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THE SHARP THRESHOLD FOR PERCOLATION ON EXPANDER GRAPHS

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ABSTRACT. We consider a random subgraph $G_n(p)$ of a finite graph family $G_n = (V_n, E_n)$ formed by retaining each edge of G_n independently with probability p. We show that if G_n is an expander graph with vertices of bounded degree, then for any $c_n \in (0,1)$ satisfying $c_n \gg 1/\sqrt{\ln n}$ and $\limsup_{n\to\infty} c_n < 1$, the property that the random subgraph contains a giant component of order $c_n|V_n|$ has a sharp threshold.

1. Introduction and results

Let $G_n = (V_n, E_n)$ be a finite graph with $|V_n| = n$ vertices and $G_n(p)$ be the spanning subgraph of G_n obtained by retaining each edge of G_n independently with probability p. When G_n is a complete graph, this model is known as the Erdős-Rényi random graph $G(n,p)$ [5,10,17], which has been extensively treated. Other examples of percolation on finite graphs are concerned with graphs of some symmetries such as regular graphs [8, 14, 15] and d-dimensional torus or box, which is closely related to percolation on corresponding infinite lattice graph $\mathbb{Z}^d$ [3, 12, 19]. Recently, percolation on general classes of finite graphs has also been investigated, see e.g. [1, 2, 4, 6, 18], where isoperimetric inequalities replacing symmetry assumptions play a key role. In this paper, following the path of Alon et al. [1] and Benjamini et al. [2], we study the sharp threshold phenomenon for percolation on finite graphs satisfying an isoperimetric inequality (called expander graphs).

For any two sets of vertices A and B in G_n, the set $E_n(A,B)$ consists of all edges with one endpoint in A and the other in B. The edge-isoperimetric

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number, $c(G_n)$, (also called the Cheeger constant) is given by

$$\min_{\substack{A\subset V_n\\ 0<|A|\le n/2}} \frac{\partial_{E_n} A}{|A|},$$

where $\partial_{E_n} A = E_n(A, V_n \backslash A)$ is the exterior edge-boundary of A. Let b and d be positive constants. A (b,d)-expander graph is a graph $G_n = (V_n, E_n)$ such that the maximal degree in G_n is not greater than d, and $c(G_n) > b$. In this paper, all asymptotics are as $n \to \infty$. We say that an event holds asymptotically almost surely (a.a.s.) if the probability that it holds tends to 1, following the notations in [10].

In [1], Alon, Benjamini and Stacey derived the precise critical probability for the emergence of a linear size giant component in expander graphs under the assumptions of regularity and high-girth:

Theorem 1.1. ([1: Theorem 3.2]) *Let $d \ge 2$ and let G_n be a sequence of d-regular (b,d)-expander graphs with girth $g_n \to \infty$.*

If $p > 1/(d-1)$, then there exists a $c > 0$ such that, asymptotically almost surely,

$G_n(p)$ *contains a component of order at least* $c|V_n|$.

If $p < 1/(d-1)$, then for any $c > 0$, asymptotically almost surely,

$G_n(p)$ *does not contain a component of order at least* $c|V_n|$.

Recently, Benjamini, Boucheron, Lugosi and Rossignol [2] are able to show that in any expander graph, every giant component of given proportion emerges in an interval of length $o(1)$ (more precisely, of order $O\left((\ln n)^{-1/3}\right)$), removing the regularity and high-girth assumptions in Theorem 1.1. Their main result may be formalized as follows.

Theorem 1.2. ([2: Theorem 1.3]) *Let G_n be a (b,d)-expander graph and let $c \in (0,1)$. There exist constants $q_1 = q_1(d) > 0$ and $q_2 = q_2(c) \in (q_1, 1)$, and $p_n^*(c) \in [q_1, q_2]$ such that, for every $\varepsilon > 0$, if $p_n \ge p_n^*(c) + \varepsilon$, then, asymptotically almost surely,*

$G_n(p_n)$ *contains a component of order at least* cn,

and if $p_n \le p_n^*(c) - \varepsilon$, then, asymptotically almost surely,

$G_n(p_n)$ *does not contain a component of order at least* cn.

Note that in Theorem 1.2, the sharp threshold $p_n^*(c)$ is dependent on the proportion, c, of the giant component in G_n. Thus, we can not assert the existence of a universal threshold function p_n^* for the emergence of a giant component.

In this paper, we move a further step beyond Theorem 1.2 by allowing more general proportions of giant components. A sharp threshold result for the events "$G_n(p_n)$ contains a component of order at least $c_n n$" for $c_n \in (0,1)$ is the following

Theorem 1.3. *Let G_n be a (b,d)-expander graph. Let $c_n \in (0,1)$ and $c_n \gg 1/\sqrt{\ln n}$. Suppose that $c := \limsup_{n\to\infty} c_n < 1$. There exist constants $q_1 = q_1(d) > 0$ and $q_2 = q_2(c) \in (q_1, 1)$, and $p_n^*(c_n) \in [q_1, q_2]$ such that, for every $\varepsilon > 0$, if $p_n \ge p_n^*(c_n) + \varepsilon$, then, asymptotically almost surely,*

$$G_n(p_n) \ \text{contains a component of order at least } c_n n,$$

and if $p_n \le p_n^*(c_n) - \varepsilon$, then, asymptotically almost surely,

$$G_n(p_n) \ \text{does not contain a component of order at least } c_n n.$$

We present a complete and self-contained proof of Theorems 1.3 in two stages: the critical probabilities (i.e., the thresholds) $p_n^*(c_n)$ are shown to be bounded away from zero and one in Section 2, and the threshold width is shown to be bounded by a function of n that tends to zero in Section 3. It is often that several key lemmas in Section 2 and Section 3 are to be found as pieces of a long proof of a big statement in [1–3] and so the validity of these technical lemmas under weaker assumptions needs to be carefully checked. We include the proofs of them, more or less as they were presented in [1–3], not only for the convenience of the reader but also to convince the reader that they do hold in our setting.

2. The threshold of giant component is bounded away from zero and one

Before proceeding, we introduce some notations that will be used throughout the paper. Let $G_n = (V_n, E_n)$ be a (b,d)-expander graph as before. Each point configuration $x \in \{0,1\}^{E_n}$ is identified with the subgraph of G_n with vertex set V_n and edge set obtained by removing from E_n all edges e such that $x(e) = 0$. For $p \in [0,1]$, we equip the space $\{0,1\}^{E_n}$ with the product probability measure $\mu_{n,p}$ under which each $x(e)$ is independently 1 with probability p and 0 with probability $1-p$. We denote by $E_{n,p}(f) = \int f(x)\,\mathrm{d}\mu_{n,p}(x)$ and $D_{n,p}(f)$ the mean and variance of random variable $f\colon \{0,1\}^{E_n} \to \mathbb{R}$, respectively. For $x \in \{0,1\}^{E_n}$, let $\mathcal{C}_n^{(1)} = \mathcal{C}_n^{(1)}(x)$ be the largest connected component in the configuration x, and let $L_n^{(1)} = L_n^{(1)}(x) = |\mathcal{C}_n^{(1)}(x)|$. Denote by $C(v)$ the connected component containing a vertex $v \in V_n$.

Note that, for fixed n and any $c_n \in (0,1)$, $\mu_{n,p}\{L_n^{(1)} \ge c_n n\}$ is a strictly increasing polynomial of p. Therefore, for any $\alpha \in [0,1]$, we define $p_{n,\alpha}(c_n)$ as the unique real number $p \in [0,1]$ such that

$$\mu_{n,p}\{L_n^{(1)} \ge c_n n\} = \alpha.$$

The threshold function in Theorem 1.3 is defined as $p_n^*(c_n) = p_{n,1/2}(c_n)$. We sometimes suppress the subscript n if no ambiguity will be caused.

Proposition 2.1. *Let $c_n \in (0,1)$ and $c_n \gg \ln n/n$. Suppose that*

$$c := \limsup_{n\to\infty} c_n < 1.$$

There exist two constants $q_1 = q_1(d) > 0$ and $q_2 = q_2(c) \in (q_1, 1)$, and $q_3(c_n)$ satisfying $q_3(c_n) \in (q_1, q_2(c))$, such that for any $\alpha \in (0,1)$, for all n large enough, $p_{n,\alpha}(c_n) \in (q_1, q_3(c_n))$.

Moreover, there are positive constants C_1 and C_2, depending only on b and d, such that for any $p_n \geq q_3(c_n)$,

$$\mu_{n,p_n}\big\{L_n^{(1)} \geq c_n n\big\} \geq 1 - C_1 \mathrm{e}^{-C_2 n}. \tag{2.1}$$

Since $q_3(c_n)$ depends on n, the introduction of a constant upper bound $q_2(c) < 1$ plays an essential role. This is different from the situation in [2], where $c_n \equiv c$ is fixed. To prove Proposition 2.1 we need the following two lemmas, the proofs of which are essentially from [1: Lemma 2.2, Proposition 3.1] and [3: Theorem 2].

Lemma 2.1. ([2: Proposition 3.1]) *There exist constants $0 < p_0(b) < 1$, $a(b) > 0$ and $C(b,d) > 0$, such that for any $p \geq p_0$ and large enough n,*

$$\mu_{n,p}\big\{L_n^{(1)} \geq an\big\} \geq 1 - \mathrm{e}^{-Cn}.$$

Lemma 2.2. *For any $a_1 \in (0,1/2)$ and $a_{2,n} \in (1/2,1)$ satisfying*

$$\limsup_{n\to\infty} a_{2,n} < 1,$$

there is $0 < q_4(a_1, a_{2,n}) < 1$ such that $\limsup_{n\to\infty} q_4(a_1, a_{2,n}) < 1$ and, for any $p_n \geq q_4(a_1, a_{2,n})$,

$$\mu_{n,p_n}\big\{G_n(p_n) \text{ contains a component of order in } [a_1 n, a_{2,n} n)\big\} \leq 4\Big(1 + \frac{1}{a_1}\Big)\mathrm{e}^{-n}.$$

Proof. From [7: pp. 68] we know that an infinite d-regular rooted tree contains $\frac{1}{(d-1)r+1}\binom{dr}{r} \leq (de)^r$ rooted subtrees of order r. Given a vertex $v \in G_n$, one may associate a subtree of the infinite d-regular tree rooted at v by considering the self-avoiding paths issued from v in G_n. Therefore, any connected subgraph of order r in G_n containing v can correspond to a different subtree of order r. Thus, the total number of connected subsets of order r in V_n is less than $n(de)^r/r$.

Thanks to the expansion property, for any subset $U \subset V_n$ of order r, the probability that all edges in $\partial_{E_n} U$ are absent is at most $(1-p_n)^{br}$ if $r \leq n/2$; and at most $(1-p_n)^{b(n-r)}$ if $r > n/2$. Hence, for any $n \in \mathbb{N}$, the probability of having a connected component of order in $[a_1 n, a_{2,n} n)$ is at most

$$\begin{aligned}
&\sum_{r=\lceil a_1 n\rceil}^{\lfloor n/2\rfloor} \frac{n(de)^r}{r}(1-p_n)^{br} + \sum_{r=\lfloor n/2\rfloor+1}^{\lfloor a_{2,n} n\rfloor} \frac{n(de)^r}{r}(1-p_n)^{b(n-r)} \\
\le\ & \frac{1}{a_1}\cdot\frac{(de(1-p_n)^b)^{a_1 n}}{1-(de(1-p_n)^b)} + 2(1-p_n)^{nb}\frac{(de(1-p_n)^{-b})^{a_{2,n}n+1}}{de(1-p_n)^{-b}-1} \\
\le\ & \frac{4}{a_1}e^{-n} + 4e^{-n},
\end{aligned}$$

provided that

$$de(1-p_n)^{-b} \ge 2, \qquad (de)^{a_{2,n}}(1-p_n)^{b(1-a_{2,n})} \le e^{-1}, \qquad (de(1-p_n)^b)^{a_1} \le e^{-1}. \tag{2.2}$$

Since $\limsup_{n\to\infty} a_{2,n} < 1$, the conditions (2.2) are satisfied if p_n is larger than some $q_4(a_1, a_{2,n})$, which is bounded away from 1. □

Now we will show Proposition 2.1 by virtue of Lemma 2.1 and Lemma 2.2.

Proof of Proposition 2.1. First, we show the lower bound of $p_{n,\alpha}(c_n)$. Fix $0 < q_1 < 1/(d-1)$ and $p \le q_1$. Consider the subcritical Galton-Watson process with the first offspring distribution $\text{Bin}(d,p)$ and other offspring distributions $\text{Bin}(d-1,p)$. Since the maximum degree of G_n is at most d, the connected component $C(v)$ containing a vertex $v \in V_n$ has order no more than S, where S is the total number of descendants of the above branching process with root v. It is well-known (e.g. [13: pp. 172]) that there are some $\lambda > 0$, $M < \infty$, depending only on d and q_1, such that, for any n and $p \le q_1$,

$$E_{n,p}(e^{\lambda S}) \le M.$$

Hence, by Markovian inequality, we have for any $t > 0$ and $p \le q_1$,

$$\mu_{n,p}\{L_n^{(1)} \ge t\} \le n\mu_{n,p}\{S \ge t\} \le \frac{nE_{n,p}(e^{\lambda S})}{e^{\lambda t}} \le nMe^{-\lambda t}.$$

Since $c_n \gg \ln n/n$, we obtain

$$\mu_{n,p}\{L_n^{(1)} \ge c_n n\} \le \mu_{n,p}\Big\{L_n^{(1)} \ge \frac{2}{\lambda}\ln(nM^{1/2})\Big\} \le \frac{1}{n}.$$

Taking into account the fact that $\mu_{n,p}\{L_n^{(1)} > c_n n\}$ is increasing with respect to p, we have $p_{n,\alpha}(c_n) > q_1$ for any $\alpha \in (0,1)$ and large enough n.

Next, the upper bound of $p_{n,\alpha}(c_n)$ can be shown by choosing (recall Lemma 2.1 and Lemma 2.2)

$$q_3(c_n) = \max\left\{q_4\left(\min\left\{\frac{1}{4}, a\right\}, \max\left\{\frac{3}{4}, c_n\right\}\right), p_0(b)\right\}$$

and

$$q_2(c) = \max\left\{q_5\left(\min\left\{\frac{1}{4}, a\right\}, \max\left\{\frac{3}{4}, c\right\}\right), p_0(b)\right\}.$$

In fact, we can show this by the reduction to absurdity. Suppose that $p_{n,\alpha}(c_n) \geq q_3(c_n)$, i.e., $p_{n,\alpha}(c_n) \geq q_4\big(\min\{1/4, a\}, \max\{3/4, c_n\}\big)$ and $p_{n,\alpha}(c_n) \geq p_0(b)$. Fix $n \in \mathbb{N}$. If $c_n < 3/4$,

$$(0,1) \ni \alpha = \mu_{n,p}\big\{L_n^{(1)} \geq c_n n\big\} \geq \mu_{n,p}\Big\{L_n^{(1)} \geq \frac{3}{4}n\Big\}, \tag{2.3}$$

and if $c_n \geq 3/4$,

$$(0,1) \ni \alpha = \mu_{n,p}\big\{L_n^{(1)} \geq c_n n\big\}. \tag{2.4}$$

Involving Lemma 2.1 and Lemma 2.2, the right-hand sides of (2.3) and (2.4) tend to 1 as $n \to \infty$, which is a contradiction. Hence, we have $p_{n,\alpha}(c_n) < q_3(c_n)$.

Finally, we show the statement (2.1). This can be proved by comparing c_n with a in Lemma 2.1. Fix $n \in \mathbb{N}$ and suppose $p_n \geq q_3(c_n)$.

Case (i): $c_n \leq a$.

By Lemma 2.1, we have

$$\mu_{n,p_n}\{L_n^{(1)} \geq c_n n\} \geq \mu_{n,p_n}\{L_n^{(1)} \geq an\} \geq 1 - \mathrm{e}^{-Cn}.$$

Case (ii): $c_n > a \geq 1/2$.

Choosing $a_1 = 1/4$ and $a_{2,n} = c_n$, we have by Lemma 2.1 and Lemma 2.2,

$$\begin{aligned}\mu_{n,p_n}\{L_n^{(1)} \geq c_n n\} &= \mu_{n,p_n}\{L_n^{(1)} \geq an\} - \mu_{n,p_n}\{an \leq L_n^{(1)} < c_n n\}\\ &\geq 1 - \mathrm{e}^{-Cn} - 4\left(1 + \frac{1}{a_1}\right)\mathrm{e}^{-n}\\ &= 1 - \mathrm{e}^{-Cn} - 20\mathrm{e}^{-n}.\end{aligned}$$

Case (iii): $c_n > 1/2 > a$.

Choosing $a_1 = a$ and $a_{2,n} = c_n$, we have by Lemma 2.1 and Lemma 2.2,

$$\begin{aligned}\mu_{n,p_n}\{L_n^{(1)} \geq c_n n\} &= \mu_{n,p_n}\{L_n^{(1)} \geq an\} - \mu_{n,p_n}\{an \leq L_n^{(1)} < c_n n\}\\ &\geq 1 - \mathrm{e}^{-Cn} - 4\left(1 + \frac{1}{a}\right)\mathrm{e}^{-n}.\end{aligned}$$

Case (iv): $1/2 \geq c_n > a$.

Choosing $a_1 = a$ and $a_{2,n} = 3/4$, we have by Lemma 2.1 and Lemma 2.2,

$$\begin{aligned}\mu_{n,p_n}\{L_n^{(1)} \geq c_n n\} &= \mu_{n,p_n}\{L_n^{(1)} \geq an\} - \mu_{n,p_n}\{an \leq L_n^{(1)} < c_n n\}\\ &\geq 1 - \mathrm{e}^{-Cn} - 4\left(1 + \frac{1}{a}\right)\mathrm{e}^{-n}.\end{aligned}$$

The proof of Proposition 2.1 is thus complete. □

3. The bound for threshold width

In this section, we prove our main result, Theorem 1.3. The main step is a threshold width result stated below in Proposition 3.1.

Proposition 3.1. *Let $\alpha < 1/2$ and $c_n \in (0,1)$. Suppose that $c_n \gg \ln n/n$ and $c := \limsup_{n\to\infty} c_n < 1$. There is a positive constant C_3, depending only on α, b and d, such that for any $n \in \mathbb{N}$,*

$$p_{n,(1-\alpha)}(c_n) - p_{n,\alpha}(c_n) \leq \frac{C_3}{(\ln n)^{1/3} \cdot \min\left\{c_n^{2/3}, (1-c_n)^{2/3}, \left(\left(1-q_3(c_n)\right)^d - \sqrt{\frac{d\ln(2/\alpha)}{2n}}\right)^{2/3}\right\}}. \tag{3.1}$$

where $q_3(c_n)$ is defined in Proposition 2.1.

Recall that $p_n^*(c_n) = p_{n,1/2}(c_n)$. Proposition 3.1, together with Proposition 2.1 implies our main result, Theorem 1.3. To see this, we note that if $c' := \liminf_{n\to\infty} c_n > 0$, then the right-hand side of (3.1) is equal to $C_4/(\ln n)^{1/3}$, where C_4 is some positive constant depending only on α, b, d, c and c'; if $c' = 0$, then the right-hand side of (3.1) also becomes $o(1)$ using the assumption $c_n \gg 1/\sqrt{\ln n}$. We mention that the threshold widths for $c' > 0$ and $c_n \equiv c'$ (as is the case treated in [2]) have the same order $O\left((\ln n)^{-1/3}\right)$.

Now, we turn to the proof of Proposition 3.1, which relies on a series of lemmas. For $y \in \mathbb{R}$, denote by $y_- = \max\{0, -y\}$ the negative part of y. When $x, x' \in \{0,1\}^{E_n}$ are chosen independently according to $\mu_{n,p}$, and $e \in E_n$, we denote by $x^{(e)}$ the random configuration obtained from x by replacing $x(e)$ by $x'(e)$. The relationship of the variance and mean of $L_n^{(1)}$ is collected in the following lemma, where (3.2) is a generalization of Russo's lemma [16].

Lemma 3.1. ([2: Lemma 4.4]) *There is a constant $C(b,d) < \infty$ such that, for any p and n,*

$$D_{n,p}\left(L_n^{(1)}\right) \leq C(b,d) \frac{n}{\ln n} \frac{\mathrm{d}E_{n,p}\left(L_n^{(1)}\right)}{\mathrm{d}p}$$

and

$$\frac{\mathrm{d}E_{n,p}\left(L_n^{(1)}\right)}{\mathrm{d}p} = \frac{1}{p(1-p)} \sum_{e\in E_n} E_{n,p}\left[\left(L_n^{(1)}(x) - L_n^{(1)}\left(x^{(e)}\right)\right)_-\right]. \tag{3.2}$$

Lemma 3.2. *Let $\gamma_n \in (0,1)$ and $\varepsilon \in (0,1)$. For any $\tilde{c}_n$ satisfying*

$$1-(1-\gamma_n)^d+\sqrt{\frac{d\ln(1/\varepsilon)}{2n}}\le \tilde{c}_n<1$$

and, for n large enough,

$$\mu_{n,\gamma_n}\bigl\{L_n^{(1)}\ge \tilde{c}_n n\bigr\}\le \varepsilon.$$

Proof. Let N be the number of isolated vertices in G_n. Denote by X_v the indicator function of the event that $v\in V_n$ is isolated. Note that X_v and $X_{v'}$ are independent as soon as $d(v,v')\ge 2$, where $d(v,v')$ is the distance of vertices v and v' according to the shortest path metric in G_n. Thus, the maximal degree in a dependency graph of $(X_v)_{v\in V_n}$ is less than d. Recall that a dependency graph of the random variables $(X_v)_{v\in V_n}$ is given by the vertex set V_n and the edge set satisfying that if for two disjoint sets of vertices A and B there is no edge between A and B then the families $(X_v)_{v\in A}$ and $(X_v)_{v\in B}$ are independent. Therefore, by [9: Theorem 2.1], for any $t>0$ and $p_n\in[0,1]$,

$$\mu_{n,p_n}\bigl\{N<E_{n,p_n}(N)-t\bigr\}\le \mathrm{e}^{-\frac{2t^2}{nd}}.$$

Notice that $L_n^{(1)}\le n-N$ and $N=\sum\limits_{v\in V_n}X_v$. Hence, we have $E_{n,\gamma_n}(N)\ge (1-\gamma_n)^d n$, and for any $\tilde{c}_n>1-(1-\gamma_n)^d$,

$$\begin{aligned}\mu_{n,\gamma_n}\bigl\{L_n^{(1)}\ge \tilde{c}_n n\bigr\} &\le \mu_{n,\gamma_n}\bigl\{N<(1-\tilde{c}_n)n\bigr\}\\ &\le \mu_{n,\gamma_n}\bigl\{N\le E_{n,\gamma_n}(N)-(1-\gamma_n)^d n+(1-\tilde{c}_n)n\bigr\}\\ &\le \mathrm{e}^{-\frac{2n}{d}((1-\gamma_n)^d-(1-\tilde{c}_n))^2}.\end{aligned}$$

Using the assumption $1-(1-\gamma_n)^d+\sqrt{d\ln(1/\varepsilon)/2n}\le \tilde{c}_n<1$, we yield the desired result. □

The next lemma concerns the growth rate of the mean.

Lemma 3.3. *Let $\alpha\in(0,1)$ and $c_n\in(0,1)$ satisfying $c_n\gg \ln n/n$ and $c=\limsup\limits_{n\to\infty}c_n<1$. Then, for every $p\in[p_{n,\alpha}(c_n),p_{n,(1-\alpha)}(c_n)]$, and for n large enough,*

$$\frac{\mathrm{d}E_{n,p}(L_n^{(1)})}{\mathrm{d}p}\ge \frac{\alpha b}{2}n\cdot\min\left\{c_n,1-c_n,(1-q_3(c_n))^d-\sqrt{\frac{d\ln(2/\alpha)}{2n}}\right\},$$

where $q_3(c_n)$ is defined in Proposition 2.1.

Proof. Given $n\in\mathbb{N}$, fix $0<c_n\le\tilde{c}_n<1$. From (3.2) and the expansion property, we obtain

$$\frac{\mathrm{d}E_{n,p}(L_n^{(1)})}{\mathrm{d}p} = \frac{1}{p(1-p)}\sum_{e\in E_n}E_{n,p}\Bigl[\Bigl(L_n^{(1)}(x)-L_n^{(1)}(x^{(e)})\Bigr)_-\Bigr]$$

$$\begin{aligned} &\geq \frac{1}{1-p} E_{n,p}\left(|\partial_{E_n}\mathcal{C}_n^{(1)}|\right) \\ &\geq \frac{b}{1-p} E_{n,p}\left(L_n^{(1)} 1_{\{L_n^{(1)}\leq n/2\}} + (n-L_n^{(1)}) 1_{\{L_n^{(1)}>n/2\}}\right) \\ &\geq bn\min\{c_n, 1-\tilde{c}_n\}\cdot \mu_{n,p}\{L_n^{(1)}\in[c_n n, \tilde{c}_n n)\}, \qquad (3.3) \end{aligned}$$

where the last inequality (3.3) can be easily proved by dividing into three cases:

(i) $c_n \leq \tilde{c}_n < 1/2$,

(ii) $c_n \leq 1/2 \leq \tilde{c}_n$ and

(iii) $1/2 < c_n \leq \tilde{c}_n$.

Now, by Proposition 2.1, there exists a $q_3(c_n) < 1$ such that for n large enough, $p_{n,(1-\alpha)}(c_n) \leq q_3(c_n)$. Thus, applying Lemma 3.2 with $\gamma_n = q_3(c_n)$ and $\varepsilon = \alpha/2$, there is some $\tilde{c}_n(c_n) = \max\left\{c_n, 1-(1-q_3(c_n))^d + \sqrt{\frac{d\ln(1/\varepsilon)}{2n}}\right\}$ such that, for n large enough, $p_{n,(1-\alpha)}(c_n) \leq q_3(c_n) \leq p_{n,\alpha/2}(\tilde{c}_n)$.

Hence, for any $p \in \left[p_{n,\alpha}(c_n), p_{n,(1-\alpha)}(c_n)\right]$, we obtain $\mu_{n,p}\{L_n^{(1)} \geq c_n n\} \geq \alpha$ and $\mu_{n,p}\{L_n^{(1)} \geq \tilde{c}_n n\} \leq \alpha/2$. Combining the above comments with (3.3), we finally have

$$\begin{aligned} \frac{\mathrm{d}E_{n,p}(L_n^{(1)})}{\mathrm{d}p} &\geq bn\min\{c_n, 1-\tilde{c_n}(c_n)\}\cdot\frac{\alpha}{2} \\ &= \frac{\alpha b}{2} n\cdot\min\left\{c_n, 1-c_n, (1-q_3(c_n))^d - \sqrt{\frac{d\ln(2/\alpha)}{2n}}\right\}, \end{aligned}$$

for large enough n. □

P r o o f o f P r o p o s i t i o n 3.1. Let $0 < \alpha < 1/2$ and $c_n \in (0,1)$ satisfying $c_n \gg \ln n/n$. We will show that there is some constant $C_5 = C_5(\alpha, b, d)$, such that if

$$\varepsilon_n = \frac{C_5}{(\ln n)^{1/3}\cdot\min\left\{c_n^{2/3}, (1-c_n)^{2/3}, \left((1-q_3(c_n))^d - \sqrt{\frac{d\ln(2/\alpha)}{2n}}\right)^{2/3}\right\}},$$

then $p_{n,1/2}(c_n) - p_{n,\alpha}(c_n) \leq \varepsilon_n$. The proof that $p_{n,(1-\alpha)}(c_n) - p_{n,1/2}(c_n) \leq \varepsilon_n$ is similar.

Applying the trivial bound $L_n^{(1)} \leq n$ and Lemma 3.1, we know that no matter how ε_n is chosen,

$$\int_{p_{n,1/2}(c_n)-\varepsilon_n}^{p_{n,1/2}(c_n)-\frac{3\varepsilon_n}{4}} D_{n,p}(L_n^{(1)})\,\mathrm{d}p \leq \frac{Cn^2}{\ln n},$$

where $C = C(b, d)$ is defined in Lemma 3.1. Hence, by virtue of the mean value theorem for integration there is some $q_{1,n} \in [p_{n,1/2}(c_n) - \varepsilon_n, p_{n,1/2}(c_n) - 3\varepsilon_n/4]$ such that

$$D_{n,q_{1,n}}(L_n^{(1)}) \leq \frac{4Cn^2}{\varepsilon_n \ln n}. \tag{3.4}$$

Likewise, there is some $q_{2,n} \in [p_{n,1/2}(c_n) - \varepsilon_n/2, p_{n,1/2}(c_n) - \varepsilon_n/4]$ such that

$$D_{n,q_{2,n}}(L_n^{(1)}) \leq \frac{4Cn^2}{\varepsilon_n \ln n}. \tag{3.5}$$

Now, it suffices to prove that $q_{1,n} \leq p_{n,\alpha}(c_n)$. To this end, we will use the method of reduction to absurdity. Suppose that $p_{n,\alpha}(c_n) < q_{1,n}$. Since $q_{1,n} + \varepsilon_n/4 \leq q_{2,n} \leq p_{n,1/2}(c_n)$, by Lemma 3.3 and Lagrange's mean value theorem, for n large enough, we have

$$\begin{aligned} & E_{n,q_{2,n}}(L_n^{(1)}) - E_{n,q_{1,n}}(L_n^{(1)}) \\ \geq & \frac{\varepsilon_n \alpha b n}{8} \cdot \min\left\{ c_n, 1 - c_n, (1 - q_3(c_n))^d - \sqrt{\frac{d \ln(2/\alpha)}{2n}} \right\}. \end{aligned}$$

On the other hand, let M_{p_n} be the median of $L_n^{(1)}$ under μ_{n,p_n} (we assume the form of $k + 1/2$ with $k \in \mathbb{N}$, which ensures its uniqueness). By the definition of median and the fact that M_p is increasing with p,

$$c_n n \geq M_{p_{n,1/2}(c_n)} - \frac{1}{2} \geq M_{q_{2,n}} - \frac{1}{2}.$$

Using Levy's inequality [11] and (3.5), it follows that

$$|E_{n,q_{2,n}}(L_n^{(1)}) - M_{q_{2,n}}| \leq \sqrt{D_{n,q_{2,n}}(L_n^{(1)})} \leq n\sqrt{\frac{4C}{\varepsilon_n \ln n}}.$$

Wrapping up the above arguments, we derive

$$\begin{aligned} & \mu_{n,q_{1,n}}\{L_n^{(1)} \geq c_n n\} \\ = & \mu_{n,q_{1,n}}\{L_n^{(1)} - E_{n,q_{1,n}}(L_n^{(1)}) \geq c_n n - E_{n,q_{1,n}}(L_n^{(1)})\} \\ \leq & \mu_{n,q_{1,n}}\left\{L_n^{(1)} - E_{n,q_{1,n}}(L_n^{(1)}) \geq M_{q_{2,n}} - \frac{1}{2} - E_{n,q_{1,n}}(L_n^{(1)})\right\} \\ = & \mu_{n,q_{1,n}}\Big\{L_n^{(1)} - E_{n,q_{1,n}}(L_n^{(1)}) \geq M_{q_{2,n}} - E_{n,q_{2,n}}(L_n^{(1)}) \\ & \qquad + E_{n,q_{2,n}}(L_n^{(1)}) - \frac{1}{2} - E_{n,q_{1,n}}(L_n^{(1)})\Big\} \end{aligned}$$

$$\leq \mu_{n,q_{1,n}}\left\{L_n^{(1)} - E_{n,q_{1,n}}(L_n^{(1)}) \geq \frac{\varepsilon_n \alpha b n}{8} \cdot \min\left\{c_n, 1-c_n, (1-q_3(c_n))^d - \sqrt{\frac{d\ln(2/\alpha)}{2n}}\right\} - \frac{1}{2} - n\sqrt{\frac{4C}{\varepsilon_n \ln n}}\right\}$$

Now, choosing $C_5 = (400C)^{1/3}/(\alpha b^{2/3})$, we have

$$\varepsilon_n = \frac{(400C)^{1/3}}{\alpha b^{2/3}(\ln n)^{1/3} \cdot \min\left\{c_n^{2/3}, (1-c_n)^{2/3}, \left((1-q_3(c_n))^d - \sqrt{\frac{d\ln(2/\alpha)}{2n}}\right)^{2/3}\right\}}.$$

Since for n large enough,

$$\begin{aligned} &\frac{\varepsilon_n \alpha b n}{8} \cdot \min\left\{c_n, 1-c_n, (1-q_3(c_n))^d - \sqrt{\frac{d\ln(2/\alpha)}{2n}}\right\} - \frac{1}{2} - n\sqrt{\frac{4C}{\varepsilon_n \ln n}} \\ &\geq \frac{\varepsilon_n \alpha b n}{10} \cdot \min\left\{c_n, 1-c_n, (1-q_3(c_n))^d - \sqrt{\frac{d\ln(2/\alpha)}{2n}}\right\}, \end{aligned}$$

by Chebyshev's inequality and (3.4), we obtain

$$\begin{aligned} &\mu_{n,q_{1,n}}\{L_n^{(1)} \geq c_n n\} \\ \leq\ & \frac{100 D_{n,q_{1,n}}(L_n^{(1)})}{\alpha^2 b^2 \varepsilon_n^2 n^2 \min\left\{c_n^2, (1-c_n)^2, \left((1-q_3(c_n))^d - \sqrt{\frac{d\ln(2/\alpha)}{2n}}\right)^2\right\}} \\ \leq\ & \frac{400 C n^2}{\alpha^2 b^2 \varepsilon_n^3 n^2 \ln n \cdot \min\left\{c_n^2, (1-c_n)^2, \left((1-q_3(c_n))^d - \sqrt{\frac{d\ln(2/\alpha)}{2n}}\right)^2\right\}} \\ =\ & \alpha. \end{aligned}$$

Thus, we deduce $q_{1,n} \leq p_{n,\alpha}(c_n)$, which is a contradiction to our previous assumption. The proof of Proposition 3.1 is complete. □

Although a more general threshold phenomenon for the appearance of giant component has been shown in this paper, the interesting conjecture (see [2: Conjecture 1.2]) that a giant component emerges in an interval of length $o(1)$ in any expander remains open.

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