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FUZZY FILTERS OF BE-ALGEBRAS

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ABSTRACT. Characterizations of fuzzy filters in a BE-algebra are established. Conditions for a fuzzy set to be a fuzzy filter are given. For a fuzzy set μ the least fuzzy filter containing μ is constructed. The homomorphic properties of fuzzy filters of a BE-algebra are provided. Finally, characterizations of Noetherian BE-algebras and Artinian BE-algebras via fuzzy filters are obtained.

1. Introduction

In 1966, Y. Imai and K. Iséki [3] introduced the notion of a BCK-algebra. There exist several generalizations of BCK-algebras, such as BCI-algebras [4], BCH-algebras [2], BCC-algebras [8], BH-algebras [5], d-algebras [10], etc. In [6], H. S. Kim and Y. H. Kim introduced the notion of a BE-algebra as a dualization of a generalization of a BCK-algebra. They defined and studied the concept of a filter in BE-algebras. This concept was also investigated in [9] and [7]. Ideals and fuzzy ideals in BE-algebras were considered in [1] and [11].

In this paper, we develop the theory of fuzzy filters in BE-algebras. In Section 3 we give characterizations of fuzzy filters and provide conditions for a fuzzy set to be a fuzzy filter. Given a fuzzy set μ, we make the least fuzzy filter containing μ. This leads us to showing that the set of fuzzy filters of a BE-algebra is a complete lattice. The homomorphic properties of fuzzy filters are provided. Finally, characterizations of Noetherian BE-algebras and Artinian BE-algebras in terms of fuzzy filters are given in Section 4. For the convenience of the reader, in Section 2 we give the necessary material needed in sequel, thus making our exposition self-contained. In the sequel, let $\mathbb{N}$ denote the set of all positive integers.

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2. Preliminaries

An algebra $(A;*,1)$ of type $(2,0)$ is called a *BE-algebra* ([6]) if for all $x,y,z \in A$ the following identities hold:

(BE1) $x * x = 1$,

(BE2) $x * 1 = 1$,

(BE3) $1 * x = x$,

(BE4) $x * (y * z) = y * (x * z)$.

We introduce a binary relation $\leq$ on A by $x \leq y$ if and only if $x * y = 1$. It is easy to see that for any $x \in A$,

$$1 \leq x \implies x = 1. \tag{1}$$

A BE-algebra A is said to be *transitive* ([1]) if for all $x,y,z \in A$,

$$y * z \leq (x * y) * (x * z).$$

Proposition 2.1. ([9]) *If a BE-algebra A is transitive, then for all $x,y,z \in A$, $y \leq z$ implies $x * y \leq x * z$.*

Following [6], a *filter* of a BE-algebra A is a subset F of A such that for all $x,y \in A$:

(F1) $1 \in F$,

(F2) if $x * y \in F$ and $x \in F$, then $y \in F$.

We will denote by $\mathrm{Fil}(A)$ the set of all filters in a BE-algebra A. It is easy to see that $\{1\}, A \in \mathrm{Fil}(A)$.

Example 2.2. ([7]) Let $A = \{1,a,b,c\}$ and define binary operation $*$ on A by the following Cayley table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	b
b	1	a	1	a
c	1	1	1	1

Then $(A;*,1)$ is a BE-algebra and $\mathrm{Fil}(A) = \big\{\{1\},\{1,a\},\{1,b\},A\big\}$.

Remark 2.3. It is easy to prove that the intersection of an arbitrary number of filters of a BE-algebra A is a filter of A. It is also not hard to show that the union of an ascending sequence of filters of A is a filter of A.

Lemma 2.4. *Let A be a transitive BE-algebra. If*

$$\begin{aligned} a*(b*x) &= 1,\\ a_n*(\cdots*(a_2*(a_1*a))\dots) &= 1,\\ b_m*(\cdots*(b_2*(b_1*b))\dots) &= 1 \end{aligned}$$

in A, then

$$b_m*(\cdots*(b_1*(a_n*(\cdots*(a_1*x)\dots)))\dots)=1.$$

Proof. Since $a*(b*x)=1$, we have $a\le b*x$. Then, by Proposition 2.1, $a_1*a\le a_1*(b*x)$ and hence again by Proposition 2.1, we obtain $a_2*(a_1*a)\le a_2*(a_1*(b*x))$. Repeating the process we have

$$a_n*(\cdots*(a_2*(a_1*a))\dots)\le a_n*(\cdots*(a_2*(a_1*(b*x)))\dots).$$

Thus, by assumption and (1), $a_n*(\cdots*(a_2*(a_1*(b*x)))\dots)=1$. Applying (BE4) we deduce that $b*(a_n*(\cdots*(a_2*(a_1*x))\dots)=1$, that is, $b\le a_n*(\cdots*(a_2*(a_1*x))\dots)$. Hence, by Proposition 2.1,

$$b_m*(\cdots*(b_1*b))\dots)\le b_m*(\cdots*(b_1*(a_n*(\cdots*(a_2*(a_1*x))\dots)))\dots).$$

Therefore, $b_m*(\cdots*(b_1*(a_n*(\cdots*(a_2*(a_1*x))\dots)))\dots)=1$. □

We now review some fuzzy logic concepts. First, for $\Gamma\subseteq[0;1]$ we define $\bigwedge\Gamma=\inf\Gamma$ and $\bigvee\Gamma=\sup\Gamma$. Obviously, if $\Gamma=\{\alpha,\beta\}$, then $\alpha\wedge\beta=\min\{\alpha,\beta\}$ and $\alpha\vee\beta=\max\{\alpha,\beta\}$. Recall that a *fuzzy set* in A is a function $\mu\colon A\to[0;1]$.

For any fuzzy sets μ and ν in A, we define

$$\mu\le\nu \iff \big[(\forall x\in A)\big(\mu(x)\le\nu(x)\big)\big].$$

It is easy to check that this relation is an order relation in the set of fuzzy sets in A.

Let A and B be any two sets, μ be any fuzzy set in A and $f\colon A\to B$ be any function. Set $f^{\leftarrow}(y)=\big\{x\in A:\ f(x)=y\big\}$ for $y\in B$. The fuzzy set ν in B defined by

$$\nu(y)=\begin{cases}\bigvee\big\{\mu(x):\ x\in f^{\leftarrow}(y)\big\} & \text{if } f^{\leftarrow}(y)\ne\emptyset,\\ 0 & \text{otherwise}\end{cases}$$

for all $y\in B$, is called the *image* of μ under f and is denoted by $f(\mu)$.

Also, for A and B being any two sets, $f\colon A\to B$ being any function and ν being any fuzzy set in $f(A)$, the fuzzy set μ in A defined by

$$\mu(x)=\nu(f(x))\qquad\text{for all}\quad x\in A$$

is called the *preimage* of ν under f and is denoted by $f^{\leftarrow}(\nu)$.

3. Fuzzy filters

We give the definition of a fuzzy filter in a BE-algebra.

A fuzzy set μ in a BE-algebra A is called a *fuzzy filter* of A if it satisfies for all $x, y \in A$:

(d1) $\mu(1) \geq \mu(x)$,

(d2) $\mu(x) \geq \mu(y * x) \wedge \mu(y)$.

Proposition 3.1. *Let μ be a fuzzy filter of a BE-algebra A. Then, for any $x, y \in A$, if $x \leq y$, then $\mu(x) \leq \mu(y)$.*

Proof. If $x \leq y$, then $x*y = 1$. Hence, by (d2), we have $\mu(y) \geq \mu(x*y) \wedge \mu(x) = \mu(1) \wedge \mu(x) = \mu(x)$. □

Denote by $\mathcal{F}\mathrm{Fil}(A)$ the set of all fuzzy filters of a BE-algebra A.

Example 3.2. Let A be the BE-algebra from Example 2.2. Let $0 \leq \alpha_3 < \alpha_2 < \alpha_1 \leq 1$. Define a fuzzy set μ in A by

$$\mu(x) = \begin{cases} \alpha_1 & \text{if } x = 1, \\ \alpha_2 & \text{if } x = a, \\ \alpha_3 & \text{if } x \in \{b, c\}. \end{cases}$$

It is easily checked that μ satisfies (d1) and (d2). Thus $\mu \in \mathcal{F}\mathrm{Fil}(A)$.

Example 3.3. Let F be a filter of a BE-algebra A and let $\alpha, \beta \in [0; 1]$, with $\alpha \geq \beta$. Define $\mu_F^{\alpha,\beta}$ as follows:

$$\mu_F^{\alpha,\beta}(x) = \begin{cases} \alpha & \text{if } x \in F, \\ \beta & \text{otherwise.} \end{cases}$$

We denote $\mu_F^{\alpha,\beta} = \mu$. Since $1 \in F$, $\mu(1) = \alpha \geq \mu(x)$ for all $x \in A$. To prove (d2), let $x, y \in A$. If $x \in F$, then $\mu(x) = \alpha \geq \mu(y * x) \wedge \mu(y)$. Suppose now that $x \notin F$. By the definition of a filter, $y * x \notin F$ or $y \notin F$. Therefore, $\mu(y * x) \wedge \mu(y) = \beta = \mu(x)$. Thus μ is a fuzzy filter of A.

In particular, the characteristic function χ_F of F:

$$\chi_F(x) = \begin{cases} 1 & \text{if } x \in F, \\ 0 & \text{otherwise} \end{cases}$$

is a fuzzy filter of A.

Proposition 3.4. *A fuzzy set μ in a BE-algebra A is a fuzzy filter of A if and only if it satisfies* (d1) *and*

(d3) *for all $x, y, z \in A$, if $x * (y * z) = 1$, then $\mu(z) \geq \mu(x) \wedge \mu(y)$.*

Proof. Let $\mu \in \mathcal{F}\mathrm{Fil}(A)$ and let $x, y, z \in A$. Suppose that $x * (y * z) = 1$. Since μ is a fuzzy filter, we have $\mu(y * z) \geq \mu(x * (y * z)) \wedge \mu(x) = \mu(1) \wedge \mu(x) = \mu(x)$ and $\mu(z) \geq \mu(y * z) \wedge \mu(y)$. Therefore, $\mu(z) \geq \mu(x) \wedge \mu(y)$.

Conversely, let μ satisfy (d3). From (BE1) we have $(y * x) * (y * x) = 1$. By (d3), $\mu(x) \geq \mu(y * x) \wedge \mu(y)$. Then μ satisfies (d2) and hence $\mu \in \mathcal{F}\mathrm{Fil}(A)$. □

It is easy to prove by induction the following.

Proposition 3.5. *Let μ be a fuzzy set satisfying* (d1) *in a BE-algebra A. Then μ is a fuzzy filter if and only if for any $a_1, \dots, a_n \in A$ $(n \geq 2)$,*

$$a_n * (\cdots * (a_1 * x) \dots) = 1 \implies \mu(x) \geq \mu(a_1) \wedge \dots \wedge \mu(a_n).$$

Theorem 3.6. *Let μ be a fuzzy set in a BE-algebra A. Then $\mu \in \mathcal{F}\mathrm{Fil}(A)$ if and only if its nonempty level subset*

$$U(\mu; \alpha) = \big\{x \in A : \ \mu(x) \geq \alpha\big\}$$

is a filter of A for all $\alpha \in [0; 1]$.

Proof. Assume that $\mu \in \mathcal{F}\mathrm{Fil}(A)$ and let $\alpha \in [0; 1]$ be such that $U(\mu; \alpha) \neq \emptyset$. Then $\mu(x_0) \geq \alpha$ for some $x_0 \in A$. Since $\mu(1) \geq \mu(x_0)$, we have $1 \in U(\mu; \alpha)$. Let $x, y \in A$ be such that $x * y, x \in U(\mu; \alpha)$. Then $\mu(x * y) \geq \alpha$ and $\mu(x) \geq \alpha$. It follows from (d2) that

$$\mu(y) \geq \mu(x * y) \wedge \mu(x) \geq \alpha$$

so that $y \in U(\mu; \alpha)$. Therefore $U(\mu; \alpha)$ is a filter of A.

Conversely, suppose that for each $\alpha \in [0; 1]$, $U(\mu; \alpha) = \emptyset$ or $U(\mu; \alpha)$ is a filter of A. If (d1) is not valid, then there exists $x_0 \in A$ such that $\mu(1) < \mu(x_0) = \beta$. Then $U(\mu; \beta) \neq \emptyset$ and by assumption, $U(\mu; \beta)$ is a filter of A. Hence $1 \in U(\mu; \beta)$ and consequently, $\mu(1) \geq \beta$. This is a contradiction and (d1) is valid. Now assume that (d2) does not hold. Then there are $a, b \in A$ such that $\mu(a) < \mu(b * a) \wedge \mu(b)$. Taking

$$\beta = \frac{1}{2}(\mu(a) + \mu(b * a) \wedge \mu(b)),$$

we get $\mu(a) < \beta < \mu(b * a) \wedge \mu(b) \leq \mu(b * a)$ and $\beta < \mu(b)$. Therefore $b * a, b \in U(\mu; \beta)$ but $a \notin U(\mu; \beta)$. This is impossible, and μ is a fuzzy filter of A. □

Corollary 3.7. *If μ is a fuzzy filter of a BE-algebra A, then the set*

$$A_b = \big\{x \in A : \ \mu(x) \geq \mu(b)\big\}$$

is a filter of A for every $b \in A$.

By Corollary 3.7, we have the following.

Corollary 3.8. *If μ is a fuzzy filter of a BE-algebra A, then the set*

$$A_\mu = \{x \in A : \ \mu(x) = \mu(1)\}$$

is a filter of A.

The following example shows that the converse of Corollary 3.8 does not hold.

Example 3.9. Let A be a BE-algebra. Define a fuzzy set μ in A by

$$\mu(x) = \begin{cases} 0.4 & \text{if } x = 1, \\ 0.6 & \text{if } x \neq 1. \end{cases}$$

Then $A_\mu = \{1\}$ is the filter of A but $\mu \notin \mathcal{F}\mathrm{Fil}(A)$.

Example 3.10. Let μ be as in Example 3.2. One can easily check that for all $\alpha \in [0;1]$ we have

$$U(\mu;\alpha) = \begin{cases} \emptyset & \text{if } \alpha > \alpha_1, \\ \{1\} & \text{if } \alpha_2 < \alpha \leq \alpha_1, \\ \{1,a\} & \text{if } \alpha_3 < \alpha \leq \alpha_2, \\ A & \text{if } \alpha \leq \alpha_3. \end{cases}$$

Since $\{1\}, \{1,a\}$ and A are filters of A, this is an another proof (by Theorem 3.6) that μ is a fuzzy filter of A.

Lemma 3.11. *Let $F_1 \subset F_2 \subset \cdots \subset F_n \subset \ldots$ be a strictly ascending sequence of filters in a BE-algebra A and (t_n) be a strictly decreasing sequence in $(0;1)$. Let μ be the fuzzy set in A defined by*

$$\mu(x) = \begin{cases} 0 & \text{if } x \notin F_n \text{ for each } n \in \mathbb{N}, \\ t_n & \text{if } x \in F_n - F_{n-1} \text{ for } n = 1,2,\ldots, \end{cases}$$

where $F_0 = \emptyset$. Then μ is a fuzzy filter of A.

Proof. Let $F = \bigcup_{n\in\mathbb{N}} F_n$. By Remark 2.3, F is a filter of A. Obviously, $\mu(1) = t_1 \geq \mu(x)$ for all $x \in A$, that is, (d1) holds. Now we show that μ satisfies (d2). Let $x, y \in A$. We have two cases.

Case 1: $x \notin F$.
Then $y * x \notin F$ or $y \notin F$. Therefore $\mu(y * x) \wedge \mu(y) = 0 = \mu(x)$.

Case 2: $x \in F_n - F_{n-1}$ for some $n = 1,2,\ldots$.
Then $y * x \notin F_{n-1}$ or $y \notin F_{n-1}$. Hence $\mu(y * x) \leq t_n$ or $\mu(y) \leq t_n$. Therefore $\mu(y * x) \wedge \mu(y) \leq t_n = \mu(x)$.

Thus (d2) is also satisfied and consequently, μ is a fuzzy filter of A. □

Let $\mu_t \in \mathcal{F}\mathrm{Fil}(A)$ for $t \in T$. The meet $\bigwedge_{t\in T} \mu_t$ of fuzzy filters μ_t of A is defined as follows:

$$\Big(\bigwedge_{t\in T} \mu_t\Big)(x) = \bigwedge\{\mu_t(x) : \ t \in T\}.$$

Theorem 3.12. *Let $\mu_t \in \mathcal{F}\mathrm{Fil}(A)$ for $t \in T$. Then $\bigwedge_{t\in T} \mu_t \in \mathcal{F}\mathrm{Fil}(A)$.*

Proof. Let $\mu = \bigwedge_{t\in T} \mu_t$. Then, by (d1),

$$\mu(1) = \bigwedge\{\mu_t(1) :\ t \in T\} \geq \bigwedge\{\mu_t(x) :\ t \in T\} = \mu(x)$$

for all $x \in A$. Let $x, y \in A$. Since $\mu_t \in \mathcal{F}\mathrm{Fil}(A)$, we have $\mu_t(x) \geq \mu_t(y*x) \wedge \mu_t(y)$. Hence

$$\begin{aligned}\bigwedge\{\mu_t(x) :\ t \in T\} &\geq \bigwedge\{\mu_t(y * x) \wedge \mu_t(y) :\ t \in T\}\\ &= \bigwedge\{\mu_t(y * x) :\ t \in T\} \wedge \bigwedge\{\mu_t(y) :\ t \in T\}.\end{aligned}$$

Consequently, $\mu(x) \geq \mu(y * x) \wedge \mu(y)$ and therefore, $\mu \in \mathcal{F}\mathrm{Fil}(A)$. □

Let f be a fuzzy set in A. A fuzzy filter μ of A is said to be *generated* by f if $f \leq \mu$ and for any fuzzy filter ν of A, $f \leq \nu$ implies $\mu \leq \nu$. The fuzzy filter generated by f will be denoted by $[f)$. The fuzzy filter $[f)$ we can define equivalently as follows:

$$[f) = \bigwedge\{\nu \in \mathcal{F}\mathrm{Fil}(A) :\ \nu \geq f\}.$$

We have simple theorem.

Theorem 3.13. *Let f and g be fuzzy sets in A. The following properties hold:*

(a) *$f \leq g$ implies $[f) \leq [g)$,*

(b) *if $f \in \mathcal{F}\mathrm{Fil}(A)$, then $[f) = f$.*

Theorem 3.14. *Let f be a fuzzy set in a transitive BE-algebra A and let μ be a fuzzy set in A defined for all $x \in A$ by*

$$\mu(x) = \bigvee\{f(a_1) \wedge \cdots \wedge f(a_n) :\ a_n * (\cdots * (a_1 * x)\dots) = 1 \text{ and } a_1, \dots, a_n \in A\}.$$

Then $\mu = [f)$.

Proof. It is easy to see that $\mu(1) \geq \mu(x)$ for all $x \in A$. Now we prove that μ satisfies (d3). Suppose that $a * (b * x) = 1$, where $x, a, b \in A$. Let $k \in \mathbb{N}$. By the definition of μ, we can select $a_1, \dots, a_n, b_1, \dots, b_m \in A$ such that

$$\begin{aligned} a_n * (\cdots * (a_2 * (a_1 * a))\dots) &= 1,\\ b_m * (\cdots * (b_2 * (b_1 * b))\dots) &= 1,\\ f(a_1) \wedge \cdots \wedge f(a_n) &> \mu(a) - \frac{1}{k},\\ f(b_1) \wedge \cdots \wedge f(b_m) &> \mu(b) - \frac{1}{k}.\end{aligned}$$

From Lemma 2.4 it follows that $b_m * (\cdots * (b_1 * (a_n * (\cdots * (a_1 * x) \dots))) \dots) = 1$. Then

$$\begin{aligned}\mu(x) &\geq f(a_1) \wedge \cdots \wedge f(a_n) \wedge f(b_1) \wedge \cdots \wedge f(b_m)\\ &> \left(\mu(a) - \frac{1}{k}\right) \wedge \left(\mu(b) - \frac{1}{k}\right).\end{aligned}$$

Hence $\mu(x) \geq \mu(a) \wedge \mu(b)$ and by Proposition 3.4, $\mu \in \mathcal{F}\mathrm{Fil}(A)$.

Since $x * x = 1$, we see that $f(x) \leq \mu(x)$. Thus $f \leq \mu$. Finally, suppose ν is a fuzzy filter in A such that $f \leq \nu$. Then for any $x \in A$ we obtain

$$\begin{aligned}\mu(x) &= \bigvee\{f(a_1) \wedge \cdots \wedge f(a_n) :\ a_n * (\cdots * (a_1 * x) \dots) = 1\\ &\qquad\qquad \text{and } a_1, \dots, a_n \in A\}\\ &\leq \bigvee\{\nu(a_1) \wedge \cdots \wedge \nu(a_n) :\ a_n * (\cdots * (a_1 * x) \dots) = 1\\ &\qquad\qquad \text{and } a_1, \dots, a_n \in A\}\end{aligned}$$

and by Proposition 3.5,

$$\bigvee\{\nu(a_1) \wedge \cdots \wedge \nu(a_n) :\ a_n * (\cdots * (a_1 * x) \dots) = 1 \text{ and } a_1, \dots, a_n \in A\} \leq \nu(x).$$

Therefore $\mu(x) \leq \nu(x)$ for all $x \in A$. Consequently, $\mu \leq \nu$. Thus μ is the fuzzy filter generated by f, that is, $\mu = [f)$. □

Example 3.15. Let A be the BE-algebra from Example 2.2. It is easy to see that A is transitive. Define a fuzzy set f in A by

$$f(x) = \begin{cases} 0.7 & \text{if } x = 1,\\ 0.3 & \text{if } x \in \{a, b\},\\ 0 & \text{if } x = c.\end{cases}$$

Then the fuzzy filter $\mu = [f)$ generated by f is as follows:

$$\mu(x) = \begin{cases} 0.7 & \text{if } x = 1,\\ 0.3 & \text{if } x \in \{a, b, c\}.\end{cases}$$

For $\mu, \nu \in \mathcal{F}\mathrm{Fil}(A)$ let $\mu \vee \nu$ denote the join of μ and ν, that is, $\mu \vee \nu = [f)$, where f is the fuzzy set in A defined by $f(x) = \mu(x) \vee \nu(x)$ for all $x \in A$.

From Theorem 3.12 we obtain

Theorem 3.16. *Let A be a BE-algebra. Then $(\mathcal{F}\mathrm{Fil}(A); \wedge, \vee)$ is a complete lattice.*

The following two theorems give the homomorphic properties of fuzzy filters.

Theorem 3.17. *Let A and B be BE-algebras and let $f\colon A \to B$ be a homomorphism and $\nu \in \mathcal{F}\mathrm{Fil}(B)$. Then $f^{\leftarrow}(\nu) \in \mathcal{F}\mathrm{Fil}(A)$.*

Proof. Let $x \in A$. Since $f(x) \in B$ and $\nu \in \mathcal{F}\,\mathrm{Fil}(B)$, we have $\nu(1) \geq \nu(f(x)) = (f^{\leftarrow}(\nu))(x)$, but $\nu(1) = \nu(f(1)) = (f^{\leftarrow}(\nu))(1)$. Thus we get $(f^{\leftarrow}(\nu))(1) \geq (f^{\leftarrow}(\nu))(x)$ for any $x \in A$, that is, $f^{\leftarrow}(\nu)$ satisfies (d1).

Now let $x, y \in A$. Since $\nu \in \mathcal{F}\mathrm{Fil}(B)$, we have

$$\begin{aligned}\nu(f(x)) &\geq \nu(f(y) * f(x)) \wedge \nu(f(y)) \\ &= \nu(f(y * x)) \wedge \nu(f(y))\end{aligned}$$

and hence $f^{\leftarrow}(\nu)(x) \geq f^{\leftarrow}(\nu)(y * x) \wedge f^{\leftarrow}(\nu)(y)$. Consequently, $f^{\leftarrow}(\nu) \in \mathcal{F}\mathrm{Fil}(A)$. □

Lemma 3.18. *Let A and B be BE-algebras and let $f \colon A \to B$ be a homomorphism and $\mu \in \mathcal{F}\mathrm{Fil}(A)$. Then, if μ is constant on $\ker(f) = f^{\leftarrow}(1)$, then $f^{\leftarrow}(f(\mu)) = \mu$.*

Proof. Let $x \in A$ and $f(x) = y$. Hence

$$\begin{aligned}(f^{\leftarrow}(f(\mu)))(x) &= (f(\mu))(f(x)) = (f(\mu))(y) \\ &= \bigvee\{\mu(a) :\ a \in f^{\leftarrow}(y)\}.\end{aligned}$$

For all $a \in f^{\leftarrow}(y)$, we have $f(x) = f(a)$. Hence $f(x * a) = 1$, i.e., $x * a \in \ker(f)$. Thus $\mu(x * a) = \mu(1)$. Therefore, $\mu(a) \geq \mu(x * a) \wedge \mu(x) = \mu(1) \wedge \mu(x) = \mu(x)$. Similarly, $\mu(x) \geq \mu(a)$. Hence $\mu(x) = \mu(a)$. Thus

$$(f^{\leftarrow}(f(\mu)))(x) = \bigvee\{\mu(a) :\ a \in f^{\leftarrow}(y)\} = \mu(x),$$

i.e., $f^{\leftarrow}(f(\mu)) = \mu$. □

Theorem 3.19. *Let A and B be BE-algebras and let $f \colon A \to B$ be a surjective homomorphism and $\mu \in \mathcal{F}\mathrm{Fil}(A)$ be such that $A_\mu \supseteq \ker(f)$. Then $f(\mu) \in \mathcal{F}\mathrm{Fil}(B)$.*

Proof. Since μ is a fuzzy filter of A and $1 \in f^{\leftarrow}(1)$, we have

$$(f(\mu))(1) = \bigvee\{\mu(a) :\ a \in f^{\leftarrow}(1)\} = \mu(1) \geq \mu(x)$$

for any $x \in A$. Hence,

$$(f(\mu))(1) \geq \bigvee\{\mu(x) :\ x \in f^{\leftarrow}(y)\} = (f(\mu))(y)$$

for any $y \in B$. Thus, $f(\mu)$ satisfies (d1). Suppose that

$$f(\mu)(x_B) < f(\mu)(y_B * x_B) \wedge f(\mu)(y_B)$$

for some $x_B, y_B \in B$. Since f is surjective, there are $x_A, y_A \in A$ such that $f(x_A) = x_B$ and $f(y_A) = y_B$. Hence,

$$f(\mu)(f(x_A)) < f(\mu)(f(y_A * x_A)) \wedge f(\mu)(f(y_A)).$$

Therefore,

$$f^{\leftarrow}(f(\mu))(x_A) < f^{\leftarrow}(f(\mu))(y_A * x_A)) \wedge f^{\leftarrow}(f(\mu))(y_A)).$$

Since $A_\mu \supseteq \ker(f)$, μ is constant on $\ker(f)$. Hence, by Lemma 3.18, we get

$$\mu(x_A) < \mu(y_A * x_A) \wedge \mu(y_A),$$

which is a contradiction with the fact that μ is a fuzzy filter. Thus $f(\mu) \in \mathcal{F}\mathrm{Fil}(B)$. □

4. Fuzzy characterizations of Noetherian and Artinian BE-algebras

In this section we characterize Noetherian BE-algebras and Artinian BE-algebras using some fuzzy concepts, in particular, fuzzy filters. In the beginning we recall some definitions.

A BE-algebra A is called *Noetherian* ([9]) if for every ascending sequence $F_1 \subseteq F_2 \subseteq \ldots$ of filters of A there exists $k \in \mathbb{N}$ such that $F_n = F_k$ for all $n \geq k$. A BE-algebra A is called *Artinian* ([9]) if for every descending sequence $F_1 \supseteq F_2 \supseteq \ldots$ of filters of A there exists $k \in \mathbb{N}$ such that $F_n = F_k$ for all $n \geq k$.

Theorem 4.1. *Let A be a BE-algebra. The following statements are equivalent:*

(a) *A is Noetherian,*

(b) *for each fuzzy filter μ of A, $\mathrm{Im}(\mu) = \{\mu(x) : x \in A\}$ is a well-ordered set.*

Proof.

$(a) \implies (b)$: Assume that A is Noetherian and μ is a fuzzy filter of A such that $\mathrm{Im}(\mu)$ is not a well-ordered subset of $[0;1]$. Then there exists a strictly decreasing sequence $(\mu(x_n))$, where $x_n \in A$. Let $t_n = \mu(x_n)$ and $U_n = U(\mu; t_n) = \{x \in A : \mu(x) \geq t_n\}$. Then, by Theorem 3.6, U_n is a filter of A for every $n \in \mathbb{N}$. So $U_1 \subset U_2 \subset \ldots$ is a strictly ascending sequence of filters of A. This contradicts the assumption that A is Noetherian. Therefore $\mathrm{Im}(\mu)$ is a well-ordered set for each fuzzy filter μ of A.

$(b) \implies (a)$: Assume that (b) is true. Suppose that A is not Noetherian. Then there exists a strictly ascending sequence $F_1 \subset F_2 \subset \cdots \subset F_n \subset \ldots$ of filters of A. Let μ be a fuzzy set in A such that

$$\mu(x) = \begin{cases} 0 & \text{if } x \notin F_n \text{ for each } n \in \mathbb{N}, \\ \frac{1}{n} & \text{if } x \in F_n - F_{n-1} \text{ for } n = 1, 2, \ldots, \end{cases}$$

where $F_0 = \emptyset$. By Lemma 3.11, $\mu \in \mathcal{F}\mathrm{Fil}(A)$, but $\mathrm{Im}(\mu)$ is not a well-ordered set, which is a contradiction. Therefore A is Noetherian and the proof is complete. □

Corollary 4.2. *Let A be a BE-algebra. If for every fuzzy filter μ of A, $\mathrm{Im}(\mu)$ is a finite set, then A is Noetherian.*

Theorem 4.3. *Let A be a BE-algebra and let $T = \{t_1, t_2, \ldots\} \cup \{0\}$ where (t_n) is a strictly decreasing sequence in $(0;1)$. Then the following conditions are equivalent:*

(a) *A is Noetherian,*

(b) *for each fuzzy filter μ of A, if $\mathrm{Im}(\mu) \subseteq T$, then there exists $k \in \mathbb{N}$ such that $\mathrm{Im}(\mu) \subseteq \{t_1, t_2, \ldots, t_k\} \cup \{0\}$.*

Proof.

$(a) \implies (b)$: Assume that A is Noetherian. Let μ be a fuzzy filter of A such that $\mathrm{Im}\,(\mu) \subseteq T$. From Theorem 4.1 we know that $\mathrm{Im}(\mu)$ is a well-ordered subset of $[0;1]$. Thus there exists $k \in \mathbb{N}$ such that $\mathrm{Im}(\mu) \subseteq \{t_1, t_2, \ldots, t_k\} \cup \{0\}$.

$(b) \implies (a)$: Assume that (b) is true. Suppose that A is not Noetherian. Then there exists a strictly ascending sequence $F_1 \subset F_2 \subset \ldots$ of filters of A. Define a fuzzy set μ in A by

$$\mu(x) = \begin{cases} 0 & \text{if } x \notin F_n \text{ for each } n \in \mathbb{N}, \\ t_n & \text{if } x \in F_n - F_{n-1} \text{ for } n = 1, 2, \ldots, \end{cases}$$

where $F_0 = \emptyset$. By Lemma 3.11, μ is a fuzzy filter of A. This contradicts our assumption. Thus A is Noetherian. □

Theorem 4.4. *Let A be a BE-algebra and let $T = \{t_1, t_2, \ldots\} \cup \{0, 1\}$, where (t_n) is a strictly increasing sequence in $(0;1)$. Then the following conditions are equivalent:*

(a) *A is Artinian,*

(b) *for each fuzzy filter μ of A, if $\mathrm{Im}(\mu) \subseteq T$, then there exists $k \in \mathbb{N}$ such that $\mathrm{Im}(\mu) \subseteq \{t_1, t_2, \ldots, t_k\} \cup \{0, 1\}$.*

Proof.

$(a) \implies (b)$: On the contrary assume that $t_{i_1} < t_{i_2} < \cdots < t_{i_m} < \ldots$ is a strictly increasing sequence of elements of $\mathrm{Im}(\mu)$. Let $U_m = U\,(\mu; t_{i_m})$ for $m = 1, 2, \ldots$. It is easy to see that $U_1 \supset U_2 \supset \cdots \supset U_m \supset \ldots$ is a strictly descending sequence of filters of A. This contradicts the assumption that A is Artinian.

$(b) \implies (a)$: Assume that (b) is true. Suppose that A is not Artinian. Then there exists a strictly descending sequence $F_1 \supset F_2 \supset \cdots \supset F_n \supset \ldots$ of filters of A. Define a fuzzy set μ in A by

$$\mu(x) = \begin{cases} 0 & \text{if } x \notin F_1, \\ t_n & \text{if } x \in F_n - F_{n+1} \text{ for } n = 1, 2, \ldots, \\ 1 & \text{if } x \in \bigcap\{F_n : n \in \mathbb{N}\}. \end{cases}$$

Obviously, $\mu(1) = 1 \geq \mu(x)$ for all $x \in A$, that is, (d1) holds. Now we show that μ satisfies (d2). Let $x, y \in A$. We have three cases.

Case 1: $x \notin F_1$.
Then $y * x \notin F_1$ or $y \notin F_1$. Therefore $\mu(y * x) \wedge \mu(y) = 0 = \mu(x)$.

Case 2: $x \in F_n - F_{n+1}$ for some $n = 1, 2, \ldots$.
Then $y * x \notin F_{n+1}$ or $y \notin F_{n+1}$. Hence $\mu(y * x) \leq t_n$ or $\mu(y) \leq t_n$. Therefore $\mu(y * x) \wedge \mu(y) \leq t_n = \mu(x)$.

Case 3: $x \in \bigcap\{F_n : n \in \mathbb{N}\}$.
Obvious.

Thus μ is a fuzzy filter of A. This contradicts our assumption. Thus A is Artinian. □

Corollary 4.5. *Let A be a BE-algebra. If for every fuzzy filter μ of A,* $\mathrm{Im}(\mu)$ *is a finite set, then A is Artinian.*

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