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ON ASSOCIATED AND CO-RECURSIVE d-ORTHOGONAL POLYNOMIALS

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ABSTRACT. The associated sequence of order r for a given d-OPS (i.e. a sequence of orthogonal polynomials satisfying a $(d+1)$-order recurrence relation), is again a d-OPS. In this paper we are interested in the determination of the corresponding dual sequence. The explicit form of the dual sequence of the first associated sequence and the corresponding formal Stieltjes function are given. Indeed, we construct by recurrence the dual sequence of the r-associated sequence and we give some properties of the corresponding Stieltjes function. Second, we give the definition of co-recursive polynomials of dimension d and some relations in the particular cases $d=3$ and $d=4$. Some properties of the dual sequence as well as of the corresponding Stieltjes functions are given.

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1. Preliminaries and notations

Let $\mathcal{P}$ be the linear space of complex polynomials in one variable and $\mathcal{P}'$ its topological dual space. We denote by $\langle u, p\rangle$ the action of $u \in \mathcal{P}'$ on $p \in \mathcal{P}$. In particular, we denote by $(u)_n = \langle u, x^n\rangle$, $n \geq 0$, the moments of u.

Let $\{P_n\}_{n\geq 0}$ be a sequence of monic polynomials with $\deg P_n = n$, $n \geq 0$. Its dual sequence $\{u_n\}_{n\geq 0}$, $u_n \in \mathcal{P}'$, is defined by $\langle u_n, P_m\rangle := \delta_{n,m}$, $n, m \geq 0$. Let us recall the following result [6–8]

Lemma 1. *For any $u \in \mathcal{P}'$ and any integer $p \geq 1$, the following statements are equivalent:*

(a) $\langle u, P_{p-1}\rangle \neq 0$, $\langle u, P_n\rangle = 0$, $n \geq p$,

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(b) $\exists \lambda_\nu \in \mathbb{C},\ \ 0 \le \nu \le p-1,\ \ \lambda_{p-1} \neq 0$ *such that*

$$u = \sum_{\nu=0}^{p-1} \lambda_\nu u_\nu .$$

The form u is called regular if we can associate with it a sequence $\{P_n\}_{n\ge 0}$ such that

$$\langle u, P_m P_n \rangle := r_n \delta_{n,m}, \qquad n,m \ge 0,\ \ r_n \neq 0,\ \ n \ge 0.$$

The sequence $\{P_n\}_{n\ge 0}$ is said to be orthogonal with respect to u. Necessarily, $u = \lambda u_0$, $\lambda \neq 0$. In this case, we have [7]

$$u_n = \left(\left\langle u_0, P_n^2 \right\rangle\right)^{-1} P_n u_0, \qquad n \ge 0. \tag{1}$$

For a linear form u, let $S(u)$ be its Stieltjes function defined by

$$S(u)(z) = -\sum_{n\ge 0} \frac{(u)_n}{z^{n+1}}.$$

For any polynomial π and any $c \in \mathbb{C}$, we can define the following forms $Du = u'$, πu and δ_c by

$$\langle u', f \rangle := -\langle u, f' \rangle, \quad \langle \pi u, f \rangle := \langle u, \pi f \rangle, \quad \langle \delta_c, f \rangle := f(c), \qquad f \in \mathcal{P}.$$

If u is regular and A is a polynomial such that $Au = 0$, then $A = 0$ [3,8].

Next, for each $P(x) = \sum\limits_{\nu=0}^{m} a_\nu x^\nu$, we recall the right-multiplication of a form by a polynomial

$$\begin{aligned}(uP)(x) &= \sum_{n=0}^{m} \left(\sum_{\nu=n}^{m} a_\nu (u)_{\nu-n} \right) x^n \\ &= \left\langle u, \frac{xP(x) - \xi P(\xi)}{x-\xi} \right\rangle, \quad u \in \mathcal{P}'.\end{aligned} \tag{2}$$

The transposed $P \to uP$ allows us to define the product of two forms by

$$\langle uv, P \rangle = \langle u, vP \rangle, \qquad u, v \in \mathcal{P}',\ \ P \in \mathcal{P}. \tag{3}$$

Notice that the sequence of moments of uv is the discrete convolution of the sequence of moments of u and v.

The functional u has an inverse element u^{-1} if and only if $(u)_0 \neq 0$ and then $uu^{-1} = \delta$. In this case, if we put $v = u^{-1}$, then

$$(uv)_n = (u)_0 (v)_n + \sum_{\nu=1}^{n} (u)_\nu (v)_{n-\nu} = 0, \qquad n \ge 1$$

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we get

$$(v)_n = -\frac{1}{(u)_0} \sum_{\nu=1}^{n} (u)_\nu (v)_{n-\nu}, \qquad n \geq 1. \tag{4}$$

The division of a form by polynomials $\pi^{-1}u$, is defined by

$$\langle \pi^{-1}u, q \rangle = \left\langle u, \frac{q(x) - L(x;q)}{\pi(x)} \right\rangle, \qquad q \in \mathcal{P},$$

where $L(x;q)$ denotes the interpolation polynomial of q in the zeros of π taking into account the multiplicity.

For each $\lambda \in \mathbb{C}$, let us consider the operator θ_λ defined by

$$(\theta_\lambda p)(x) = \frac{p(x) - p(\lambda)}{x - \lambda}, \qquad p \in \mathcal{P}. \tag{5}$$

The following results are fundamental.

Lemma 2. ([7–9]) *For any $p \in \mathcal{P}$ and any $u, v \in \mathcal{P}'$, and $\lambda \in \mathbb{C}$, we have*

(i) $p(x)(uv) = (p(x)u)v + x(u\theta_0 p)(x)v,$

(ii) $p\left((x-\lambda)^{-1}u\right) = p(\lambda)\left((x-\lambda)^{-1}u\right) + (\theta_\lambda p)u,$

(iii) $x^{-n}u = (-1)^n (\delta')^n u, \ \ n \geq 0.$

For the Stieltjes function, we recall some of its properties.

Lemma 3. ([3,7,9]) *For any $p \in \mathcal{P}$ and any $u, v \in \mathcal{P}'$, we have*

(a) $S(uv)(z) = -zS(u)(z)S(v)(z),$

(b) $S(pu)(z) = p(z)S(u)(z) + (u\theta_0 p)(z),$

(c) $S(x^{-n}u)(z) = z^{-n}S(u)(z)$ *for all* $n \geq 1.$

2. d-orthogonality

Let us consider d forms $\Gamma^1, \Gamma^2, \ldots, \Gamma^d$ $(d \geq 1)$.

Definition 4. ([6]) The sequence $\{P_n\}_{n\geq 0}$ is d-orthogonal polynomials sequence, or simply d-OPS with respect to $\Gamma = (\Gamma^1, \Gamma^2, \ldots, \Gamma^d)^T$ if

$$\begin{aligned} \langle \Gamma^\alpha, P_m P_n \rangle &= 0, \qquad n \geq md + \alpha, \quad m \geq 0, \\ \langle \Gamma^\alpha, P_m P_{md+\alpha-1} \rangle &\neq 0, \qquad m \geq 0, \end{aligned} \tag{6}$$

for each $1 \leq \alpha \leq d$.

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Remark 5.

(a) In this case, the d-dimensional form Γ is called regular.

(b) Γ is not unique. We deduce from Lemma 1

$$\Gamma^{\alpha} = \sum_{\nu=0}^{\alpha-1} \lambda_{\nu} u_{\nu}, \qquad \text{with} \quad \lambda_{\alpha-1}^{\alpha} \neq 0, \quad 1 \leq \alpha \leq d$$

or, equivalently

$$u_{\nu} = \sum_{\sigma=1}^{\nu} \zeta_{\sigma}^{\nu} \Gamma^{\sigma}, \qquad \text{with} \quad \zeta_{\sigma}^{\sigma} \neq 0, \quad 0 \leq \nu \leq d-1.$$

Therefore, from now on, we shall work uniquely with the dual functionals $U = (u_0, \ldots, u_{d-1})^T$. In this case, by [5, 6], the sequence $\{P_n\}_{n\geq 0}$ is d-OPS if and only if

$$\langle u_{\alpha}, x^m P_n(x) \rangle = 0, \qquad n \geq md + \alpha + 1, \quad m \geq 0,$$
$$\langle u_{\alpha}, x^m P_{md+\alpha}(x) \rangle \neq 0, \qquad m \geq 0,$$

for each $0 \leq \alpha \leq d-1$.

Let us now recall some characterizations which we need below.

Theorem 6. ([6]) *Let $\{P_n\}_{n\geq 0}$ be a monic sequence of polynomials, then the following statements are equivalent*

(a) *The sequence $\{P_n\}_{n\geq 0}$ is d-OPS with respect to $\mathcal{U} = (u_0, \ldots, u_{d-1})^T$.*

(b) *The sequence $\{P_n\}_{n\geq 0}$ satisfies a recurrence relation of order $d+1$ $(d \geq 1)$:*

$$\begin{aligned} &P_{m+d+1}(x) \\ &= (x - \beta_{m+d}) P_{m+d}(x) - \sum_{\nu=0}^{d-1} \gamma_{m+d-\nu}^{d-1-\nu} P_{m+d-1-\nu}(x), \qquad m \geq 0, \end{aligned} \tag{7}$$

with the initial data

$$\begin{aligned} &P_0(x) = 1, \quad P_1(x) = x - \beta_0 \\ &P_m(x) = (x - \beta_{m-1}) P_{m-1}(x) - \sum_{\nu=0}^{m-2} \gamma_{m-1-\nu}^{d-1-\nu} P_{m-2-\nu}(x), \qquad 2 \leq m \leq d \end{aligned} \tag{8}$$

and the regularity conditions $\gamma_{m+1}^{0} \neq 0, \quad m \geq 0$.

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(c) *For each* (n,ν), $n\geq 0$, $0\leq \nu\leq d-1$, *there exist* d *polynomials* $\Lambda^{\mu}(n,\nu)$, $0\leq \mu\leq d-1$ *such that*

$$u_{nd+\nu}=\sum_{\mu=0}^{d-1}\Lambda^{\mu}(n,\nu)\,u_{\mu},\qquad n\geq 0,\quad 0\leq \nu\leq d-1, \tag{9}$$

and verifying

$$\begin{aligned}
&\deg\Lambda^{\nu}(n,\nu)=n, && 0\leq\nu\leq d-1, && \textit{and if}\ \ d\geq 2,\\
&\deg\Lambda^{\mu}(n,\nu)\leq n, && 0\leq\mu\leq\nu-1, && \textit{if}\ \ 1\leq\nu\leq d-1\\
&\deg\Lambda^{\mu}(n,\nu)\leq n-1, && \nu+1\leq\mu\leq d-1, && \textit{if}\ \ 0\leq\nu\leq d-2.
\end{aligned} \tag{10}$$

Notice that the coefficients $\{\beta_m\}$ and $\left\{\gamma^{i}_{m+1}\right\}$ are given by [6]

$$\begin{aligned}
\beta_{\upsilon}&=\langle u_{\upsilon},xP_{\upsilon}\rangle, && 0\leq\upsilon\leq d-1,\\
\gamma^{\upsilon}_{n+1+\upsilon}&=\langle u_{n+\upsilon},xP_{n+d}\rangle, && 0\leq\upsilon\leq d-1,\quad n\geq 0.
\end{aligned} \tag{11}$$

Using the recurrence relations (7)–(8) we obtain by direct calculations the following lemma

Lemma 7. *For any d-OPS satisfying the recurrence relations* (7)–(8)*, the first moments are*

$$\begin{aligned}
(u_n)_{n+1}&=\beta_0+\beta_1+\cdots+\beta_n,\qquad n\geq 0,\\
(u_1)_3&=\beta_0^2+\beta_1^2+\beta_0\beta_1+\gamma_1^{d-1}+\gamma_2^{d-1},\\
(u_0)_3&=\beta_0^3+\gamma_1^{d-1}(2\beta_0+\beta_1)+\gamma_1^{d-2}
\end{aligned}$$

and

$$(u_0)_2=\beta_0^2+\gamma_1^{d-1}.$$

Proposition 8. ([8]) *If* (a) *or* (b) *or* (c) *of the previous* Theorem 6 *is fulfilled, then*

$$xu_n=u_{n-1}+\beta_n u_n+\sum_{\nu=0}^{d-1}\gamma_{n+1}^{d-1-\nu}u_{n+1+\nu},\qquad n\geq 0\quad (u_{-1}=0). \tag{12}$$

The dual elements of the relation (9) can be explicitly determined. In fact, expanding (9) in term of elements of the basis and using the fact that if U is regular and A is a polynomial matrix such that $AU=0$, then $A=0$, we get a system of recurrence allow us to describe explicitly the polynomials $\Lambda^{\mu}(n,\nu)$ in the relation (9).

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3. Associated sequences

Now we recall the definition of the associated sequence and some of properties.

Definition 9. ([6,8]) We call the associated sequence of $\{P_n\}_{n\geq 0}$ (with respect to u_0), the sequence $\{P_n^{(1)}\}_{n\geq 0}$ defined by

$$P_n^{(1)}(x) = \left\langle u_0, \frac{P_{n+1}(x) - P_{n+1}(\xi)}{x-\xi} \right\rangle, \qquad n \geq 0. \tag{13}$$

$P_n^{(1)}$ is a monic polynomial of degree n. Furthermore, from (2), we have

$$P_n^{(1)}(x) = (u_0\theta_0 P_{n+1})(x), \qquad n \geq 0. \tag{14}$$

When $\{P_n\}_{n\geq 0}$ is orthogonal with respect to u_0, it verifies the recurrence relation

$$\begin{aligned} &P_0(x) = 1, \quad P_1(x) = x - \beta_0, \\ &P_{n+2}(x) = (x-\beta_{n+1})\,P_{n+1}(x) - \gamma_{n+1}P_n(x), \qquad n \geq 0 \end{aligned} \tag{15}$$

with the regularity condition $\gamma_{n+1} \neq 0$, $n \geq 0$, in this case the sequence $\{P_n^{(1)}\}_{n\geq 0}$ verifies the recurrence relation

$$\begin{aligned} &P_0^{(1)}(x) = 1, \quad P_1^{(1)}(x) = x - \beta_1, \\ &P_{n+2}^{(1)}(x) = (x-\beta_{n+2})\,P_{n+1}^{(1)}(x) - \gamma_{n+2}P_n^{(1)}(x), \qquad n \geq 0. \end{aligned} \tag{16}$$

Let us denote by $\{u_n^{(1)}\}_{n\geq 0}$ the dual sequence of $\{P_n^{(1)}\}_{n\geq 0}$. In this case, $u_n^{(1)}$ satisfies the following statements

Lemma 10. ([8])

(1) $u_n^{(1)} = (xu_{n+1})\,u_0^{-1}$, $n \geq 0$.

(2) $\gamma_1 u_0^{(1)} = -x^2 u_0^{-1}$,

(3) $\gamma_1 S\left(u_0^{(1)}\right)(z) = -\dfrac{1}{S(u_0)(z)} - (z-\beta_0)$.

Now the successive associated sequences are defined by the recurrence [8]

$$P_n^{(r+1)}(x) = \left(P_n^{(r)}(x)\right)^{(1)} \quad \text{and} \quad u_n^{(r+1)} = \left(u_n^{(r)}\right)^{(1)}, \qquad n, r \geq 0, \tag{17}$$

with $P_n^{(0)} = P_n$ and $u_0^{(0)} = u_0$. That is to say

$$\begin{aligned} P_n^{(r+1)}(x) &= \left(u_0^{(r)}\theta_0 P_{n+1}^{(r)}\right)(x); \\ u_n^{(r+1)} &= \left(xu_{n+1}^{(r)}\right)\left(u_0^{(r)}\right)^{-1}, \qquad n, r \geq 0, \end{aligned} \tag{18}$$

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$$\beta_n^{(r)} = \beta_{n+r}, \quad \gamma_{n+1}^{(r)} = \gamma_{n+1+r}, \qquad n, r \geq 0. \tag{19}$$

The sequence of polynomials and the corresponding associated sequence are connected by [8]

$$P_n^{(r+1)}(x) = (u_r \theta_0 P_{n+r+1})(x), \qquad n, r \geq 0. \tag{20}$$

When $\{P_n\}_{n\geq 0}$ is orthogonal, by (1) we have $r_n u_n = P_n u_0$, $n \geq 0$ with $r_n = \gamma_0 \ldots \gamma_n$ ($\gamma_0 = 1$) and then [1, 2]

$$\left(\prod_{\nu=0}^{r} \gamma_\nu\right) P_n^{(r+1)}(x) = \left\langle u_0, P_r(\xi) \frac{P_{n+r+1}(x) - P_{n+r+1}(\xi)}{x - \xi} \right\rangle. \tag{21}$$

Likewise, by [5], if $\{P_n\}_{n\geq 0}$ is d-OPS, it satisfies (7)–(8). Then $\{P_n^{(1)}\}_{n\geq 0}$ also satisfies the $(d+1)$-order recurrence relation

$$\begin{aligned} &P_{m+d+1}^{(1)}(x) \\ &= (x - \beta_{m+d+1}) P_{m+d}^{(1)}(x) - \sum_{\nu=0}^{d-1} \gamma_{m+d+1-\nu}^{d-1-\nu} P_{m+d-1-\nu}^{(1)}(x), \qquad m \geq 0, \end{aligned} \tag{22}$$

with the initial conditions

$$\begin{aligned} &P_0^{(1)}(x) = 1, \\ &P_1^{(1)}(x) = x - \beta_1, \\ &P_m^{(1)}(x) = (x - \beta_m) P_{m-1}^{(1)}(x) - \sum_{\nu=0}^{m-2} \gamma_{m-\nu}^{d-1-\nu} P_{m-2-\nu}^{(1)}(x), \quad 2 \leq m \leq d. \end{aligned} \tag{23}$$

Thus, we have the following generalization of the Maroni result given in [8].

Theorem 11. *When $\{P_n\}_{n\geq 0}$ is d-OPS with respect to $\mathcal{U} = (u_0, \ldots, u_{d-1})^T$, then $\{P_n^{(1)}\}_{n\geq 0}$ is d-OPS with respect to $\mathcal{U}^{(1)} = (u_0^{(1)}, \ldots, u_{d-1}^{(1)})^T$ where*

$$\begin{cases} u_\nu^{(1)} = x\left(u_{\nu+1} u_0^{-1}\right), \qquad 0 \leq \nu \leq d-2, \quad d \geq 2, \\ \gamma_1^0 u_{d-1}^{(1)} = -x^2 u_0^{-1} - \sum_{\nu=0}^{d-2} \gamma_1^{d-1-\nu} x\left(u_{\nu+1} u_0^{-1}\right). \end{cases} \tag{24}$$

Proof. From the first assertion of Lemma 10, we have

$$u_\nu^{(1)} = (x u_{\nu+1}) u_0^{-1}, \quad 0 \leq \nu \leq d-2, \qquad \text{and} \qquad u_{d-1}^{(1)} = (x u_d) u_0^{-1},$$

and from the equation (12), we obtain for $n = 0$

$$\gamma_1^0 u_d = P_1(x) u_0 - \sum_{\nu=0}^{d-2} \gamma_1^{d-1-\nu} u_{\nu+1},$$

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then

$$\gamma_1^0 u_{d-1}^{(1)} = \left[x \left(P_1(x) u_0 - \sum_{\nu=0}^{d-2} \gamma_1^{d-1-\nu} u_{\nu+1} \right) \right] u_0^{-1}.$$

Using the first assertion of Lemma 2, to get

$$\gamma_1^0 u_{d-1}^{(1)} = x P_1(x) \delta - x \left(u_0 \theta_0 \xi P_1(\xi) \right)(x) u_0^{-1} - \sum_{\nu=0}^{d-2} \gamma_1^{d-1-\nu} x \left(u_{\nu+1} u_0^{-1} \right),$$

so by (2) and (5)) we obtain $(u_0 \theta_0 \xi P_1(\xi)) = x$ and then (24). □

An equivalent result of the Theorem 11 with the aid of the formal Stieltjes function is the following.

Theorem 12. *For $d \geq 2$, we have*

$$S\left(u_\nu^{(1)}\right)(z) = -\frac{S(u_{\nu+1})(z)}{S(u_0)(z)}, \qquad 0 \leq \nu \leq d-2, \tag{25}$$

$$\gamma_1^0 S\left(u_{d-1}^{(1)}\right)(z) = \sum_{\nu=0}^{d-2} \frac{\gamma_1^{d-1-\nu} S(u_{\nu+1})(z)}{S(u_0)(z)} - \frac{1}{S(u_0)(z)} - P_1(z). \tag{26}$$

P r o o f. For $0 \leq \nu \leq d-2$, $d \geq 2$, we have from (24) and the Lemma 3

$$\begin{aligned} S\left(u_\nu^{(1)}\right)(z) &= S\left(z\left(u_{\nu+1}u_0^{-1}\right)\right)(z) \\ &= zS\left(u_{\nu+1}u_0^{-1}\right)(z) + \left(u_{\nu+1}u_0^{-1}\theta_0 z\right)(z) \\ &= -z^2 S(u_{\nu+1})(z) S\left(u_0^{-1}\right)(z) \\ &= -\frac{S(u_{\nu+1})(z)}{S(u_0)(z)}, \qquad 0 \leq \nu \leq d-2, \end{aligned}$$

because $\left(u_{\nu+1}u_0^{-1}\right)_0 = 0$ and $S(\delta)(z) = -\dfrac{1}{z} = -zS(u_0)(z) S\left(u_0^{-1}\right)(z)$. And for $\nu = d-1$, we have

$$\begin{aligned} \gamma_1^0 S\left(u_{d-1}^{(1)}\right)(z) &= -S\left(z^2 u_0^{-1}\right)(z) - \sum_{\nu=0}^{d-2} \gamma_1^{d-1-\nu} S\left(z\left(u_{\nu+1}u_0^{-1}\right)\right)(z) \\ &= -z^2 S\left(u_0^{-1}\right)(z) - \left(u_0^{-1}\theta_0 z^2\right)(z) + \sum_{\nu=0}^{d-2} \gamma_1^{d-1-\nu} \frac{S(u_{\nu+1})(z)}{S(u_0)(z)} \\ &= -\frac{1}{S(u_0)(z)} - (z - \beta_0) + \sum_{\nu=0}^{d-2} \gamma_1^{d-1-\nu} \frac{S(u_{\nu+1})(z)}{S(u_0)(z)}, \end{aligned}$$

because of the relations (2) and (4) we have

$$\left(u_0^{-1}\theta_0 z^2\right) = \left(u_0^{-1} z\right) = z + \left(u_0^{-1}\right)_1 = z - \beta_0.$$ □

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Consequently, by recurrence, we can define the successive associated sequences of dimension d, $\{P_n^{(r)}\}_{n\geq 0}$, $r \geq 1$ by [8]

$$\begin{aligned} &P_{m+d+1}^{(r)}(x) \\ &= (x-\beta_{m+d+r}) P_{m+d}^{(r)}(x) - \sum_{\nu=0}^{d-1} \gamma_{m+d+r-\nu}^{d-1-\nu} P_{m+d-1-\nu}^{(r)}(x), \qquad m \geq 0, \end{aligned} \tag{27}$$

with the initial conditions

$$P_0^{(r)}(x) = 1, \quad P_1^{(r)}(x) = x - \beta_r,$$

$$P_m^{(r)}(x) = (x-\beta_{m+r-1}) P_{m-1}^{(r)}(x) - \sum_{\nu=0}^{m-2} \gamma_{m+r-1-\nu}^{d-1-\nu} P_{m-2-\nu}^{(r)}(x), \quad 2 \leq m \leq d, \tag{28}$$

and we have the following recursive result between the elements of the dual sequence.

Theorem 13. *When $\{P_n\}_{n\geq 0}$ is d-OPS with respect to $\mathcal{U} = (u_0, \ldots, u_{d-1})^T$, then $\{P_n^{(r)}\}_{n\geq 0}$ is d-OPS with respect to $\mathcal{U}^{(r)} = (u_0^{(r)}, \ldots, u_{d-1}^{(r)})^T$ given by* (18)–(20). *In addition $u_n^{(r+1)}$ satisfies the following relation*

$$u_n^{(r+1)} = x^{r+1}\left(u_{n+r+1}u_0^{-1}\right)\left(u_0^{(1)}u_0^{(2)}\ldots u_0^{(r)}\right)^{-1}, \qquad n, r \geq 0. \tag{29}$$

Proof. From (18), we have for $r = 1$

$$u_n^{(2)}u_0^{(1)} = xu_{n+1}^{(1)}, \qquad n \geq 0.$$

From the first statement of Proposition 10 and Lemma 2 we get

$$\begin{aligned} u_n^{(2)}u_0^{(1)} &= xu_{n+1}^{(1)} = x\left[(xu_{n+2})\, u_0^{-1}\right] = x^2\left(u_{n+2}u_0^{-1}\right) \qquad \text{or} \\ u_n^{(2)} &= x^2\left(u_{n+2}u_0^{-1}\right)\left(u_0^{(1)}\right)^{-1}. \end{aligned} \tag{30}$$

For $r = 2$, we have from (18)

$$u_n^{(3)}u_0^{(2)} = xu_{n+1}^{(2)}, \qquad n \geq 0.$$

(30) yields

$$u_n^{(3)}u_0^{(2)}u_0^{(1)} = x\left[x^2\left(u_{n+3}u_0^{-1}\right)\right] = x^3\left(u_{n+3}u_0^{-1}\right), \qquad n \geq 0.$$

Thus we obtain (29) by recurrence. □

Remark 14. Using Lemma 2 and (23), we can write relation (29) as

$$u_n^{(r+1)} = x^r\left(u_{n+r}^{(1)}\right)\left(u_0^{(1)}u_0^{(2)}\ldots u_0^{(r)}\right)^{-1}, \qquad n, r \geq 0. \tag{31}$$

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Or in the following form

$$u_n^{(r+1)} = x^{r+1}\left(u_{n+r+1}\right)\left(u_0^{(0)}u_0^{(1)}u_0^{(2)}\dots u_0^{(r)}\right)^{-1}, \qquad n, r \geq 0. \tag{32}$$

In terms of the formal Stieltjes function we have.

Corollary 15. *For all n, $r \geq 0$, the Stieltjes function of the associated sequence satisfies*

$$S\left(u_n^{(r+1)}\right)(z) = -\frac{S\left(u_{n+1}^{(r)}\right)(z)}{S\left(u_0^{(r)}\right)(z)}, \tag{33}$$

$$(-1)^{r+1} S\left(u_n^{(r+1)}\right)(z) = \frac{S\left(u_{n+r+1}\right)(z)}{S\left(u_0\right)(z)\, S\left(u_0^{(1)}\right)(z)\, S\left(u_0^{(2)}\right)(z)\dots S\left(u_0^{(r)}\right)(z)}. \tag{34}$$

P r o o f. Using (18) and Lemma 3, we obtain (33). Applying the formal Stieltjes function to the form

$$u_n^{(r+1)}u_0^{(1)}u_0^{(2)}\dots u_0^{(r)} = x^{r+1}\left(u_{n+r+1}u_0^{-1}\right).$$

Then, the left hand side becomes

$$S\left(u_n^{(r+1)}u_0^{(1)}u_0^{(2)}\dots u_0^{(r)}\right)(z) = (-z)^r\, S\left(u_n^{(r+1)}\right)(z)\prod_{k=1}^{r} S\left(u_0^{(k)}\right)(z),$$

and the right yields

$$S\left(z^{r+1}\left(u_{n+r+1}u_0^{-1}\right)\right)(z) = -z^r\frac{S\left(u_{n+r+1}\right)(z)}{S\left(u_0\right)(z)}.$$

Whence (34). □

If we substitute the relation (33) in the denominator of (34), we get by recurrence the following relation

Corollary 16. *For all n, $r \geq 0$, the Stieltjes function of the associated sequence satisfies*

$$S\left(u_n^{(r+1)}\right)(z) = -\frac{S\left(u_{n+r+1}\right)(z)}{S\left(u_r\right)(z)}. \tag{35}$$

Using the relation (32), we can determine the elements of the dual sequence explicitly as in Theorem 11. In fact

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Corollary 17. *For $r=1$ and $d\geq 3$ we obtain*

$$\begin{aligned} u_{\upsilon}^{(2)}u_0^{(1)} &= \left(x^2u_{\upsilon+2}\right)u_0^{-1}, \qquad 0\leq \upsilon\leq d-3,\\ \gamma_1^0u_{d-2}^{(2)}u_0^{(1)} &= -x\left(x^2+\gamma_1^{d-1}\right)u_0^{-1}-\left[\sum_{\nu=0}^{d-2}\gamma_1^{d-1-\nu}\left(x^2u_{\nu+1}\right)\right]u_0^{-1},\\ \gamma_1^0\gamma_2^0u_{d-1}^{(2)}u_0^{(1)} &= \gamma_1^0x^2P_1^{(1)}\left(u_1u_0^{-1}\right)+\gamma_2^1x\left(x^2+\gamma_1^{d-1}\right)u_0^{-1}\\ &\quad+\gamma_2^1\left[\sum_{\nu=0}^{d-2}\gamma_1^{d-1-\nu}\left(x^2u_{\nu+1}\right)\right]u_0^{-1}\\ &\quad-\gamma_1^0\left[\sum_{\nu=0}^{d-3}\gamma_2^{d-1-\nu}\left(x^2u_{\nu+2}\right)\right]u_0^{-1}. \end{aligned} \tag{36}$$

P r o o f. For $r=1$, we obtain from (32)

$$u_{\upsilon}^{(2)}u_0^{(1)}=x^2\left(u_{\upsilon+2}u_0^{-1}\right), \qquad \text{for} \quad 0\leq\upsilon\leq d-3$$

and when $\upsilon=d-2$, using the relation (12) with $n=0$, we get

$$\gamma_1^0u_{d-2}^{(2)}u_0^{(1)}=x^2P_1\delta-x\left(u_0\theta_0\xi^2P_1\left(\xi\right)\right)(x)\,u_0^{-1}-\left[\sum_{\nu=0}^{d-2}\gamma_1^{d-1-\nu}\left(x^2u_{\nu+1}\right)\right]u_0^{-1},$$

and by (2) and Lemma 7 we have

$$\left(u_0\theta_0\xi^2P_1\left(\xi\right)\right)(x)=x^2+\gamma_1^{d-1}.$$

For $\upsilon=d-1$, we have

$$u_{d-1}^{(2)}u_0^{(1)}=\left(x^2u_{d+1}\right)u_0^{-1},$$

and for $n=1$ we obtain from (12)

$$\gamma_2^0u_{d+1}=P_1^{(1)}u_1-u_0-\gamma_2^1u_d-\sum_{\nu=0}^{d-3}\gamma_2^{d-1-\nu}u_{\nu+2}$$

$$\gamma_1^0\gamma_2^0u_{d+1}=\gamma_1^0P_1^{(1)}u_1-\gamma_1^0u_0-\gamma_2^1\left(\gamma_1^0u_d\right)-\gamma_1^0\left[\sum_{\nu=0}^{d-3}\gamma_2^{d-1-\nu}u_{\nu+2}\right],$$

hence

$$\begin{aligned}\gamma_1^0\gamma_2^0u_{d-1}^{(2)}u_0^{(1)} &= \gamma_1^0x^2\left[P_1^{(1)}u_1-u_0\right]u_0^{-1}-\gamma_2^1x^2\left[P_1u_0-\sum_{\nu=0}^{d-2}\gamma_1^{d-1-\nu}u_{\nu+1}\right]u_0^{-1}\\ &\quad-\gamma_1^0\left[\sum_{\nu=0}^{d-3}\gamma_2^{d-1-\nu}\left(x^2u_{\nu+2}\right)\right]u_0^{-1},\end{aligned}$$

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where

$$x^2\left[P_1^{(1)}u_1\right]u_0^{-1}=x^2P_1^{(1)}\left(u_1u_0^{-1}\right)-x\left(u_1\theta_0\xi^2P_1^{(1)}(\xi)\right)(x)\,u_0^{-1}$$

$$\left(x^2u_0\right)u_0^{-1}=x^2\left(u_0u_0^{-1}\right)-x\left(u_0\theta_0\xi^2\right)(x)\,u_0^{-1}=-x\left(u_0x\right)u_0^{-1},$$

and

$$\left(u_1\theta_0\xi^2P_1^{(1)}(\xi)\right)=\left(u_1xP_1^{(1)}(x)\right)=\sum_{n=0}^{2}\left[\sum_{\nu=n}^{2}a_\nu\left(u_1\right)_{\nu-n}\right]x^n=x+\beta_0$$

$$\left(u_0x\right)=a_1\left(u_0\right)_1+a_1\left(u_0\right)_0x=x+\beta_0.$$

Moreover, we obtain $u_{d-1}^{(2)}$ by substitution. □

Remark 18. Using the fact $x\left(u_1u_0^{-1}\right)=u_0^{(1)}$ in the previous corollary we get

$$\begin{aligned}\gamma_1^0\gamma_2^0u_{d-1}^{(2)}&=\gamma_2^1x\left[x^2+\gamma_1^{d-1}\right]\left(u_0u_0^{(1)}\right)^{-1}\\&+\gamma_2^1\left[\sum_{\nu=0}^{d-2}\gamma_1^{d-1-\nu}\left(x^2u_{\nu+1}\right)\right]\left(u_0u_0^{(1)}\right)^{-1}\\&-\gamma_1^0\left[\sum_{\nu=0}^{d-3}\gamma_2^{d-1-\nu}\left(x^2u_{\nu+2}\right)\right]\left(u_0u_0^{(1)}\right)^{-1}.\end{aligned}$$

4. Co-recursive sequences

Let now turn to the so called co-recursive sequences. We call co-recursive sequence of the orthogonal sequence $\{P_n\}_{n\geq 0}$, any sequence $\{Q_n\}_{n\geq 0}$ given by the following recurrence

$$\begin{aligned}&Q_0(x)=1,\qquad Q_1(x)=x-\beta_0-\mu_0,\\&Q_{n+2}(x)=\left(x-\beta_{n+1}\right)Q_{n+1}(x)-\gamma_{n+1}Q_n(x),\qquad n\geq 0.\end{aligned}\tag{37}$$

It is known [4], that $Q_n(x)=P_n(x)-\mu_0P_{n-1}^{(1)}(x)$.

From now on, let denote by $\{{}^cu_n\}_{n\geq 0}$ the dual sequence of the co-recursive polynomials $\{Q_n\}_{n\geq 0}$. In this case, we have

$$S\left({}^cu_0\right)(z)=\left[\frac{1}{S\left(u_0\right)(z)}+\mu_0\right]^{-1}=\frac{S\left(u_0\right)(z)}{1+\mu_0S\left(u_0\right)(z)},\tag{38}$$

and $\left({}^cu_0\right)^{(1)}=u_0^{(1)}$.

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Using Lemma 10, we obtain from the last expression and from the relation (4) and third statement of Lemma 2 the following result

$$ {}^{c}u_0 = \left[u_0^{-1} + \mu_0 \delta'\right]^{-1} = u_0 \left[\delta - \mu_0 x^{-1} u_0\right]^{-1}. \tag{39}$$

Maroni [8] gives the definition of the co-recursive 2-orthogonal polynomials. The initial conditions of the sequence are given by

$$\begin{aligned} &Q_0(x) = 1, \\ &Q_1(x) = x - \beta_0 - \mu_0, \\ &Q_2(x) = (x - \beta_1 - \mu_1) Q_1(x) - \left(\gamma_1^1 + \mu_1^1\right), \end{aligned} \tag{40}$$

and the following result

$$Q_n(x) = P_n(x) - \mu_0 P_{n-1}^{(1)}(x) - \left\{\mu_1 (x - \beta_0 - \mu_0) + \mu_1^1\right\} P_{n-2}^{(2)}(x), \qquad n \geq 0. \tag{41}$$

In a general framework, the co-recursive sequence $\{Q_n\}_{n\geq 0}$ of the d-orthogonal polynomials $\{P_n\}_{n\geq 0}$ is defined by the following initial conditions

$$\begin{aligned} Q_0(x) &= 1, \\ Q_1(x) &= x - \beta_0 - \mu_0, \\ Q_m(x) &= (x - \beta_{m-1} - \mu_{m-1}) Q_{m-1}(x) \\ &\quad - \sum_{\nu=0}^{m-2} \left(\gamma_{m-1-\nu}^{d-1-\nu} + \mu_{m-1-\nu}^{d-1-\nu}\right) Q_{m-2-\nu}(x), \qquad 2 \leq m \leq d. \end{aligned} \tag{42}$$

In view of Chihara and Maroni results, we can state the definition of the co-recursive d-orthogonal polynomials as follow

Definition 19. Let $\{P_n\}_{n\geq 0}$ be a sequence of d-orthogonal polynomials. The sequence $\{Q_n\}_{n\geq 0}$ given by the following linear combination

$$Q_n(x) = P_n(x) - \sum_{k=1}^{d} A_{k-1}^k(x) P_{n-k}^{(k)}(x), \qquad d \geq 1, \quad n \geq 0, \tag{43}$$

where A_i^k is a polynomial with $\deg A_i^k = i$ for $i \geq 0$, is called a co-recursive sequence.

Remark 20. A way to determine the polynomials $\left\{A_{k-1}^k\right\}_{k=1}^d$, $d \geq 1$, in (43) is the initial conditions (42). For example, the particular cases $d = 3$ and $d = 4$, respectively, imply

$$\begin{aligned} Q_n(x) &= P_n(x) - \mu_0 P_{n-1}^{(1)}(x) - \left\{\mu_1 (x - \beta_0 - \mu_0) + \mu_1^2\right\} P_{n-2}^{(2)}(x) \\ &\quad - \left\{\mu_2 \left[(x - \beta_1 - \mu_1)(x - \beta_0 - \mu_0) - \left(\gamma_1^2 + \mu_1^2\right)\right]\right. \\ &\quad \left. + \mu_2^2 (x - \beta_0 - \mu_0) + \mu_1^1\right\} P_{n-3}^{(3)}(x), \qquad n \geq 0, \end{aligned} \tag{44}$$

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and

$$\begin{aligned}Q_n(x) = {}& P_n(x) - \mu_0 P^{(1)}_{n-1}(x) - \left\{\mu_1(x-\beta_0-\mu_0)+\mu_1^3\right\} P^{(2)}_{n-2}(x) \\ & - \left\{\left\{\mu_2\left[(x-\beta_1-\mu_1)(x-\beta_0-\mu_0)-\left(\gamma_1^3+\mu_1^3\right)\right]\right\}\right. \\ & \left. + \mu_2^3(x-\beta_0-\mu_0)+\mu_1^2\right\} P^{(3)}_{n-3}(x) \\ & - \left\{\mu_3\left[(x-\beta_2-\mu_2)\left\{(x-\beta_1-\mu_1)(x-\beta_0-\mu_0)-\left(\gamma_1^3+\mu_1^3\right)\right\}\right.\right. \\ & \left. - \left(\gamma_1^2+\mu_1^2\right) - (x-\beta_0-\mu_0)\left(\gamma_2^3+\mu_2^3\right)\right] \\ & + \mu_3^3\left[(x-\beta_1-\mu_1)(x-\beta_0-\mu_0)-\left(\gamma_1^3+\mu_1^3\right)\right] \\ & \left. + \mu_2^2(x-\beta_0-\mu_0)+\mu_1^1\right\} P^{(4)}_{n-4}(x), \qquad n \geq 0.\end{aligned} \tag{45}$$

Remark 21. From the definition of the associated sequence (13) and by Lemma 7, we can prove by recurrence that if $\{P_n\}_{n\geq 0}$ is a d-OPS where $\{Q_n\}_{n\geq 0}$ is the corresponding d-co-recursive sequence, then

$$Q_n^{(d)}(x) = P_n^{(d)}(x), \qquad d \geq 1, \quad n \geq 0. \tag{46}$$

Corollary 22. *We have*

$$\left({}^c u_\upsilon\right)^{(d)} = \alpha_\upsilon \left(u_\upsilon\right)^{(d)}, \qquad 0 \leq \upsilon \leq d-1 \tag{47}$$

where $\alpha_\upsilon = \left(\left({}^c u_\upsilon\right)^{(d)}\right)_\upsilon \left[\left(\left(u_\upsilon\right)^{(d)}\right)_\upsilon\right]^{-1}$.

P r o o f. Since $Q_\upsilon^{(d)}(x) = P_\upsilon^{(d)}(x)$, for $\upsilon \geq 0$, then there exist a constant α_υ such that

$$\left({}^c u_\upsilon\right)^{(d)} = \alpha_\upsilon \left(u_\upsilon\right)^{(d)}, \qquad \upsilon \geq 0,$$

where

$$\alpha_\upsilon = \left\langle \left({}^c u_\upsilon\right)^{(d)}, x^\upsilon \right\rangle \left[\left\langle \left(u_\upsilon\right)^{(d)}, x^\upsilon \right\rangle\right]^{-1} = 1.$$

□

We end our work with the following Theorem which gives another representation of the corresponding dual sequence of the co-recursive form.

Theorem 23. *The dual sequence* $\{{}^c u_\upsilon\}_{\upsilon\geq d}$ *of the d-co-recursive polynomials satisfies*

$${}^c u_{n+d}\left[{}^c u_0 \left({}^c u_0\right)^{(1)} \ldots \left({}^c u_0\right)^{(d-1)}\right]^{-1} = u_{n+d}\left[u_0 \left(u_0\right)^{(1)} \ldots \left(u_0\right)^{(d-1)}\right]^{-1}, \tag{48}$$

for $d \geq 1$ *and* $n \geq 0$.

Before giving the proof, we need the following lemma.

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Lemma 24. *For any linear form u, we have*

$$x^{-n}\left(x^{n}u\right)=u-\sum_{\upsilon=0}^{n-1}\frac{(-1)^{\upsilon}}{\upsilon!}(u)_{\upsilon}\,\delta^{(\upsilon)},\qquad n\geq 0. \tag{49}$$

Proof of Theorem 23. Since $\left({}^{c}u_{n}\right)^{(d)}=\left(u_{n}\right)^{(d)}$, $n\geq 0$, then, by relation (32) we obtain

$$x^{d}\left\{\left({}^{c}u_{n+d}\right)\left[\left({}^{c}u_{0}\right)\dots\left({}^{c}u_{0}\right)^{(d-1)}\right]^{-1}-\left(u_{n+d}\right)\left[\left(u_{0}\right)\dots\left(u_{0}\right)^{(d-1)}\right]^{-1}\right\}=0. \tag{50}$$

Therefore, we have by (49)

$$\begin{aligned}&\left({}^{c}u_{n+d}\right)\left[\left({}^{c}u_{0}\right)\dots\left({}^{c}u_{0}\right)^{(d-1)}\right]^{-1}\\&=\left(u_{n+d}\right)\left[\left(u_{0}\right)\dots\left(u_{0}\right)^{(d-1)}\right]^{-1}-\sum_{\upsilon=0}^{d-1}\frac{(-1)^{\upsilon}}{\upsilon!}\left\{\left({}^{c}u_{n+d}\right)\left[\left({}^{c}u_{0}\right)\dots\left({}^{c}u_{0}\right)^{(d-1)}\right]^{-1}\right.\\&\qquad\left.-\left(u_{n+d}\right)\left[\left(u_{0}\right)\dots\left(u_{0}\right)^{(d-1)}\right]^{-1}\right\}_{\upsilon}\delta^{(\upsilon)}.\end{aligned} \tag{51}$$

Taking into account all the moments vanish, then the statement follows. □

In term of the formal Stieltjes function

Corollary 25. *We have*

$$S\left({}^{c}u_{n+d}\right)(z)=S\left(u_{n+d}\right)(z)\,\frac{S\left({}^{c}u_{0}\right)(z)\,S\left(\left({}^{c}u_{0}\right)^{(1)}\right)(z)\dots S\left(\left({}^{c}u_{0}\right)^{(d-1)}\right)(z)}{S\left(u_{0}\right)(z)\,S\left(\left(u_{0}\right)^{(1)}\right)(z)\dots S\left(\left(u_{0}\right)^{(d-1)}\right)(z)}, \tag{52}$$

for $d\geq 1$ and $n\geq 0$.

Proof. Applying the formal Stieltjes function to the form

$${}^{c}u_{n+d}\left[u_{0}\left(u_{0}\right)^{(1)}\dots\left(u_{0}\right)^{(d-1)}\right]=u_{n+d}\left[{}^{c}u_{0}\left({}^{c}u_{0}\right)^{(1)}\dots\left({}^{c}u_{0}\right)^{(d-1)}\right],$$

and making use of the Lemma 3 we complete the proof. □

Corollary 26. *The formal Stieltjes function satisfies*

$$S\left({}^{c}u_{\upsilon+d}\right)(z)=S\left(u_{\upsilon+d}\right)(z)\,\frac{S\left({}^{c}u_{d-1}\right)(z)}{S\left(u_{d-1}\right)(z)},\qquad d\geq 1,\quad \upsilon\geq 0. \tag{53}$$

Proof. Making use of the relation (33) to get the result by recurrence. □

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