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COEFFICIENT INEQUALITIES FOR UNIVALENT STARLIKE FUNCTIONS

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ABSTRACT. Let $\mathcal{A}$ be the class of analytic functions in the unit disk $\mathbb{D}$ with the normalization $f(0) = f'(0) - 1 = 0$. In this paper the authors discuss necessary and sufficient coefficient conditions for $f \in \mathcal{A}$ of the form

$$\left(\frac{z}{f(z)}\right)^{\mu} = 1 + b_1 z + b_2 z^2 + \dots$$

to be starlike in $\mathbb{D}$ and more generally, starlike of some order β, $0 \le \beta < 1$. Here μ is a suitable complex number so that the right hand side expression is analytic in $\mathbb{D}$ and the power is chosen to be the principal power. A similar problem for the class of convex functions of order β is open.

1. Introduction and main results

Let $\mathcal{A}$ denote the family of all normalized analytic functions f $(f(0) = 0 = f'(0) - 1)$ defined on the unit disk $\mathbb{D} = \{z : \ |z| < 1\}$ of the complex plane $\mathbb{C}$, and

$$\mathcal{S} = \{f \in \mathcal{A} : \ f \text{ is one-to-one in } \mathbb{D}\}.$$

A function $f \in \mathcal{S}$ is called starlike (with respect to 0), denoted by $f \in \mathcal{S}^*$, if $tw \in f(\mathbb{D})$ whenever $w \in f(\mathbb{D})$ and $t \in [0, 1]$. A function $f \in \mathcal{S}$ that maps $\mathbb{D}$ onto a convex domain, denoted by $f \in \mathcal{K}$, is called a convex function. The

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analytic condition for the class $\mathcal{S}^\star(\beta)$ of starlike functions f of order β can be written in the form

$$\mathrm{Re}\Big(\frac{zf'(z)}{f(z)}\Big) > \beta, \qquad z \in \mathbb{D},$$

where $0 \le \beta < 1$. It is well-known that $\mathcal{S}^\star(0) \equiv \mathcal{S}^\star$. We denote by $\mathcal{S}_p$ the class of spirallike functions. Analytically, the class $\mathcal{S}_p$ is characterized as follows:

$$\mathcal{S}_p := \Big\{ f \in \mathcal{A} : \ \mathrm{Re}\Big(\mathrm{e}^{\mathrm{i}\theta} \tfrac{zf'(z)}{f(z)}\Big) > 0, \ \ z \in \mathbb{D}, \ \text{ for some } \ \theta \in (-\pi/2, \pi/2) \Big\}.$$

Clearly $\mathcal{S}^\star \subset \mathcal{S}_p$. A spirallike function is not necessarily close-to-convex and hence need not belong to $\mathcal{S}^\star$. Moreover, close-to-convex function is not necessarily spirallike, see [3, 4]. The above analytic characterizations are deduced by using the corresponding geometric criterion for $f(z)$ in the disk $|z| < r$. Various aspects of these and many other special subclasses of $\mathcal{S}$ are presented in [3, 4]. Finally, we now introduce the classes

$$\mathcal{U}(\lambda, \mu) := \Big\{ f \in \mathcal{A} : \ \tfrac{f(z)}{z} \neq 0 \ \text{ and } \ \big|f'(z)\big(\tfrac{z}{f(z)}\big)^{\mu+1} - 1\big| < \lambda, \ \ z \in \mathbb{D} \Big\}$$

and $\mathcal{U}(\lambda) := \mathcal{U}(\lambda, 1)$. It is known [2, 7] (see also [10]) that functions in $\mathcal{U}(\lambda)$ are univalent if $0 < \lambda \le 1$ but not necessarily univalent if $\lambda > 1$. It is also known [5] that $\mathcal{U}(1, -1) \not\subset \mathcal{S}^\star$ and V. Singh [12] gave an estimate for the radius of starlikeness (which is surprisingly close to unity) of $\mathcal{U}(1, -1)$. More recently, Obradović [8] and, Ponnusamy and Singh [9] proved that

$$\mathcal{U}(\lambda, \mu) \subset \mathcal{S}^\star \qquad \text{if} \quad \mu < 1 \quad \text{and} \quad 0 \le \lambda \le \frac{1-\mu}{\sqrt{(1-\mu)^2 + \mu^2}}.$$

In a recent paper, R. Fournier and S. Ponnusamy [6] extended it as follows:

Lemma A. *Let* $\mu \in \mathbb{C}$ *with* $\mathrm{Re}(\mu) < 1$. *Then*

(a) $\mathcal{U}(\lambda, \mu) \subset \mathcal{S}^\star$ iff $0 \le \lambda \le \frac{|1-\mu|}{\sqrt{|1-\mu|^2 + |\mu|^2}}$

(b) $\mathcal{U}(\lambda, \mu) \subset \mathcal{S}_p$ iff $0 \le \lambda \le \min\Big(1, \frac{|1-\mu|}{|\mu|}\Big)$.

It is worth to recalling the following implications from the last two results:

(c) $\mathcal{U}(1, \mu) \subset \mathcal{S}^\star$ iff $\mu = 0$

(d) $\mathcal{U}(1, \mu) \subset \mathcal{S}_p$ iff $\mathrm{Re}\,(\mu) \le \frac{1}{2}$.

Moreover, each $f \in \mathcal{S}$ can be written in the form

$$\frac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \dots, \qquad z \in \mathbb{D}. \tag{1.1}$$

Many interesting results concerning functions of this form are known in the literature, see for instance [1,11]. We begin the discussion by recalling the well-known Area Theorem [3: Theorem 11, p. 193, Vol. 2] and the case $\mu = 1$ of this result has a special role in geometric function theory.

Lemma B. *If $f \in \mathcal{S}$, $\mu > 0$, and if*

$$\left(\frac{z}{f(z)}\right)^{\mu} = 1 + b_1 z + b_2 z^2 + \dots, \tag{1.2}$$

then

$$\sum_{k=1}^{\infty} (k - \mu)|b_k|^2 \leq \mu.$$

In [11], Reade et. al. have obtained the following sufficient condition for a function f of the form (1.1) to be in $\mathcal{S}^\star(\beta)$.

Lemma C. *Suppose that f is of the form* (1.1)*. If*

$$\sum_{k=1}^{\infty} (k - 1 + \beta)|b_k| \leq \begin{cases} (1-\beta) - (1-2\beta)|b_1| & \text{for } 0 \leq \beta \leq \frac{1}{2} \\ 1 - \beta & \text{for } \frac{1}{2} < \beta < 1, \end{cases}$$

then $f \in \mathcal{S}^\star(\beta)$.

In this article, we generalize this result in the context of Lemma B (see Corollary 3 with $\mu = 1$). We now state our first result which provides a necessary condition for f of the form (1.2) to belong to $\mathcal{S}^\star(\beta)$. Unlike the original proof of Lemma B, proof in this case is a bit straightforward.

Theorem 1. *Every $f \in \mathcal{S}^\star(\beta)$ which has the form* (1.2) *with $\mu > 0$ necessarily satisfies the coefficient inequality*

$$\sum_{k=1}^{\infty} (k - (1-\beta)\mu)|b_k|^2 \leq (1-\beta)\mu. \tag{1.3}$$

P r o o f. Let $f \in \mathcal{S}^\star(\beta)$. Next, we observe that

$$\frac{zf'(z)}{f(z)} = \frac{\left(\frac{z}{f(z)}\right)^{\mu} - \frac{1}{\mu} z \left(\left(\frac{z}{f(z)}\right)^{\mu}\right)'}{\left(\frac{z}{f(z)}\right)^{\mu}}$$

and the inequality $|\zeta - 1| < |\zeta + 1 - 2\beta|$ is equivalent to $\operatorname{Re}\zeta > \beta$. In view of this observation, it can be easily seen that $f \in \mathcal{S}^\star(\beta)$ is equivalent to the inequality

$$\left|\frac{1}{\mu} z \left(\left(\frac{z}{f(z)}\right)^{\mu}\right)'\right| < \left|2(1-\beta)\left(\frac{z}{f(z)}\right)^{\mu} - \frac{1}{\mu} z \left(\left(\frac{z}{f(z)}\right)^{\mu}\right)'\right|.$$

Using the representation (1.2), the last inequality takes the form

$$\left|\sum_{k=1}^{\infty} kb_k z^k\right| < \left|2(1-\beta)\mu - \sum_{k=1}^{\infty}(k-2(1-\beta)\mu)b_k z^k\right|. \tag{1.4}$$

Therefore, with $z = re^{i\theta}$ for $r \in (0,1)$ and $0 \le \theta \le 2\pi$, the inequality (1.4) gives

$$\frac{1}{2\pi}\int_0^{2\pi} \Big|\sum_{k=1}^{\infty} kb_k z^k\Big|^2 \,\mathrm{d}\theta < \frac{1}{2\pi}\int_0^{2\pi} \Big|2(1-\beta)\mu - \sum_{k=1}^{\infty}(k-2(1-\beta)\mu)b_k z^k\Big|^2 \,\mathrm{d}\theta$$

or equivalently,

$$\sum_{k=1}^{\infty} k^2|b_k|^2 r^{2k} < 4(1-\beta)^2\mu^2 + \sum_{k=1}^{\infty}(k-2(1-\beta)\mu)^2|b_k|^2 r^{2k}$$

which, upon simplification, implies that

$$\sum_{k=1}^{\infty}(k-(1-\beta)\mu)|b_k|^2 r^{2k} < (1-\beta)\mu.$$

Allowing $r \to 1^-$, we obtain the desired inequality. □

Using the method of proof of Theorem 1, we can easily establish the following general result, and so we omit the details.

Corollary 1. *Every $f \in \mathcal{S}^\star(\beta)$ which has the form* (1.2) *with* $\operatorname{Re}\mu > 0$ *necessarily satisfies the coefficient inequality*

$$\sum_{k=1}^{\infty}(k\operatorname{Re}\mu - (1-\beta)|\mu|^2)|b_k|^2 \le (1-\beta)|\mu|^2. \tag{1.5}$$

There is some difficulty in obtaining an analog of Theorem 1 for functions $f \in \mathcal{K}(\beta)$ of the form (1.2) and hence, this remains an open problem. Here $\mathcal{K}(\beta)$, $0 \le \beta < 1$, denotes the class of convex functions f of order β defined by the analytic condition

$$\operatorname{Re}\Big(1 + \frac{zf''(z)}{f'(z)}\Big) > \beta, \qquad z \in \mathbb{D},$$

so that $\mathcal{K} := \mathcal{K}(0)$. However, if $\beta = 0$ in Theorem 1, then the coefficient inequality (1.3) is same as in Lemma B for the case of univalent functions. At this place it is worth remembering that for this problem, the Koebe function is extremal for both $\mathcal{S}$ and $\mathcal{S}^\star$. However, for all other values of $\beta > 0$, Theorem 1 provides an improved inequality. For instance, we have

Corollary 2. *Every $f \in \mathcal{S}^\star(1/2)$ which has the form (1.2) with $\mu > 0$ necessarily satisfies the coefficient inequality*

$$\sum_{k=1}^{\infty}(2k-\mu)|b_k|^2 \le \mu.$$

In connection with Lemma B, it is natural to ask for a sufficient condition in terms of the coefficients b_k implying the univalence of the corresponding f in the unit disk.

Theorem 2. *Let $\mu \in \mathbb{C}$ with $\mathrm{Re}\,(\mu) < 1$. Assume that $f \in \mathcal{A}$, $\frac{f(z)}{z} \neq 0$, and has the form (1.2). If*

$$\sum_{k=1}^{\infty} |k-\mu|\,|b_k| \le \lambda|\mu|.$$

Then we have the following

(a) $f \in \mathcal{S}^\star$ *if* $0 \le \lambda \le \frac{|1-\mu|}{\sqrt{|1-\mu|^2+|\mu|^2}}$

(b) $f \in \mathcal{S}_p$ *if* $0 \le \lambda \le \min\Big(1, \frac{|1-\mu|}{|\mu|}\Big)$.

P r o o f. We observe that

$$\left(\frac{z}{f(z)}\right)^{\mu} - \left(\frac{z}{f(z)}\right)^{1+\mu} f'(z) = \frac{1}{\mu} z\left(\left(\frac{z}{f(z)}\right)^{\mu}\right)' = \frac{1}{\mu}\sum_{k=1}^{\infty} k b_k z^k$$

so that

$$\left(\frac{z}{f(z)}\right)^{1+\mu} f'(z) - 1 = -\sum_{k=1}^{\infty} \frac{k-\mu}{\mu} b_k z^k$$

and therefore,

$$\begin{aligned}\left|\left(\frac{z}{f(z)}\right)^{1+\mu} f'(z) - 1\right| &= \left|\sum_{k=1}^{\infty} \frac{k-\mu}{\mu} b_k z^k\right| \\ &\le \sum_{k=1}^{\infty} |(k-\mu)/\mu|\,|b_k|, \qquad z \in \mathbb{D}.\end{aligned}$$

The desired conclusion follows from Lemma A and the hypotheses. □

Theorem 3. *Assume that $f \in \mathcal{A}$, $\frac{f(z)}{z} \neq 0$, and has the form (1.2) with $\mu > 0$. If*

$$\sum_{k=1}^{\infty}(|k-2(1-\beta)\mu| + k)\,|b_k| \le 2(1-\beta)\mu, \tag{1.6}$$

then $f \in \mathcal{S}^\star(\beta)$.

P r o o f. Without loss of generality assume that $b_k \neq 0$ at least for a $k \geq 1$. In order to prove (1.6) implies $f \in \mathcal{S}^\star(\beta)$, it suffices to show that (1.4) holds. By (1.6), it follows that

$$\begin{aligned}\left|\sum_{k=1}^{\infty} kb_k z^k\right| &< \sum_{k=1}^{\infty} k|b_k| \\ &\leq 2(1-\beta)\mu - \sum_{k=1}^{\infty} |k - 2(1-\beta)\mu|\,|b_k| \\ &\leq \left|2(1-\beta)\mu - \sum_{k=1}^{\infty}(k - 2(1-\beta)\mu)b_k z^k\right|\end{aligned}$$

and therefore the inequality (1.4) holds. Thus, $f \in \mathcal{S}^\star(\beta)$. □

If $0 < \mu \leq 1$, then Theorem 3 can be simplified. A simplification gives

Corollary 3. *Suppose that f is of the form* (1.2) *with $0 < \mu \leq 1$. If*

$$\sum_{k=1}^{\infty}(k-(1-\beta)\mu)|b_k| \leq \begin{cases}(1-\beta)\mu - (1-2\beta)\mu|b_1| & \text{for } 1-(1/\mu) \leq \beta \leq 1 - 1/(2\mu) \\ (1-\beta)\mu & \text{for } 1 - 1/(2\mu) \leq \beta < 1,\end{cases}$$

then $f \in \mathcal{S}^\star(\beta)$.

Clearly the choice $\lambda = 1$ of Theorem 2 yields that $\mu = 0$ and hence, we have only a trivial function, namely $f(z) = z$. On the other hand, in the case of $0 < \mu \leq 1/2$, one can refine Theorem 2(b) through coefficient condition as follows. Note that spiral-like function is not necessarily starlike.

Corollary 4. *Suppose that f is of the form* (1.2) *with $0 < \mu \leq 1/2$. If*

$$\sum_{k=1}^{\infty}(k-\mu)|b_k| \leq \mu,$$

then $f \in \mathcal{S}^\star$.

P r o o f. Set $\beta = 0$ in Corollary 3. □

Because of its natural form, the coefficient inequalities of Corollaries 3 and 4 are easy to apply in many special situations.

In order to state our next result we need to introduce the definition of Hadamard product or convolution $f \star g$ of two convergent power series $f(z) := \sum_{n=0}^{\infty} a_n(f)z^n$ and $g(z) := \sum_{n=0}^{\infty} a_n(g)z^n$ in the unit disk $|z| < 1$. This is defined by the power series

$$f \star g(z) := \sum_{n=0}^{\infty} a_n(f)a_n(g)z^n.$$

It is clear that $f \star g$ is also a member of the class of analytic functions in $\mathbb{D}$. It is possible to formulate our next result.

Theorem 4. *Let $0 < \mu \leq 1/2$, $f, g \in \mathcal{S}$ such that $\left(\frac{z}{f(z)}\right)^{\mu} \star \left(\frac{z}{g(z)}\right)^{\mu} \neq 0$ for $z \in \mathbb{D}$. Then the function F defined by*

$$\frac{z}{F(z)} = \left(\frac{z}{f(z)}\right)^{\mu} \star \left(\frac{z}{g(z)}\right)^{\mu}$$

belongs to the class $\mathcal{S}^{\star}(1-\mu)$.

Especially, for $\mu = 1/2$ we have

$$f, g \in \mathcal{S} \quad \Longrightarrow \quad F(z) = \frac{z}{\left(\frac{z}{f(z)}\right)^{\frac{1}{2}} \star \left(\frac{z}{g(z)}\right)^{\frac{1}{2}}} \in \mathcal{S}^{\star}(1/2)$$

whenever $\left(\frac{z}{f(z)}\right)^{\frac{1}{2}} \star \left(\frac{z}{g(z)}\right)^{\frac{1}{2}} \neq 0$ for $z \in \mathbb{D}$.

P r o o f. By hypothesis, we may let

$$\begin{aligned} \left(\frac{z}{f(z)}\right)^{\mu} &= 1 + b_1 z + b_2 z^2 + \dots \\ \left(\frac{z}{g(z)}\right)^{\mu} &= 1 + c_1 z + c_2 z^2 + \dots . \end{aligned} \tag{1.7}$$

Then by Lemma B we have

$$\sum_{k=1}^{\infty} (k-\mu)|b_k|^2 \leq \mu \qquad \text{and} \qquad \sum_{k=1}^{\infty} (k-\mu)|c_k|^2 \leq \mu. \tag{1.8}$$

As

$$\begin{aligned} \frac{z}{F(z)} &= \left(\frac{z}{f(z)}\right)^{\mu} \star \left(\frac{z}{g(z)}\right)^{\mu} \\ &= 1 + (b_1 c_1)z + (b_2 c_2)z^2 + \cdots , \end{aligned}$$

from (1.8) we get, by means of the Cauchy-Schwarz inequality

$$\sum_{k=1}^{\infty}(k-\mu)|b_kc_k| \leq \left(\sum_{k=1}^{\infty}(k-\mu)|b_k|^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{\infty}(k-\mu)|c_k|^2\right)^{\frac{1}{2}}$$

$$\leq \sqrt{\mu}\sqrt{\mu} = \mu.$$

Since F is of the form (1.1) (with the new coefficients b_kc_k in place of b_k's), according to Lemma C with $\beta = 1-\mu$, the last inequality implies that $F \in \mathcal{S}^\star(1-\mu)$. □

Example 1. Let $f, g \in \mathcal{S}^\star(1/2)$ and have the form (1.1). Then, according to Corollary 2 with $\mu = 1$, we have

$$\sum_{k=1}^{\infty}(2k-1)|b_k|^2 \leq 1$$

and

$$\sum_{k=1}^{\infty}(2k-1)|c_k|^2 \leq 1.$$

Because $f, g \in \mathcal{S}^\star(1/2)$, by the Marx-Strohhäcker theorem

$$\mathrm{Re}\left(\frac{f(z)}{z}\right) > \frac{1}{2} \quad \text{and} \quad \mathrm{Re}\left(\frac{g(z)}{z}\right) > \frac{1}{2}, \qquad z \in \mathbb{D},$$

and thus, $f(z)+g(z) \neq 0$ for $z \in \mathbb{D}\backslash\{0\}$. Now, we consider the function

$$F(z) = \frac{2f(z)g(z)}{f(z)+g(z)}.$$

Then $F \in \mathcal{A}$. For $0 < r \leq 1$, we introduce $G(z) = r^{-1}F(rz)$. We are interested in determining range values of r for which $G \in \mathcal{S}^\star(1/2)$. In order to solve this, we form

$$\frac{z}{G(z)} = \frac{rz}{F(rz)} = \frac{1}{2}\left(\frac{rz}{f(rz)} + \frac{rz}{g(rz)}\right)$$

$$= 1 + \sum_{k=1}^{\infty}\frac{b_k+c_k}{2}r^kz^k.$$

Then, according to Corollary 3 (with $\mu = 1$ and $\beta = 1/2$), it suffices to show that

$$\sum_{k=1}^{\infty}(2k-1)|b_k+c_k|r^k \leq 2.$$

Again, in view of the Cauchy-Schwarz inequality, we see that

$$\begin{aligned}
&\sum_{k=1}^{\infty}(2k-1)\,|b_k+c_k|\,r^k\\
&\quad\le \sum_{k=1}^{\infty}(2k-1)|b_k|r^k+\sum_{k=1}^{\infty}(2k-1)|c_k|r^k\\
&\quad\le \left(\left(\sum_{k=1}^{\infty}(2k-1)|b_k|^2\right)^{\frac{1}{2}}+\left(\sum_{k=1}^{\infty}(2k-1)|c_k|^2\right)^{\frac{1}{2}}\right)\left(\sum_{k=1}^{\infty}\frac{r^{2k}}{2k-1}\right)^{\frac{1}{2}}\\
&\quad\le 2\left(\sum_{k=1}^{\infty}\frac{r^{2k}}{2k-1}\right)^{\frac{1}{2}}=2\left(\frac{r}{2}\log\frac{1+r}{1-r}\right)^{\frac{1}{2}}\le 2
\end{aligned}$$

for $0<r\le r_0$, where $r_0\approx 0.83356$ is the root of the equation

$$\frac{r}{2}\log\frac{1+r}{1-r}=1$$

in the interval $(0,1)$. Thus, $G\in\mathcal{S}^{\star}(1/2)$ for $0<r\le r_0$ and hence, we conclude that $F\in\mathcal{S}^{\star}\,(1/2)$ in the disk $|z|<r_0$.

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