a good pseudo-hoop with* (SO) *property and τ be a state operator on A. If s is a Riečan state on* $\mathrm{Reg}(A)$*, then the mapping $\tilde{s}_\tau\colon A \to [0,1]$ defined by $\tilde{s}_\tau(x) = s(\tau(x^{-\sim}))$ is a Riečan state on A.*

P r o o f. Since s is Riečan state on $\mathrm{Reg}(A)$, according to [8: Prop. 5.13], $s(x^{-\sim}) = s(x)$ for all $x \in \mathrm{Reg}(A)$. Obviously, $\tilde{s}_\tau(1) = 1$, $\tilde{s}_\tau(x \oplus 0) = \tilde{s}_\tau(x) + \tilde{s}_\tau(0)$ and $\tilde{s}_\tau(0 \oplus x) = \tilde{s}_\tau(0) + \tilde{s}_\tau(x)$.

Consider $x, y \in A$ such that $x \neq 0$, $y \neq 0$ and $x \perp y$. It follows that $x^{-\sim} \perp y^{-\sim}$ (Proposition 2.12(6)). Hence by (SO) property, $x^{-\sim} \perp_{no} y^{-\sim}$.

Since $(x \oplus y)^{-\sim} = x^{-\sim} \oplus y^{-\sim}$ (Proposition 2.3(5)), by Proposition 3.2(10) we get:

$$\tilde{s}_\tau(x \oplus y) = s(\tau((x \oplus y)^{-\sim})) = s(\tau(x^{-\sim} \oplus y^{-\sim})) = s(\tau(x^{-\sim}) \oplus \tau(y^{-\sim}))$$
$$= s(\tau(x^{-\sim})) \oplus s(\tau(y^{-\sim})) = \tilde{s}_\tau(x) + \tilde{s}_\tau(y)$$

(since s is a Riečan state and from $x^{-\sim} \perp y^{-\sim}$ it follows that $\tau(x^{-\sim}) \perp \tau(y^{-\sim})$). Thus $\tilde{s}_\tau$ is a Riečan state on A. □

7. State operators and generalized states on pseudo-hoops

Starting from the observation that in the definition of Bosbach states there intervenes the standard MV-algebra structure of $[0, 1]$, for the case of the residuated lattices the notion of a state was generalized as a function with values in a residuated lattice ([12], [13]). Recently, this concept was extended to the case of pseudo-BCK algebras and pseudo-hoops ([14]). Properties of generalized states are useful for the development of an algebraic theory of probabilistic models for non-commutative fuzzy logics.

Let A be a bounded pseudo-hoop and $s\colon A \to A$ an arbitrary function such that $s(0) = 0$ and $s(x \vee_1 y) = s(y \vee_2 x)$ for all $x, y \in A$. Then s is said to be a *generalized Bosbach state of type I* or a *type I state* if it satisfies one of the following equivalent conditions:

(bsI_1) for all $x, y \in A$ with $x \geq y$, $s(x \to y) = s(x) \to s(y)$ and $s(x \rightsquigarrow y) = s(x) \rightsquigarrow s(y)$;

(bsI_2) for all $x, y \in A$, $s(x \vee_1 y) = s(x \to y) \rightsquigarrow s(y)$ and $s(x \vee_2 y) = s(x \rightsquigarrow y) \to s(y)$;

(bsI_3) for all $x, y \in A$, $s(x \to y) \rightsquigarrow s(y) = s(y \rightsquigarrow x) \to s(x)$ and $s(1) = 1$;

(bsI_4) for all $x, y \in A$ with $x \geq y$, $s(x) = s(x \to y) \rightsquigarrow s(y) = s(x \rightsquigarrow y) \to s(y)$;

(bsI_5) for all $x, y \in A$, $s(x \to y) = s(x \vee_1 y) \to s(y)$ and $s(x \rightsquigarrow y) = s(x \vee_2 y) \rightsquigarrow s(y)$;

(bsI_6) for all $x, y \in A$, $s(x \to y) = s(x) \to s(x \wedge y)$ and $s(x \rightsquigarrow y) = s(x) \rightsquigarrow s(x \wedge y)$.

Proposition 7.1. *Let A be a bounded pseudo-hoop and $s\colon A \to A$ an order-preserving type I state on A. Then the following hold for all $a, b \in A$:*

(1) $s(a \odot b) = s(b \to a \odot b) \odot s(b) = s(a) \odot s(a \rightsquigarrow a \odot b)$;

(2) $s(a) \odot s(b) \leq s(a \odot b)$.

P r o o f.

(1) Since $a \odot b \leq a, b$, applying (bsI_1) we have:

$$s(b \to a \odot b) \odot s(b) = (s(b) \to s(a \odot b)) \odot s(b) = s(b) \wedge s(a \odot b) = s(a \odot b) \text{ and}$$
$$s(a) \odot s(a \rightsquigarrow a \odot b) = s(a) \odot (s(a) \rightsquigarrow s(a \odot b)) = s(a) \wedge s(a \odot b) = s(a \odot b).$$

(2) By Proposition 2.2(9), $a \leq b \to a \odot b$, so $s(a) \leq s(b \to a \odot b)$. Applying (1) we get: $s(a) \odot s(b) \leq s(b \to a \odot b) \odot s(b) = s(a \odot b)$. □

Let A be a bounded pseudo-hoop and $s\colon A \to A$ an arbitrary function such that $s(0) = 0$ and $s(x \vee_1 y) = s(y \vee_2 x)$ for all $x, y \in A$. The function s is said to be a *generalized Bosbach state of type II* or a *type II state* if it satisfies one of the following equivalent conditions:

(bsII_1) for all $x, y \in A$ with $x \geq y$, $s(x \to y) = s(x) \rightsquigarrow s(y)$ and $s(x \rightsquigarrow y) = s(x) \to s(y)$;

(bsII_2) for all $x, y \in A$, $s(x \vee_1 y) = s(x \to y) \to s(y)$ and $s(x \vee_2 y) = s(x \rightsquigarrow y) \rightsquigarrow s(y)$;

(bsII_3) for all $x, y \in A$, $s(x \to y) \to s(y) = s(y \rightsquigarrow x) \rightsquigarrow s(x)$ and $s(1) = 1$;

(bsII_4) for all $x, y \in A$ with $x \geq y$, $s(x) = s(x \to y) \to s(y) = s(x \rightsquigarrow y) \rightsquigarrow s(y)$;

(bsII_5) for all $x, y \in A$, $s(x \to y) = s(x \vee_1 y) \rightsquigarrow s(y)$ and $s(x \rightsquigarrow y) = s(x \vee_2 y) \to s(y)$;

(bsII_6) for all $x, y \in A$, $s(x \to y) = s(x) \rightsquigarrow s(x \wedge y)$ and $s(x \rightsquigarrow y) = s(x) \to s(x \wedge y)$.

Let A be a bounded Wajsberg pseudo-hoop and $s\colon A \to A$ be a mapping satisfying $s(0) = 0$, $s(1) = 1$ and $s(x \vee_1 y) = s(y \vee_2 x)$. Then:

(1) s is a type I state iff

$$s(d_1(x, y)) = s(x \vee y) \to s(x \wedge y) \quad \text{and} \quad s(d_2(x, y)) = s(x \vee y) \rightsquigarrow s(x \wedge y);$$

(2) s is a type II state iff

$$s(d_1(x, y)) = s(x \vee y) \rightsquigarrow s(x \wedge y) \quad \text{and} \quad s(d_2(x, y)) = s(x \vee y) \to s(x \wedge y).$$

Let A be a bounded pseudo-hoop. An endomorphism $h\colon A \to A$ satisfying the condition $h(x \vee_1 y) = h(y \vee_2 x)$ for all $x, y \in A$ is called a *generalized state-morphism.* If, moreover, $h(x \to y) = h(x \rightsquigarrow y)$ for all $x, y \in A$, then h is a *strong* generalized state-morphism.

A mapping $m\colon A \to A$ is called *generalized Riečan state* iff the following conditions are satisfied for all $x, y \in A$:

(rs_1) $m(1) = 1$;

(rs_2) for all $x, y \in A$, if $x \perp y$, then $m(x) \perp m(y)$ and $m(x \oplus y) = m(x) \oplus m(y)$.

Proposition 7.2. *Every C-state operator on a bounded pseudo-hoop is a type I state.*

Proof. Let σ be a state operator on a bounded pseudo-hoop A. From (S_1) we have $\sigma(0) = 0$. By condition (C) and Proposition 3.7 we get $\sigma(x \vee_1 y) = \sigma(y \vee_2 x)$. Since condition (S_2) in the definition of a state operator is condition (bsI_6), it follows that σ is a type I state on A. □

Corollary 7.3. *Every perfect pseudo-hoop admits a type I state on it.*

Proof. According to Theorem 5.5 and Remark 5.6, every perfect pseudo-hoop has a C-state operator, hence by Proposition 7.2 every perfect pseudo-hoop admits a type I state on it. □

Proposition 7.4. *Let (A, σ) be an idempotent state pseudo-hoop such that $\sigma(x \vee_1 y) = \sigma(y \vee_2 x)$. Then σ is a generalized state-morphism on A.*

Proof. It follows by Propositions 4.6 and Proposition 3.4. □

Proposition 7.5. *Let (A, σ) be a linearly ordered state pseudo-hoop such that $\sigma(x \vee_1 y) = \sigma(y \vee_2 x)$. Then σ is a generalized state-morphism on A.*

Proof. It follows by Corollary 3.3 and Proposition 3.4. □

Proposition 7.6. *If (A, σ) is a good state pseudo-hoop satisfying* (SO) *property, then σ is a generalized Riečan state on A.*

Proof. From Proposition 3.2(1) we have $\sigma(1) = 1$, that is (rs_1). It is easy to check that $\sigma(x \oplus 0) = \sigma(x) \oplus \sigma(0)$ and $\sigma(0 \oplus x) = \sigma(0) \oplus \sigma(x)$.

Consider $x, y \in A$ such that $x \neq 0$, $y \neq 0$ and $x \perp y$. From (SO) property we have $x \perp_{no} y$ and applying Proposition 3.2(10), we get $\sigma(x \oplus y) = \sigma(x) \oplus \sigma(y)$, so (rs_2) is verified too. Thus σ is a generalized Riečan state on A. □

Theorem 7.7. *If A is a linearly ordered bounded pseudo-hoop and $s\colon A \to A$ is an order-preserving type I state such that $s^2(x) = s(x) \leq x$ for all $x \in A$, then s is a state operator on A.*

Proof. Applying the hypothesis and the definition of a type I state, we will check the axioms (S_1)–(S_5) from the definition of a state operator.

(S_1) $s(0) = 0$: It follows from the definition of a type I state.

(S_2) $s(a \to b) = s(a) \to s(a \wedge b)$ and $s(a \rightsquigarrow b) = s(a) \rightsquigarrow s(a \wedge b)$: It is the condition (bsI_6).

(S_3) $s(a \odot b) = s(a) \odot s(a \rightsquigarrow a \odot b) = s(b \to a \odot b) \odot s(b)$:
It follows from Proposition 7.1(1).

(S_4) $s(s(a) \odot s(b)) = s(a) \odot s(b)$: Since $s(x) \leq x$ for all $x \in A$ we have $s(s(a) \odot s(b)) \leq s(a) \odot s(b)$.

On the other hand, from Proposition 7.1(2), replacing a with $s(a)$ and b with $s(b)$ we get $s^2(a) \odot s^2(b) \leq s(s(a) \odot s(b))$, that is $s(a) \odot s(b) \leq s(s(a) \odot s(b))$. Thus $s(s(a) \odot s(b)) = s(a) \odot s(b)$.

(S_5) $s(s(a) \to s(b)) = s(a) \to s(b)$ and $s(s(a) \rightsquigarrow s(b)) = s(a) \rightsquigarrow s(b)$: Since A is linearly ordered we consider the cases:

(a) $b \leq a$, so $s(b) \leq s(a)$. According to condition (bsI_1) we get

$$s(s(a) \to s(b)) = s^2(a) \to s^2(b) = s(a) \to s(b).$$

(b) $a \leq b$, so $s(a) \leq s(b)$. It follows that $s(a) \to s(b) = 1$, thus

$$s(s(a) \to s(b)) = s(a) \to s(b) = s(1) = 1.$$

Similarly, $s(s(a) \rightsquigarrow s(b)) = s(a) \rightsquigarrow s(b)$. We conclude that s is a state operator on A. □

Theorem 7.8. *If A is a linearly ordered bounded pseudo-hoop and $s\colon A \to A$ is an order-preserving type I state such that $s^2 = s$ and $s(x \odot y) = s(x) \odot s(y)$ for all $x, y \in A$, then s is a weak state-morphism operator on A.*

Proof. The axioms (S_1), (S_2), (S_3) and (S_5) are verified in a similar way as in the case of Theorem 7.7.

For axiom (S_4) we have: $s(s(a) \odot s(b)) = s^2(a) \odot s^2(b) = s(a) \odot s(b)$. Since $s(x \odot y) = s(x) \odot s(y)$ for all $x, y \in A$, it follows that s is a weak state-morphism operator on A. □

Theorem 7.9. *Let σ be a C-state operator on the bounded pseudo-hoop A preserving $\to$ or $\rightsquigarrow$. If $s\colon A \to A$ is a type I (type II) state on A, then $s_\sigma\colon A \to A$ defined by $s_\sigma(x) = s(\sigma(x))$ is a type I (type II) state on A.*

Proof. Obviously, $s_\sigma(0) = s(\sigma(0)) = s(0) = 0$. We remark again that, if σ preserves one of the arrows $\to$, $\rightsquigarrow$, then by Proposition 3.2(12) the second one is also preserved.

Since σ is a C-state operator on A, applying Corollary 3.8 we have:

$$s_\sigma(x \vee_1 y) = s(\sigma(x \vee_1 y)) = s(\sigma(x \vee_2 y)) = s_\sigma(x \vee_2 y).$$

On the other hand, from $\sigma(x \vee_2 y) = \sigma(y \vee_2 x)$, we get $s_\sigma(x \vee_2 y) = s_\sigma(y \vee_2 x)$. Hence $s_\sigma(x \vee_1 y) = s_\sigma(y \vee_2 x)$. Let s be a type I state on A, so it satisfies (bsI_1).

Consider $y \leq x$. It follows that $\sigma(y) \leq \sigma(x)$ and taking into consideration that σ preserves $\to$, we get:

$$\begin{aligned} s_\sigma(x \to y) &= s(\sigma(x \to y)) = s(\sigma(x) \to \sigma(y)) \\ &= s(\sigma(x)) \to s(\sigma(y)) = s_\sigma(x) \to s_\sigma(y). \end{aligned}$$

Similarly, $s_\sigma(x \rightsquigarrow y) = s_\sigma(x) \rightsquigarrow s_\sigma(y)$ for all $x, y \in A$. Hence s_σ satisfies (bsI_1), thus it is a type I state on A. Consider s to be a type II state on A, so it satisfies (bsII_1). Assume $y \leq x$, so that $\sigma(y) \leq \sigma(x)$. Since σ preserves $\to$, we get:

$$\begin{aligned} s_\sigma(x \to y) &= s(\sigma(x \to y)) = s(\sigma(x) \to \sigma(y)) \\ &= s(\sigma(x)) \rightsquigarrow s(\sigma(y)) = s_\sigma(x) \rightsquigarrow s_\sigma(y). \end{aligned}$$

Similarly, $s_\sigma(x \rightsquigarrow y) = s_\sigma(x) \to s_\sigma(y)$ for all $x, y \in A$. Thus s_σ satisfies (bsII_1), hence it is a type II state on A. □

Corollary 7.10. *Let (A, σ) be a linearly ordered C-state pseudo-hoop and s be a type I (type II) state on A. Then the mapping $s_\sigma \colon A \to A$ defined by $s_\sigma(x) = s(\sigma(x))$ is a type I (type II) state on A.*

Proof. According to Corollary 3.3, σ preserves $\to$ and $\rightsquigarrow$, hence by Theorem 7.9, s_σ is a type I (type II) state on A. □

Corollary 7.11. *Let (A, σ) be an idempotent C-state pseudo-hoop and s be a Bosbach state on A. Then the mapping $s_\sigma \colon A \to A$ defined by $s_\sigma(x) = s(\sigma(x))$ is a type I (type II) state on A.*

Proof. By Proposition 4.6 we have $\sigma(x \wedge y) = \sigma(x) \wedge \sigma(y)$. According to Proposition 3.4, σ preserves $\to$ and $\rightsquigarrow$, hence by Theorem 7.9, s_σ is a type I (type II) state on A. □

Remark 7.12. The state operator σ from Corollaries 7.10 and 7.11 is an endomorphism satisfying condition (C). Moreover, $\sigma(A)$ is a Wajsberg sub-pseudo-hoop of A.

8. Concluding remarks

We suggest further directions of research, as the above topics are of current interest.

1. The state operators investigated in this paper can be extended to other non-commutative structures such as pseudo-BCK algebras.

2. A lot of work has been done regarding the relationship between the existence of states and the existence of maximal normal filters of non-commutative fuzzy structures. For the case of state Rℓ-monoids (M, σ) the notion of σ-filter was introduced in [28]. One can try to investigate the correspondence between the existence of state operators and the maximal and normal σ-filters on state Rℓ-monoids and state pseudo-hoops.

3. Classes of state-morphism MV-algebras and varieties of MV-algebras with internal states have been studied by Di Nola and Dvurečenskij in [17] and respectively [18]. One can try to approach these topics for the case of state operators on pseudo-hoops and bounded Rℓ-monoids.

4. Dvurečenskij has investigated in [22] subdirectly irreductible state-morphism BL-algebras. A further research topic could be to investigate similar results for the case of state-morphism Rℓ-monoids and state-morphism pseudo-hoops.

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