Mathematica Slovaca

DOI: 10.2478/s12175-013-0151-0
Math. Slovaca **63** (2013), No. 5, 1007–1024

ČEBYŠEV-GRÜSS-TYPE INEQUALITIES REVISITED

Heiner Gonska* — Ioan Raşa** — Maria-Daniela Rusu*

(*Communicated by Ján Borsík*)

ABSTRACT. We generalize and improve several inequalities of the Čebyšev-Grüss-type using least concave majorants of the moduli of continuity of the functions involved. Our focus is on normalized positive linear functionals. We discuss a problem posed by the two Gavreas and also give the solution of a stronger one. In a section about the non-multiplicativity of positive linear operators it is demonstrated that the previous use of second moments is not quite the right choice. This is documented in the case of the classical Hermite-Fejér and de La Vallée Poussin convolution operators.

1. Introduction

The present paper is motivated by two classical results which we cite below.

Theorem 1.1. (Čebyšev, 1882, see [2]) *Let $f, g\colon [a,b] \to \mathbb{R}$ be absolutely continuous functions. If $f', g' \in L_\infty([a,b])$, then*

$$\left|\frac{1}{b-a}\int_a^b f(x)\cdot g(x)\,\mathrm{d}x - \left(\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right)\cdot\left(\frac{1}{b-a}\int_a^b g(x)\,\mathrm{d}x\right)\right|$$
$$\le \frac{1}{12}(b-a)^2\cdot\|f'\|_\infty\cdot\|g'\|_\infty\,.$$

We also mention:

2010 Mathematics Subject Classification: Primary 26D05, 41A25, 41A36, 47A63, 47B38.
Keywords: Čebyšev inequality, Čebyšev-Grüss-type inequality, least concave majorant of the modulus of continuity, Hermite-Fejér interpolation operator, convolution-type operators, Gavrea's problem.

Theorem 1.2. (Grüss, 1935, see [2]) *Let f, g be integrable functions from $[a,b]$ into $\mathbb{R}$, such that $m \le f(x) \le M$, $\rho \le g(x) \le \sigma$, for all $x \in [a,b]$, where $m, M, \rho, \sigma \in \mathbb{R}$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(x)\cdot g(x)\,\mathrm{d}x - \left(\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right)\cdot\left(\frac{1}{b-a}\int_a^b g(x)\,\mathrm{d}x\right)\right| \le \frac{1}{4}(M-m)(\rho-\sigma).$$

Another related inequality of Čebyšev is given as follows.

Theorem 1.3. (see [8]) *Let $f, g\colon [a,b] \to \mathbb{R}$ be integrable functions, both increasing or both decreasing. Furthermore, let $p\colon [a,b] \to \mathbb{R}_0^+$ be an integrable function. Then*

$$\int_a^b p(x)\,\mathrm{d}x \int_a^b p(x)\cdot f(x)\cdot g(x)\,\mathrm{d}x \ge \int_a^b p(x)\cdot f(x)\,\mathrm{d}x \int_a^b p(x)\cdot g(x)\,\mathrm{d}x. \tag{1.1}$$

If one of the functions f or g is nonincreasing and the other nondecreasing, then inequality (1.1) *is reversed.*

If $p(x) = 1$ for $a \le x \le b$, then inequality (1.1) reduces to

$$\frac{1}{b-a}\int_a^b f(x)\cdot g(x)\,\mathrm{d}x \ge \frac{1}{(b-a)^2}\int_a^b f(x)\,\mathrm{d}x \cdot \int_a^b g(x)\,\mathrm{d}x.$$

The functional L given by $L(f) := \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$ is a positive linear one satisfying $L(e_0) = 1$; here we used the notation $e_i(x) = x^i,\ i \ge 0$.

In the present note we will deal with general positive linear functionals L satisfying $L(e_0) = 1$, give bounds for the quantities

$$T(f,g) := L(f\cdot g) - L(f)\cdot L(g)$$

in terms of the least concave majorants of the moduli of continuity of the functions in question and also include a detailed discussion of a problem posed earlier by Bogdan and Ioan Gavrea.

We recall the following definition:

Definition 1.4. (see [1]) Let $f \in C[a,b]$. If for $t \in [0,\infty)$ the quantity

$$\omega(f;t) = \sup\{|f(x) - f(y)| : \ |x-y| \le t\}$$

is the usual modulus of continuity, its least concave majorant is given by

$$\widetilde{\omega}(f;t) = \begin{cases} \sup\limits_{\substack{0\le x\le t\le y\le b-a \\ x\neq y}} \frac{(t-x)\omega(f,y)+(y-t)\omega(f,x)}{y-x} & \text{if } 0 \le t \le b-a, \\ \omega(f, b-a) & \text{if } t > b-a. \end{cases}$$

In [7] the following result for the least concave majorant is proved:

$$K\left(\frac{t}{2}, f; C[a,b], C^1[a,b]\right) := \inf_{g\in C^1[a,b]} \left(\|f-g\|_\infty + \frac{t}{2}\|g'\|_\infty\right) = \frac{1}{2}\widetilde{\omega}(f;t), \quad t \ge 0.$$

If $H\colon C[a,b] \to C[a,b]$ is a positive linear operator reproducing linear functions, then for $x \in [a,b]$ fixed the functional L given by

$$L(f) = H(f;x), \qquad f \in C[a,b],$$

is of the above type. Hence in this special case one is lead to differences

$$D(f,g) := D(f,g;x) := H(f\cdot g;x) - H(f;x)\cdot H(g;x).$$

These explain the non-multiplicativity of H at the point x. A recent result is:

Theorem 1.5. (see [9]) *If $f, g \in C[a,b]$ and $x \in [a,b]$ is fixed, then the inequality*

$$|D(f,g)| \le \frac{1}{4}\widetilde{\omega}\left(f; 2\sqrt{H((e_1-x)^2;x)}\right)\cdot \widetilde{\omega}\left(g; 2\sqrt{H((e_1-x)^2;x)}\right) \tag{1.2}$$

holds.

The latter inequality can be generalized to the situation in which $[a,b]$ is replaced by a compact metric space (X,d) as shown in Rusu's paper [9].

In the present note we will replace the second moments $H((e_1-x)^2;x)$ by the smaller quantity $H(e_2;x) - H(e_1;x)^2$, proving that the first approach is not exactly the ideal choice. We then give two appropriate examples concerning the classical Hermite-Fejér interpolation operator, as well as the de La Vallée Poussin convolution operator.

2. On Čebyšev's inequality

In this section we generalize the second of Čebyšev's inequalities from above and first give bounds for $T(f,g)$ for continuously differentiable functions. Although the proposition below appears to be well-known, we were unable to locate a reference.

Proposition 2.1. *Let $L\colon C[a,b] \to \mathbb{R}$ be a positive linear functional with $L(e_0) = 1$. If $f, g \in C[a,b]$ are both increasing (decreasing) functions, then the inequality*

$$L(f \cdot g) \geq L(f) \cdot L(g)$$

holds.

P r o o f. From the monotonicity of f and g we have

$$[f(x) - f(y)] \cdot [g(x) - g(y)] \geq 0,$$

for all $x, y \in [a,b]$, i.e.,

$$f(x) \cdot g(x) - f(x) \cdot g(y) - f(y) \cdot g(x) + f(y) \cdot g(y) \geq 0.$$

Applying L with respect to the variable x to this last inequality gives

$$L^x(f \cdot g) - g(y) \cdot L^x(f) - f(y) \cdot L^x(g) + f(y) \cdot g(y) \geq 0,$$

for all $y \in [a,b]$. Here we used that $L(e_0) = 1$. If we now use L with respect to the variable y, we get

$$L^x(f \cdot g) - L^y(g) \cdot L^x(f) - L^y(f) \cdot L^x(g) + L^y(f \cdot g) \geq 0,$$

and this is equivalent to

$$L(f \cdot g) \geq L(f) \cdot L(g). \qquad \square$$

Using the above proposition we give a Grüss-type inequality for the functional L, so we now prove the following theorem (for a different proof, see [1: Theorem 12]):

Theorem 2.2. *If L given as above is a positive linear functional with $L(e_0) = 1$, then for the bilinear functional*

$$T(f,g) := L(f \cdot g) - L(f) \cdot L(g)$$

we have the inequality

$$|T(f \cdot g)| \leq \left\| \frac{f'}{h'} \right\|_\infty \cdot \left\| \frac{g'}{h'} \right\|_\infty \cdot |T(h,h)|,$$

where $f, g, h \in C^1[a,b]$ and $h'(t) \neq 0$ for each $t \in [a,b]$.

P r o o f. We may suppose that $h'(t) > 0$, $t \in [a, b]$.

Let $F := \left\|\frac{f'}{h'}\right\|_\infty$, $G := \left\|\frac{g'}{h'}\right\|_\infty$. Then all four functions $Fh \pm f$, $Gh \pm g$ are increasing. According to Čebyšev's inequality, we have

$$L[(Fh+f)\cdot(Gh+g)] \geq L(Fh+f)\cdot L(Gh+g),$$
$$L[(Fh-f)\cdot(Gh-g)] \geq L(Fh-f)\cdot L(Gh-g).$$

Adding these two inequalities yields

$$\begin{aligned}
&L[(Fh+f)\cdot(Gh+g)] + L[(Fh-f)\cdot(Gh-g)]\\
&\geq L(Fh+f)\cdot L(Gh+g) + L(Fh-f)\cdot L(Gh-g)\\
\iff\ & L(FGh^2 + Fgh + Gfh + fg) + L(FGh^2 - Fgh - Gfh + fg)\\
&\geq [L(Fh)+L(f)]\cdot[L(Gh)+L(g)] + [L(Fh)-L(f)]\cdot[L(Gh)-L(g)]\\
\iff\ & FG\cdot L(h^2) + F\cdot L(gh) + G\cdot L(fh) + L(fg) + FG\cdot L(h^2)\\
&\quad - F\cdot L(gh) - G\cdot L(fh) + L(fg)\\
&\geq FG\cdot[L(h)]^2 + L(Fh)\cdot L(g) + L(f)\cdot L(Gh) + L(f)\cdot L(g)\\
&\quad + FG\cdot[L(h)]^2 - L(Fh)\cdot L(g) - L(f)\cdot L(Gh) + L(f)\cdot L(g)\\
\iff\ & 2\cdot FG\cdot L(h^2) + 2\cdot L(f\cdot g) \geq 2\cdot FG\cdot[L(h)]^2 + 2\cdot L(f)\cdot L(g)
\end{aligned}$$

and dividing both sides by 2, we get

$$FG\cdot[L(h^2) - (Lh)^2] \geq (Lf)\cdot(Lg) - L(f\cdot g). \tag{2.1}$$

Changing now g by $-g$ in (2.1) yields

$$FG\cdot[L(h^2) - (Lh)^2] \geq -(Lf)\cdot(Lg) + L(f\cdot g). \tag{2.2}$$

From (2.1) and (2.2) we derive

$$|L(f\cdot g) - (Lf)\cdot(Lg)| \leq FG\cdot(L(h^2) - (Lh)^2),$$

i.e.,

$$|T(f\cdot g)| \leq \left\|\frac{f'}{h'}\right\|_\infty \cdot \left\|\frac{g'}{h'}\right\|_\infty \cdot |T(h,h)|\,,$$

and this is the desired inequality. □

3. A new Čebyšev-Grüss-type inequality

Next we will give a new upper bound for $|T(f,g)|$ involving $\widetilde{\omega}$.

Theorem 3.1. *If $L\colon C[a,b] \to \mathbb{R}$ is a positive linear functional with $L(e_0) = 1$, then for $f, g \in C[a,b]$ we have*

$$|T(f,g)| \le \frac{1}{4} \cdot \widetilde{\omega}\left(f; 2 \cdot \sqrt{T(e_1,e_1)}\right) \cdot \widetilde{\omega}\left(g; 2 \cdot \sqrt{T(e_1,e_1)}\right),$$

where $\widetilde{\omega}$ is the least concave majorant of the modulus of continuity, ω, and

$$T(e_1,e_1) = L(e_2) - [L(e_1)]^2.$$

Moreover,

$$T\left(\frac{e_1 - a}{b - a}, \frac{e_1 - a}{b - a}\right) \le \frac{1}{4},$$

with equality holding if and only if $L = \frac{1}{2} \cdot (\varepsilon_a + \varepsilon_b)$, where $\varepsilon_x(f) = f(x)$, $x \in \{a, b\}$.

P r o o f. First we use the Cauchy-Schwarz inequality for positive linear functionals:

$$L(f) \le L(|f|) \le \sqrt{L(f^2) \cdot L(e_0)} = \sqrt{L(f^2)},$$

so we have

$$T(f,f) = L(f^2) - L(f)^2 \ge 0,$$

for all $f \in C[a,b]$. Hence, T is a positive bilinear form on $C[a,b]$. Using Cauchy-Schwarz for T, for all $f, g \in C[a,b]$ we get

$$|T(f,g)| \le \sqrt{T(f,f) \cdot T(g,g)} \le \|f\|_\infty \cdot \|g\|_\infty .$$

For $f, g \in C[a,b]$ fixed and $r, s \in C^1[a,b]$ arbitrary, we decompose as follows:

$$\begin{aligned} |T(f,g)| &= |T(f - r + r, g - s + s)| \\ &\le |T(f - r, g - s)| + |T(f - r, s)| + |T(r, g - s)| + |T(r,s)| . \end{aligned}$$

Now $f - r, g - s \in C[a,b]$, so that

$$|T(f - r, g - s)| \le \|f - r\|_\infty \cdot \|g - s\|_\infty .$$

For the second summand we have

$$\begin{aligned} |T(f - r, s)| &\le \sqrt{T(f - r, f - r) \cdot T(s,s)} \\ &\le \|f - r\|_\infty \cdot \sqrt{T(s,s)} \\ &\le \|f - r\|_\infty \cdot \|s'\|_\infty \cdot \sqrt{T(e_1,e_1)}, \end{aligned}$$

where the last step follows from Theorem 2.2 with $f = g = s$ and $h = e_1$. Likewise,

$$|T(r, g-s)| \leq \|g-s\|_\infty \cdot \|r'\|_\infty \cdot \sqrt{T(e_1, e_1)}.$$

Finally,

$$|T(r,s)| \leq \sqrt{T(r,r)\cdot T(s,s)} \leq \|r'\|_\infty \cdot \|s'\|_\infty \cdot T(e_1, e_1),$$

by taking $f = r,\ g = s$ and $h = e_1$ in Theorem 2.2. Hence,

$$\begin{aligned}
&|T(f,g)| \\
&\leq \|f-r\|_\infty \cdot \|g-s\|_\infty + \|f-r\|_\infty \cdot \|s'\|_\infty \cdot \sqrt{T(e_1,e_1)} \\
&\quad + \|g-s\|_\infty \cdot \|r'\|_\infty \cdot \sqrt{T(e_1,e_1)} + \|r'\|_\infty \cdot \|s'\|_\infty \cdot T(e_1,e_1) \\
&\leq \left(\|f-r\|_\infty + \|r'\|_\infty \cdot \sqrt{T(e_1,e_1)}\right) \cdot \left(\|g-s\|_\infty + \|s'\|_\infty \cdot \sqrt{T(e_1,e_1)}\right).
\end{aligned}$$

Passing to the infimum over r and s yields

$$\begin{aligned}
|T(f,g)| &\leq K\Big(\sqrt{T(e_1,e_1)}, f; C^0, C^1\Big) \cdot K\Big(\sqrt{T(e_1,e_1)}, g; C^0, C^1\Big) \\
&= \frac{1}{4} \cdot \widetilde{\omega}\Big(f; 2\cdot\sqrt{T(e_1,e_1)}\Big) \cdot \widetilde{\omega}\Big(g; 2\cdot\sqrt{T(e_1,e_1)}\Big).
\end{aligned}$$

Furthermore we have

$$\begin{aligned}
T\left(\frac{e_1-a}{b-a}, \frac{e_1-a}{b-a}\right) &= L\left(\left(\frac{e_1-a}{b-a}\right)^2\right) - L\left(\frac{e_1-a}{b-a}\right) \cdot L\left(\frac{e_1-a}{b-a}\right) \\
&\leq L\left(\frac{e_1-a}{b-a}\right) - L\left(\frac{e_1-a}{b-a}\right) \cdot L\left(\frac{e_1-a}{b-a}\right) \\
&= L\left(\frac{e_1-a}{b-a}\right) \cdot \left[1 - L\left(\frac{e_1-a}{b-a}\right)\right] \\
&\leq \frac{1}{4}
\end{aligned}$$

since $0 \leq L\left(\frac{e_1-a}{b-a}\right) \leq 1$. Equality holds if and only if

$$L\left(\frac{e_1-a}{b-a}\right) = \frac{1}{2}.$$

Clearly, if $L\left(\left(\frac{e_1-a}{b-a}\right)^2\right) = L\left(\frac{e_1-a}{b-a}\right) = \frac{1}{2}$, then

$$T\left(\frac{e_1-a}{b-a}, \frac{e_1-a}{b-a}\right) = \frac{1}{4}.$$

Assume now that the latter inequality holds and that

$$L\left(\left(\frac{e_1-a}{b-a}\right)^2\right) < L\left(\frac{e_1-a}{b-a}\right).$$

Then

$$\begin{aligned}\frac{1}{4} &= T\left(\frac{e_1-a}{b-a}, \frac{e_1-a}{b-a}\right)\\ &< L\left(\frac{e_1-a}{b-a}\right)\cdot\left[1-L\left(\frac{e_1-a}{b-a}\right)\right] \le \frac{1}{4},\end{aligned}$$

which is a contradiction. Thus

$$\begin{aligned}&L\left(\left(\frac{e_1-a}{b-a}\right)^2\right) = L\left(\frac{e_1-a}{b-a}\right), \qquad \text{or}\\ &L\left(\frac{e_1-a}{b-a}\right) - \left[L\left(\frac{e_1-a}{b-a}\right)\right]^2 = \frac{1}{4} \iff L\left(\frac{e_1-a}{b-a}\right) = \frac{1}{2}.\end{aligned}$$

Now, if $L = \frac{1}{2}\cdot(\varepsilon_a+\varepsilon_b)$, then

$$L\left(\left(\frac{e_1-a}{b-a}\right)^2\right) - \left[L\left(\frac{e_1-a}{b-a}\right)\right]^2 = \frac{1}{2}-\frac{1}{4} = \frac{1}{4} = T\left(\frac{e_1-a}{b-a}, \frac{e_1-a}{b-a}\right).$$

If the latter inequality holds, then we saw above that

$$L\left(\left(\frac{e_1-a}{b-a}\right)^2\right) = L\left(\frac{e_1-a}{b-a}\right),$$

which is equivalent to

$$L\left(\frac{e_1-a}{b-a} - \left(\frac{e_1-a}{b-a}\right)^2\right) = 0.$$

The function in the argument is strictly positive in (a,b). So the above inequality is equivalent to L being supported by $\{a,b\}$, i.e.,

$$L = \alpha\cdot\varepsilon_a + (1-\alpha)\cdot\varepsilon_b,$$

for some $\alpha\in[0,1]$. On the other hand,

$$L\left(\left(\frac{e_1-a}{b-a}\right)^2\right) = L\left(\frac{e_1-a}{b-a}\right) = \frac{1}{2},$$

and so $\alpha = \frac{1}{2}$ and $L = \frac{1}{2}\cdot(\varepsilon_a+\varepsilon_b)$. This concludes the proof. □

Example 3.2. If the functional $L\colon C[a,b] \to \mathbb{R}$ is again given by

$$L(f) = \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x,$$

then the inequality in Theorem 3.1 holds with

$$T(e_1, e_1) = \frac{1}{b-a} \int_a^b x^2\,\mathrm{d}x - \frac{1}{(b-a)^2} \left(\int_a^b x\,\mathrm{d}x \right)^2 = \frac{(b-a)^2}{12}.$$

This means that

$$\left| \frac{1}{b-a} \int_a^b f(x) \cdot g(x)\,\mathrm{d}x - \frac{1}{(b-a)^2} \int_a^b f(x)\,\mathrm{d}x \cdot \int_a^b g(x)\,\mathrm{d}x \right|$$
$$\leq \frac{1}{4} \cdot \widetilde{\omega}\left(f; \frac{2(b-a)}{\sqrt{12}} \right) \cdot \widetilde{\omega}\left(g; \frac{2(b-a)}{\sqrt{12}} \right), \qquad f, g \in C[a,b].$$

If f is absolutely continuous with $f' \in L_\infty([a,b])$, then for any difference $|f(x) - f(y)|$, $y \leq x$, figuring in the definition of $\omega(f;t)$ we observe that

$$\begin{aligned}
|f(x) - f(y)| &= \left| \int_a^x f'(t)\,\mathrm{d}t - \int_a^y f'(t)\,\mathrm{d}t \right| \\
&= \left| \int_y^x f'(t)\,\mathrm{d}t \right| \leq \int_y^x |f'(t)|\,\mathrm{d}t \\
&\leq \|f'\|_{L_\infty([a,b])} \cdot (x-y) \\
&\leq \|f'\|_{L_\infty([a,b])} \cdot t.
\end{aligned}$$

As a consequence, for any expression figuring in the sup defining $\widetilde{\omega}(f;t)$ we have, for $x < y$,

$$\begin{aligned}
&\frac{(t-x) \cdot \omega(f;y) + (y-t) \cdot \omega(f;x)}{y-x} \\
&\leq \frac{(t-x) \cdot \|f'\|_{L_\infty([a,b])} \cdot y + (y-t) \cdot \|f'\|_{L_\infty([a,b])} \cdot x}{y-x} \\
&= \|f'\|_{L_\infty([a,b])} \cdot t.
\end{aligned}$$

So, for $f', g' \in L_\infty([a,b])$ we obtain

$$\begin{aligned} &\frac{1}{4}\widetilde{\omega}\left(f; \frac{2(b-a)}{\sqrt{12}}\right) \cdot \widetilde{\omega}\left(g; \frac{2(b-a)}{\sqrt{12}}\right) \\ &\leq \frac{1}{4} \cdot \frac{4(b-a)^2}{12} \cdot \|f'\|_{L_\infty} \cdot \|g'\|_{L_\infty} \\ &= \frac{(b-a)^2}{12} \cdot \|f'\|_{L_\infty} \cdot \|g'\|_{L_\infty}. \end{aligned}$$

Hence our result from Theorem 3.1 is best possible since we rediscovered the Čebyšev-Grüss inequality for the integration functional in which the constant $\frac{(b-a)^2}{12}$ is best possible.

4. On the Gavrea problem

In [5] B. Gavrea and I. Gavrea raised the following problem:

Problem 4.1. Let L be a linear positive functional defined on $C[0,1]$ with $L(e_0) = 1$ and $f, g \in C[0,1]$. Do positive numbers $\delta_1 = \delta_1(f) < 1$ and $\delta_2 = \delta_2(f) < 1$ exist such that

$$|L(f \cdot g) - L(f) \cdot L(g)| \leq \frac{1}{4} \cdot \widetilde{\omega}(f; \delta_1) \cdot \widetilde{\omega}(f; \delta_2)?$$

In [1] it was shown that the answer is negative. Indeed, for

$$L(f) = B_1\left(f; \frac{1}{2}\right) = \frac{1}{2} \cdot (f(0) + f(1)),\ f \in C[0,1],$$

where B_1 is the first Bernstein operator, it was shown that

$$|L(e_2) - L(e_1) \cdot L(e_1)| = \frac{1}{4} \qquad \text{and}$$
$$\frac{1}{4} \cdot \widetilde{\omega}(e_1; t) \cdot \widetilde{\omega}(e_1; t) = \frac{1}{4} \cdot t^2 < \frac{1}{4},$$

for $0 \leq t < 1$. So in this particular case the numbers δ_1 and δ_2 do not exist.

The counter-example from above is a very singular one in the sense that in all other cases of L the answer is in the affirmative. As a consequence of the second statement in Theorem 3.1 we have the following:

Corollary 4.2 (The "positive answer" to the Gavrea problem)**.**
Let $L\colon C[0,1] \to \mathbb{R}$ be a positive linear functional, $L(e_0) = 1$, $L \neq \frac{1}{2} \cdot (\varepsilon_0 + \varepsilon_1)$. Denote $\delta := 2 \cdot \sqrt{Le_2 - (Le_1)^2}$. Then $0 \leq \delta < 1$ and for all $f, g \in C[0,1]$,

$$|L(f \cdot g) - L(f) \cdot L(g)| \leq \frac{1}{4}\widetilde{\omega}(f; \delta) \cdot \widetilde{\omega}(g; \delta).$$

Here δ is independent of f and g.

P r o o f. As we showed in Theorem 3.1

$$\begin{aligned}|T(f,g)| &= |L(f\cdot g) - L(f)\cdot L(g)|\\ &\le \frac{1}{4}\cdot\widetilde{\omega}\Big(f;2\cdot\sqrt{T(e_1,e_1)}\Big)\cdot\widetilde{\omega}\Big(g;2\cdot\sqrt{T(e_1,e_1)}\Big)\\ &=: \frac{1}{4}\cdot\widetilde{\omega}(f;2\cdot c)\cdot\widetilde{\omega}(g;2\cdot c)\end{aligned}$$

with $c < \frac{1}{2}$ because $L \neq \frac{1}{2}(\varepsilon_0 + \varepsilon_1)$. Clearly c is independent of f and g. □

The following result solves a problem which is stronger than that posed by the two Gavreas, in the sense that the least concave majorant $\widetilde{\omega}$ is now replaced by ω itself.

Theorem 4.3 (A strong form of Gavrea's problem). *Let $A\colon C[0,1]\to\mathbb{R}$ be a positive linear functional, $A(e_0)=1$, $A\neq\frac{1}{2}\cdot(\varepsilon_0+\varepsilon_1)$. Let $f,g\in C[0,1]$. Then there exist $\delta_1=\delta_1(f)<1$ and $\delta_2=\delta_2(g)<1$ such that*

$$|A(f\cdot g) - A(f)\cdot A(g)| \le \frac{1}{4}\cdot\omega(f;\delta_1)\cdot\omega(g;\delta_2). \tag{4.1}$$

P r o o f. Suppose that there exist $f,g\in C[0,1]$ such that

$$|A(f\cdot g) - A(f)\cdot A(g)| > \frac{1}{4}\cdot\omega(f;\delta_1)\cdot\omega(g;\delta_2), \tag{4.2}$$

for all $\delta_1,\delta_2<1$. Without loss of generality we may suppose that

$$\min f = \min g = 0; \qquad \max f = \max g = 1. \tag{4.3}$$

We can do this, because if $\min f\neq 0$ and/or $\min g\neq 0$, we replace f by $\tilde{f} := f-\min f$ and/or g by $\tilde{g} := g - \min g$ and notice that

$$\begin{aligned}&A(f\cdot g) - A(f)\cdot A(g)\\ &= A[(\tilde{f}+\min f)\cdot(\tilde{g}+\min g)] - A(\tilde{f}+\min f)\cdot A(\tilde{g}+\min g)\\ &= A(\tilde{f}\cdot\tilde{g} + \tilde{f}\cdot\min g + \tilde{g}\cdot\min f + \min f\cdot\min g)\\ &\quad - [A(\tilde{f}) + A(\min f)]\cdot[A(\tilde{g}) + A(\min g)]\\ &= A(\tilde{f}\cdot\tilde{g}) + A(\tilde{f}\cdot\min g) + A(\tilde{g}\cdot\min f) + A(\min f\cdot\min g)\\ &\quad - A(\tilde{f})\cdot A(\tilde{g}) - A(\tilde{f}\cdot\min g) - A(\tilde{g}\cdot\min f) - A(\min f\cdot\min g)\\ &= A(\tilde{f}\cdot\tilde{g}) - A(\tilde{f})\cdot A(\tilde{g}).\end{aligned}$$

So we may assume that $f,g\ge 0$ with $\min f = \min g = 0$. Now replace f by $\bar{f} := \frac{f}{\|f\|}$ and g by $\bar{g} := \frac{g}{\|g\|}$, assuming that $\|f\|\neq 0\neq\|g\|$, since otherwise the inequality (4.1) holds trivially. Then from the inequality

$$\begin{aligned}
&\left|A(\bar{f}\cdot\bar{g}) - A(\bar{f})\cdot A(\bar{g})\right| \\
&= \frac{1}{\|f\|\cdot\|g\|}\cdot|A(f\cdot g) - A(f)\cdot A(g)| \\
&\le \frac{1}{4}\cdot\omega(\bar{f};\delta_1)\cdot\omega(\bar{g};\delta_2) \\
&= \frac{1}{4}\cdot\frac{1}{\|f\|\cdot\|g\|}\cdot\omega(f;\delta_1)\cdot\omega(g;\delta_2)
\end{aligned}$$

we again arrive at inequality (4.1). We may thus asssume that $\|f\| = \|g\| = 1$ with $\min f = \min g = 0$, hence $f, g \ge 0$. From (4.2) and (4.3) it follows:

$$\begin{aligned}
|A(f\cdot g) - A(f)\cdot A(g)| &\ge \frac{1}{4}\cdot\lim_{\delta_1\nearrow 1}\omega(f;\delta_1)\cdot\lim_{\delta_2\nearrow 1}\omega(g;\delta_2) \\
&= \frac{1}{4}\cdot\omega(f;1)\cdot\omega(g;1) = \frac{1}{4}
\end{aligned}$$

Thus

$$[A(f^2) - A^2(f)]\cdot[A(g^2) - A^2(g)] \ge [A(f\cdot g) - A(f)\cdot A(g)]^2 \ge \frac{1}{16}.$$

By Grüss' inequality we know that

$$A(f^2) - A^2(f) \le \frac{1}{4},\ A(g^2) - A^2(g) \le \frac{1}{4},$$

so that

$$[A(f^2) - A^2(f)] = [A(g^2) - A^2(g)] = |A(f\cdot g) - A(f)\cdot A(g)| = \frac{1}{4}. \tag{4.4}$$

Now $f\cdot(1-f) \ge 0$, so

$$\begin{aligned}
0 &\le A(f\cdot(1-f)) \\
&= A(f) - A(f^2) \\
&= A(f) - \left[A^2(f) + \frac{1}{4}\right] \qquad \text{(by (4.4))} \\
&= A(f)\cdot(1 - A(f)) - \frac{1}{4}
\end{aligned}$$

which is equivalent to

$$\frac{1}{4} \le A(f)\cdot(1 - A(f)) \le \frac{1}{4},$$

since $0 \le A(f) \le 1$. It follows that

$$A(f)\cdot(1 - A(f)) = \frac{1}{4}$$

i.e.,

$$A(f \cdot (1-f)) = 0 \qquad \text{and}$$
$$A(f) = A(f^2) = \frac{1}{2},$$

and similarly for g.

From $A(f \cdot (1-f)) = 0$ and $f \cdot (1-f) \geq 0$ we infer that $f(x) \in \{0,1\}$ for all $x \in \operatorname{supp} A$, and similarly $g(x) \in \{0,1\}$ for all $x \in \operatorname{supp} A$.

We shall prove that $\operatorname{supp} A \subset \{0,1\}$. Indeed, let $x \in \operatorname{supp} A$, $0 < x < 1$. Then $f(x), g(x) \in \{0,1\}$. Let us remark that there exists $y \in \operatorname{supp} A$ such that $|f(x) - f(y)| = 1$. Indeed, otherwise $f = \text{const.}$ on $\operatorname{supp} A$, and since $A(f) = \frac{1}{2}$, it follows that $f \equiv \frac{1}{2}$ on $\operatorname{supp} A$, which is a contradiction.

Similarly, there exists $z \in \operatorname{supp} A$ such that $|g(x) - g(z)| = 1$. Now define $\delta_1 := |x-y| < 1$, $\delta_2 := |x-z| < 1$. Then $\omega(f;\delta_1) = 1$, $\omega(g;\delta_2) = 1$. By (4.2) it follows:

$$|A(f \cdot g) - A(f) \cdot A(g)| > \frac{1}{4},$$

which contradicts (4.4). So we have $\operatorname{supp} A \subset \{0,1\}$, i.e.,

$$A = (1-a) \cdot \varepsilon_0 + a \cdot \varepsilon_1,$$

for some $0 < a < 1$. From (4.4) we know that $A(f^2) - A^2(f) = \frac{1}{4}$, which leads to

$$a \cdot (1-a) \cdot [f(1) - f(0)]^2 = \frac{1}{4}.$$

Since $[f(1) - f(0)]^2 \leq 1$, we have $a \cdot (1-a) \geq \frac{1}{4}$, i.e., $a = \frac{1}{2}$. So

$$A = \frac{1}{2} \cdot \varepsilon_0 + \frac{1}{2} \cdot \varepsilon_1,$$

a contradiction. Our proof is complete. □

5. On the non-multiplicativity of positive linear operators

Next we present some consequences concerning the non-multiplicativity of positive linear operators reproducing constant functions. As an immediate corollary of Theorem 3.1 we get:

Corollary 5.1. *If $H\colon C[a,b] \to C[a,b]$ is a positive linear operator which reproduces constant functions, then for $f, g \in C[a,b]$ and $x \in [a,b]$ fixed we have the inequalities:*

$$\begin{aligned}
&|D(f,g)| = |H(f\cdot g;x) - H(f;x)\cdot H(g;x)| \\
&\le \frac{1}{4}\cdot\widetilde{\omega}\left(f; 2\cdot\sqrt{H(e_2;x)-H(e_1;x)^2}\right)\cdot\widetilde{\omega}\left(g; 2\cdot\sqrt{H(e_2;x)-H(e_1;x)^2}\right) \\
&\le \frac{1}{4}\cdot\widetilde{\omega}\left(f; 2\cdot\sqrt{H((e_1-x)^2;x)}\right)\cdot\widetilde{\omega}\left(g; 2\cdot\sqrt{H((e_1-x)^2;x)}\right).
\end{aligned}$$

P r o o f. Clearly

$$\begin{aligned}
D(e_1,e_1) &= H(e_2;x) - H(e_1;x)^2 \\
&\le H(e_2;x) - 2\cdot x\cdot H(e_1;x) + x^2\cdot H(e_0;x) \\
&= H((e_1-x)^2;x),
\end{aligned}$$

and the statement in the corollary follows from the monotonicity of $\widetilde{\omega}$ with respect to the real variable. □

Remark 5.2. If $H(e_1;x) = x$, then the equality

$$\begin{aligned}
D(e_1,e_1) &= H(e_2;x) - x^2 \\
&= H((e_1-x)^2;x)
\end{aligned}$$

holds. Thus, there is no improvement of [9: Theorem 4.1]. If however $H(e_1;x) \neq x$, then

$$\begin{aligned}
D(e_1,e_1) &= H((e_1-x)^2;x) - (H(e_1;x)-x)^2 \\
&< H((e_1-x)^2;x),
\end{aligned}$$

and vice versa.

Two examples for the latter situation are given next.

5.1. A new Čebyšev-Grüss-type inequality for the classical Hermite-Fejér interpolation operator

The classical Hermite-Fejér interpolation operator is a positive linear operator and can be written as

$$L_n(f;x) = \sum_{k=1}^{n} f(x_k)\cdot(1-x\cdot x_k)\cdot\left(\frac{T_n(x)}{n\cdot(x-x_k)}\right)^2, \qquad (5.1)$$

where $f \in C[-1,1]$ and $x_k = \cos\left(\frac{2k-1}{2n}\pi\right)$, $1 \le k \le n$, are the zeros of $T_n(x) = cos(n\cdot arccos(x))$, the n-th Čebyšev polynomial of the first kind. For this operator we have

$$L_n((e_1-x)^2;x) = \frac{1}{n}\cdot T_n^2(x).$$

It was shown in DeVore's book ([4: p. 43]) that for $x \in [-1,1]$ the following holds:

$$\begin{aligned}
|L_n(e_1 - x; x)| &= \left| -\frac{1}{n^2} \cdot (1-x^2) \cdot T_n(x) \cdot T_n'(x) - \frac{1}{n} \cdot x \cdot T_n^2(x) \right| \\
&= \frac{1}{n} \cdot |T_n(x)| \cdot \left| \frac{1}{n} \cdot (1-x^2) \cdot T_n'(x) + x \cdot T_n(x) \right| \\
&= \frac{1}{n} \cdot |T_n(x)| \cdot \Big| n^{-1} \cdot (1-x^2) \cdot n \cdot \sin(n \cdot \arccos(x)) \cdot \frac{1}{\sqrt{1-x^2}} \\
&\quad + x \cdot \cos(n \cdot \arccos(x)) \Big| \\
&= \frac{1}{n} \cdot |T_n(x)| \cdot \left| \sqrt{1-x^2} \cdot \sin(n \cdot \arccos(x)) + x \cdot \cos(n \cdot \arccos(x)) \right| \\
&= \frac{1}{n} \cdot |T_n(x)| \cdot \big| \sin(\arccos(x)) \cdot \sin(n \cdot \arccos(x)) \\
&\quad + \cos(\arccos(x)) \cdot \cos(n \cdot \arccos(x)) \big| \\
&= \frac{1}{n} \cdot |T_n(x)| \cdot |\cos((n-1) \cdot \arccos(x))| \\
&= \frac{1}{n} \cdot |T_n(x)| \cdot |T_{n-1}(x)| .
\end{aligned}$$

Hence

$$[L_n(e_1 - x; x)]^2 = \frac{1}{n^2} \cdot T_{n-1}^2(x) \cdot T_n^2(x).$$

So we have

$$\begin{aligned}
D(e_1, e_1) &= \frac{1}{n} \cdot T_n^2(x) - \frac{1}{n^2} \cdot T_n^2(x) \cdot T_{n-1}^2(x) \\
&= \frac{1}{n} \cdot T_n^2(x) \cdot \left[1 - \frac{1}{n} \cdot T_{n-1}^2(x) \right]
\end{aligned}$$

and we arrive at

$$\begin{aligned}
&|L_n(f \cdot g; x) - L_n(f; x) \cdot L_n(g; x)| \\
&\le \frac{1}{4} \cdot \widetilde{\omega} \left(f; \frac{2}{\sqrt{n}} \cdot |T_n(x)| \cdot \sqrt{1 - \frac{1}{n} \cdot T_{n-1}^2(x)} \right) \\
&\quad \cdot \widetilde{\omega} \left(g; \frac{2}{\sqrt{n}} \cdot |T_n(x)| \cdot \sqrt{1 - \frac{1}{n} \cdot T_{n-1}^2(x)} \right).
\end{aligned}$$

This improves [1: Remark 7, Inequality (14)].

5.2. A new Čebyšev-Grüss-type inequality for convolution-type operators

For the case $X = [-1,1]$, a function $f \in C[-1,1]$ and any natural number n, the convolution operator $G_{m(n)}$ is given by

$$G_{m(n)}(f,x) := \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} f(\cos(\arccos(x) + \upsilon)) \cdot K_{m(n)}(\upsilon)\,\mathrm{d}\upsilon,$$

where the kernel $K_{m(n)}$ is a positive and even trigonometric polynomial of degree $m(n)$ satisfying

$$\int_{-\pi}^{\pi} K_{m(n)}(\upsilon)\,\mathrm{d}\upsilon = \pi,$$

meaning that $G_{m(n)}(1,x) = 1$ for $x \in [-1,1]$.

It is clear that $G_{m(n)}(f,\cdot)$ is an algebraic polynomial of degree $m(n)$ and the kernel $K_{m(n)}$ has the following form:

$$K_{m(n)}(\upsilon) = \frac{1}{2} + \sum_{k=1}^{m(n)} \rho_{k,m(n)} \cdot \cos(k\upsilon),$$

for $\upsilon \in [-\pi,\pi]$.

We also need a result that goes back to H. G. Lehnhoff [6]:

Lemma 5.3. *For $x \in X$ the equality*

$$G_{m(n)}((e_1 - x)^2, x)$$
$$= x^2 \cdot \left\{ \frac{3}{2} - 2 \cdot \rho_{1,m(n)} + \frac{1}{2} \cdot \rho_{2,m(n)} \right\} + (1 - x^2) \cdot \left\{ \frac{1}{2} - \frac{1}{2} \cdot \rho_{2,m(n)} \right\}$$

holds.

The first moment of the convolution-type operator (see [3]) is given by:

$$G_{m(n)}(e_1 - x; x) = x \cdot [\rho_{1,m(n)} - 1].$$

The lemma gives the second moment of the convolution-type operator which, along with the first moment, will be needed in the sequel.

For the special case $m(n) = n \in \mathbb{N}_0$ we consider the de La Vallée Poussin kernel given by

$$V_n(\upsilon) = \frac{(n!)^2}{(2n)!} \cdot \left(2\cos\left(\frac{\upsilon}{2}\right)\right)^{2n}$$

with

$$\rho_{1,n} = \frac{n}{n+1}, \qquad \rho_{2,n} = \frac{(n-1)n}{(n+1)(n+2)}.$$

Using Lemma 5.3, we have for the second moment:

$$\begin{aligned}
&G_n\left((e_1-x)^2;x\right)\\
&\le \left|\frac{3}{2}-\frac{2n}{n+1}+\frac{1}{2}\cdot\frac{n(n-1)}{(n+1)(n+2)}\right|+\frac{1}{2}\left|1-\frac{n(n-1)}{(n+1)(n+2)}\right|\\
&\le \frac{3}{(n+1)(n+2)}+\frac{2n+1}{(n+1)(n+2)}\\
&\le \frac{2}{n+1}.
\end{aligned}$$

We also know

$$G_n(e_1;x)=\rho_{1,n}\cdot x=\frac{n}{n+1}\cdot x,$$

which implies that

$$G_n(e_1;x)-x=\frac{n}{n+1}\cdot x-x=x\cdot\left(\frac{n}{n+1}-1\right)=-x\cdot\frac{1}{n+1}.$$

So we have

$$\begin{aligned}
D(e_1,e_1)&=G_n((e_1-x)^2;x)-(G_n(e_1;x)-x)^2\\
&\le\frac{2}{n+1}-x^2\cdot\frac{1}{(n+1)^2}.
\end{aligned}$$

Taking this into account, we give the following Grüss-type inequality for the convolution operator with de La Vallée Poussin kernel:

Theorem 5.4. *For the convolution-type operator with the de La Vallée Poussin kernel, we have*

$$\begin{aligned}
|D(f,g)|&=|G_n(f\cdot g;x)-G_n(f;x)\cdot G_n(g;x)|\\
&\le\frac{1}{4}\cdot\widetilde{\omega}\left(f;2\cdot\sqrt{D(e_1,e_1)}\right)\cdot\widetilde{\omega}\left(g;2\cdot\sqrt{D(e_1,e_1)}\right)\\
&\le\frac{1}{4}\widetilde{\omega}\left(f;2\cdot\frac{\sqrt{2-\frac{x^2}{n+1}}}{\sqrt{n+1}}\right)\cdot\widetilde{\omega}\left(g;2\cdot\frac{\sqrt{2-\frac{x^2}{n+1}}}{\sqrt{n+1}}\right).
\end{aligned}$$

This result is a slight improvement of [9: Theorem 5.5].

REFERENCES

[1] ACU, A.—GONSKA, H.—RAŞA, I.: *Grüss-type and Ostrowski-type inequalities in approximation theory*, Ukraïn. Mat. Zh. **63** (2011), 723–740; Ukrainian Math. J. **63** (2011), 843–864.

[2] ANASTASSIOU, G. A.: *Advanced Inequalities*, World Scientific, Singapore, 2011.

[3] CAO, J.-D.—GONSKA, H.: *Approximation by Boolean sums of positive linear operators III: Estimates for some numerical approximation schemes*, Numer. Funct. Anal. Optim. **10** (1989), 643–672.
[4] DEVORE, R. A.: *The Approximation of Continuous Functions by Positive Linear Operators*. Lecture Notes in Math. 293, Springer, Berlin-Heidelberg-New York, 1972.
[5] GAVREA, B.—GAVREA, I.: *Ostrowski type inequalities from a linear functional point of view*, JIPAM. J. Inequal. Pure Appl. Math. **1** (2000), Article 11.
[6] LEHNHOFF, H. G.: *A simple proof of A. F. Timan's theorem*, J. Approx. Theory **38** (1983), 172–176.
[7] MITJAGIN, B. S.—SEMENOV, E. M.: *Lack of interpolation of linear operators in spaces of smooth functions*, Izv. Math. USSR **11** (1977), 1229–1266.
[8] MITRINOVIĆ, D. S.—PEČARIĆ, J. E.—FINK, A. M.: *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.
[9] RUSU, M.-D.: *On Grüss-type inequalities for positive linear operators*, Studia Univ. Babeş-Bolyai Math. **56** (2011), 551–565.

Received 30. 3. 2011
Accepted 27. 2. 2012

* *University of Duisburg-Essen*
Faculty of Mathematics
Forsthausweg 2
D–47057 Duisburg
GERMANY
E-mail: heiner.gonska@uni-due.de
maria.rusu@uni-due.de

** *Technical University*
Department of Mathematics
Str. C. Daicoviciu, 15
RO-400020 Cluj-Napoca
ROMANIA
E-mail: Ioan.Rasa@math.utcluj.ro