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A NOTE ON THE CONVEXITY OF LATTICES GENERATED BY THE SET OF NONNEGATIVE INTEGERS

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Dedicated to Professor Ján Jakubík on the occasion of his 90th birthday

(*Communicated by Miroslav Ploščica*)

ABSTRACT. A class of lattices is said to be a convexity if it is closed under homomorphic images, convex sublattices and direct products. The main aim of this paper is to show that convexity generated by nonnegative integers contains all ordinal numbers. Consequently, any two infinite ordinals generate the same convexity.

1. Introduction

The notion of convexity of lattices has been introduced by E. Fried at the Problem session of the International Conference held in memory of Wilfried Nöbauer in Krems, Austria, in 1988, cf. [15]. A class of lattices is said to be a convexity if it is closed under homomorphic images, convex sublattices and direct products. E. Fried proposed a problem concerning the "number" of convexities of lattices. He expressed the conjecture that there is no such cardinal, i.e., there are too many different convexities. This conjecture was solved affirmatively by J. Jakubík in [3]. In [3], it is also proved that the system of all convexities of lattices ordered by class-theoretical inclusion forms a complete lattice (omitting the fact that this system does not form a set) and that two-element chain generates an atom in this lattice. Hence in [3], the natural question was raised whether the convexity generated by two-element chain is the only atom in the lattice of all

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convexities of lattices. G. Czédli in [2] shed more light on this problem. Using the notion of M_3-simple semifractal lattice he showed that convexity generated by any such lattice does not contain any minimal subconvexity. Moreover, he proved that the collection of all lattice convexities which contain no minimal convexity forms a proper class, not a set. However, the question whether there exists another atom in the lattice of all convexities of lattices is still open.

Another interesting results concerning the lattice of all convexities of lattices were obtained by J. Lihová in [13]. Characterization of join of two convexities was given. Consequently, based on this characterization it was proved that the lattice of all convexities is distributive. In [13] also relations between convexities generated by some types of chains were studied. In particular, it was proved that convexities generated by nonisomorphic finite chains are different.

Convexities can be defined also for various types of algebraic structures where the notion of convex subalgebra can be applied. J. Jakubík defined and studied convexities of d-groups [4] and ℓ-groups [5, 6]. Some results concerning convexities of Riesz groups were derived by J. Lihová in [12] and the lattice of all convexities of partial monounary algebras was described by D. Jakubíková-Studenovská in [10]. Let us note, that the investigation of convexities is related to the study of other class of algebras, cf. [7–9].

The main aim of this paper is to investigate whether there exist two infinite ordinals which generate different convexities. Contrary to the above mentioned result of J. Lihová concerning finite chains (ordinals), we answer this question negatively. We prove that all infinite ordinals generate the same convexity, in particular convexity generated by the set of all nonnegative integers (first infinite ordinal ω_0).

2. Preliminaries

Let $\mathcal{L}$ be the class of all lattices. For any nonempty subclass $\mathcal{K} \subseteq \mathcal{L}$ of lattices we denote by

$\mathbf{H}(\mathcal{K})$ – the class of all homomorphic images of elements of $\mathcal{K}$;
$\mathbf{C}(\mathcal{K})$ – the class of all convex sublattices of elements of $\mathcal{K}$;
$\mathbf{P}(\mathcal{K})$ – the class of all direct products of elements of $\mathcal{K}$.

As we mentioned in the introductory section, a class $\mathcal{K}$ of lattices is said to be a *convexity* if it is closed with respect to the operators $\mathbf{H}$, $\mathbf{C}$ and $\mathbf{P}$.

We will use the following theorem [15].

Theorem 2.1. *Let $\mathcal{K} \subseteq \mathcal{L}$ be a nonempty class of lattices. Then $\mathbf{HCP}(\mathcal{K})$ is the least convexity containing $\mathcal{K}$.*

In this case, as it is usual, we will say that the class $\mathcal{K}$ generates convexity $\mathbf{HCP}(\mathcal{K})$.

Obviously, any variety of lattices is convexity. The converse does not hold in general. An example being the convexity generated by two-element chain C_2. It is easy to see that each $L \in \mathbf{HCP}(\{C_2\})$ is a relatively complemented lattice, while three-element chain fails to have this property. Consequently, convexity $\mathbf{HCP}(\{C_2\})$ is a proper subclass of the variety of all distributive lattices.

Further, we will need some notions common in set theory.

Let $(P, \leq)$ be a poset. A subset $A \subseteq P$ is *cofinal* in P if for any $x \in P$, there is an element $a \in A$ such that $x \leq a$.

Using Zorn's Lemma one can prove the following assertion.

Lemma 2.2. *Let $(S, \leq)$ be a linearly ordered set. Then there exists a well-ordered subset $W \subseteq S$ which is cofinal in S.*

For a linearly ordered set S, $\mathrm{cf}(S)$ is defined as the least ordinal α such that there exists a well-ordered subset $W \subseteq S$ of order-type α.

We will use $\aleph_\alpha$ when referring to the cardinal number, and ω_α to denote the order-type. An infinite cardinal $\aleph_\alpha$ is *regular* if $\mathrm{cf}(\omega_\alpha) = \omega_\alpha$. It is *singular* if $\mathrm{cf}(\omega_\alpha) < \omega_\alpha$. For every limit ordinal α, $\mathrm{cf}(\mathrm{cf}(\alpha)) = \mathrm{cf}(\alpha)$ and $\mathrm{cf}(\alpha)$ is a regular cardinal.

Let κ be a cardinal. If $X \subseteq \kappa$ and $|X| < \mathrm{cf}(\kappa)$ then X is bounded in κ, i.e., $\sup X < \kappa$. As a consequence we obtain that if κ is a regular cardinal then any subset $X \subseteq \kappa$ with $|X| < \kappa$ is bounded in κ. Equivalently, if any $X \subseteq \kappa$ is cofinal in κ, then $|X| = \kappa$.

The next well known result is the consequence of the Axiom of Choice, cf. [11].

Lemma 2.3. *Every $\aleph_{\alpha+1}$ is a regular cardinal.*

Hence, assuming the Axiom of Choice, if κ is a singular cardinal, then κ is limit. Let κ be a limit cardinal. If $\{\omega_\xi : \xi < \mathrm{cf}(\kappa)\}$ is cofinal in κ then $\{\omega_\xi^+ : \xi < \mathrm{cf}(\kappa)\}$ (ω_ξ^+ denotes successor cardinal) is cofinal in κ, too. Hence using Lemma 2.3 we obtain that for any limit cardinal κ there exists a $\mathrm{cf}(\kappa)$-sequence $\{\omega_\xi : \xi < \mathrm{cf}(\kappa)\}$ cofinal in κ such that ω_ξ is regular for each $\xi < \mathrm{cf}(\kappa)$.

Lemma 2.4. *Let κ be a limit cardinal and $\{\omega_\xi : \xi < \mathrm{cf}(\kappa)\}$ be a cofinal $\mathrm{cf}(\kappa)$-sequence in κ. If $\varkappa < \kappa$ is a cardinal, then*

$$|\{\xi : \omega_\xi \leq \varkappa\}| < |\mathrm{cf}(\kappa)| .$$

Proof. The set $\{\omega_\xi : \xi < \mathrm{cf}(\kappa)\}$ is cofinal in κ, hence there exists the smallest $\xi_0 < \mathrm{cf}(\kappa)$ with $\omega_{\xi_0} > \varkappa$. Since $\mathrm{cf}(\kappa)$ is cardinal and $\{\xi : \omega_\xi \leq \varkappa\} = \{\xi : \xi < \xi_0\}$ we obtain $|\{\xi : \omega_\xi \leq \varkappa\}| < |\mathrm{cf}(\kappa)|$. □

Now we prove that any well-ordered cofinal subset of a linearly ordered set is a lattice homomorphic image of this linearly ordered set.

Lemma 2.5. *Let $(S, \leq)$ be a linearly ordered set and $W \subseteq S$ be a well-ordered cofinal subset of S. Then there exist a lattice homomorphism of S onto W.*

Proof. Define $f\colon S \to W$ as follows: for $s \in S$ we put

$$f(s) = \min\{w \in W : \ w \geq s\}.$$

Since W is cofinal in S the mapping f is defined for all $s \in S$ and obviously it is surjective. Further, f is order preserving and thus a lattice homomorphism. $\square$

Finally, we briefly recall the basic notions concerning ultraproducts.

Let L_i for each $i \in I$, $I \neq \emptyset$, be a system of lattices and U be an ultrafilter over the index set I. The symbol $\prod\limits_{i\in I} L_i/U$ will be used for the ultraproduct of the sytem $(L_i : \ i \in I)$. If the system of lattices $(L_i : \ i \in I)$ contains only one lattice L, then $L^I/U = \prod\limits_{i\in I} L/U$ will be also referred to as an ultrapower of L. For an element $g \in \prod\limits_{i\in I} L_i$ the symbol $[g]/U$ will denote the congruence class containing the element g.

The famous Theorem of Łoś says that for a formula $\phi(x_1, \dots, x_n)$ and $f_1, \dots$ $\dots, f_n \in \prod\limits_{i\in I} L_i$ we have

$$\prod_{i\in I} L_i/U \models \phi\big([f_1]/U, \dots, [f_n]/U\big) \quad \text{iff} \quad \big\{i \in I : \ L_i \models \phi(f_1(i), \dots, f_n(i))\big\} \in U.$$

In particular, for all sentences ϕ we have $\prod\limits_{i\in I} L_i/U \models \phi$ if and only if $\big\{i \in I :$ $L_i \models \phi\big\} \in U$. Consequently, we obtain that an ultraproduct of ordinals is always a linearly ordered set.

It is well known (see [1]) that an ultraproduct of infinite ordinals modulo ultrafilter U is well-ordered if and only if U is σ-complete, i.e., if U is closed under countable intersections. However, the hypothesis that there exists a non-principal σ-complete ultrafilters is not provable from ZFC. The existence of such ultrafilters is closely related to the existence of large cardinals. In particular, the first cardinal κ such that there is a non-principal σ-complete ultrafilter over a set of cardinality κ is a measurable cardinal.

3. Convexity generated by nonnegative integers

In this section we prove our main result. First, recall that ultraproduct of some system $(L_i : \ i \in I)$ is defined as some quotient set modulo a congruence relation on the direct product $\prod\limits_{i\in I} L_i$. Hence, if $L_i \in \mathcal{K}$ for all $i \in I$, then $\prod_{i\in I} L_i/U \in \mathbf{HP}(\mathcal{K}) \subseteq \mathbf{HCP}(\mathcal{K})$, i.e., convexities are closed under forming ultraproducts.

Further, let $\beta \leq \alpha$ be ordinals. Since β is an initial segment of the ordinal α, it follows that $\beta \in \mathbf{C}(\{\alpha\})$. Consequently, convexities are "downward" closed, i.e., if $\beta \leq \alpha$ and $\alpha \in \mathbf{HCP}(\mathcal{K})$ then $\beta \in \mathbf{HCP}(\mathcal{K})$.

In what follows, we prove that convexities are also "upward" closed, i.e., $\beta \leq \alpha$ and $\beta \in \mathbf{HCP}(\mathcal{K})$ implies $\alpha \in \mathbf{HCP}(\mathcal{K})$. This yields that any two infinite ordinals generate the same convexity, in particular the convexity $\mathbf{HCP}(\{\omega_0\})$, i.e., the convexity generated by the set of all nonnegative integers. Let *Ord* be the class of all ordinal numbers.

Theorem 3.1. *Ord* $\subseteq \mathbf{HCP}\big(\{\omega_0\}\big)$.

Proof. Assume, that *Ord* $\nsubseteq \mathbf{HCP}\big(\{\omega_0\}\big)$, i.e, there is some ordinal $\gamma \in$ *Ord* with $\gamma \notin \mathbf{HCP}\big(\{\omega_0\}\big)$. Since convexities are downward closed, there is the smallest $\alpha \in$ *Ord* such that cardinal $\omega_\alpha \notin \mathbf{HCP}\big(\{\omega_0\}\big)$. Consequently, $\omega_\beta \in \mathbf{HCP}\big(\{\omega_0\}\big)$ for all $\beta < \alpha$.

We will consider two cases.

First, assume that ω_α is the successor cardinal of a regular cardinal ω_β, i.e., $\omega_\alpha = \omega_{\beta+1}$.

Let $I = \omega_\beta$ be the index set and U be an uniform ultrafilter over I, i.e., ultrafilter containing the set $\big\{X \subseteq I : \ |I \smallsetminus X| < \aleph_\beta\big\}$. We will show that any subset $Y \subseteq {\omega_\beta}^I/U$ with $|Y| \leq \aleph_\beta$ is bounded in ${\omega_\beta}^I/U$. Let $Y = \big\{[g_\xi]/U\big\}_{\xi<\kappa}$, $\kappa \leq \omega_\beta$ be a subset of elements of ${\omega_\beta}^I/U$. Define $g\colon I \to \omega_\beta$ for each $\gamma \in I$ as follows:

$$g(\gamma) = \sup\big\{g_\xi(\gamma) : \ \xi \leq \gamma\big\}.$$

For any $\gamma < \omega_\beta$ the value $g(\gamma)$ is the supremum of fewer than $\aleph_\beta$ elements of ω_β and since ω_β is regular, it follows that $g(\gamma) < \omega_\beta$. Hence $g \in \omega_\beta^I$. Now we show that $[g]/U$ is the upper bound of the set Y in ω_β^I/U. Let $\xi < \kappa$ be any ordinal. If $\gamma \geq \xi$, then $g(\gamma) \geq g_\xi(\gamma)$. Since U is the uniform ultrafilter over I, we obtain

$$\big\{\gamma \in I : \ g(\gamma) \geq g_\xi(\gamma)\big\} \supseteq \{\gamma \in I : \ \gamma \geq \xi\} \in U.$$

This shows that $[g]/U$ is the upper bound of the set Y in ${\omega_\beta}^I/U$. According to Lemma 2.2, the set ${\omega_\beta}^I/U$ contains well-ordered cofinal subset W. Since we have shown that any subset of ${\omega_\beta}^I/U$ with cardinality lower than $\aleph_\beta$ is not cofinal, we obtain that $|W| \geq \aleph_\alpha = \aleph_{\beta+1} > \aleph_\beta$. Consequently, the order-type of W is at least ω_α, hence according to Lemma 2.5 we obtain $W \in \mathbf{HCP}\big(\{\omega_\beta\}\big)$. Since $\omega_\alpha \in \mathbf{C}(\{W\})$, we have proved $\omega_\alpha \in \mathbf{HCP}\big(\{\omega_\beta\}\big) = \mathbf{HCP}\big(\{\omega_0\}\big)$, which yields a contradiction.

Next, assume that ω_α is the limit cardinal or the successor of a singular cardinal. As we mentioned in the previous section, singular cardinals are always limit cardinals. Let $\varkappa$ be a limit cardinal such that $\omega_\alpha = \varkappa$ or $\omega_\alpha = \varkappa^+$. There exists a $\mathrm{cf}(\varkappa)$-sequence $\{\omega_\lambda\}_{\lambda<\mathrm{cf}(\varkappa)}$ cofinal in $\varkappa$, such that ω_λ is a regular cardinal for all $\lambda < \mathrm{cf}(\varkappa)$. We put $I = \mathrm{cf}(\varkappa)$ as the index set and let U denote an uniform ultrafilter over I. As in the previous case, our aim is to show that $\mathrm{cf}\Big(\prod\limits_{\lambda\in I}\omega_\lambda/U\Big) \geq \omega_\alpha$.

Let $Y = \big\{[g_\xi]/U\big\}_{\xi<\kappa}$, $\kappa < \varkappa$ be a subset of $\prod\limits_{\lambda\in I}\omega_\lambda/U$, we prove that Y is bounded in $\prod\limits_{\lambda\in I}\omega_\lambda/U$. Define $g \in \prod\limits_{\lambda\in I}\omega_\lambda$ for all $\lambda \in I$ as follows:

$$g(\lambda) = \begin{cases} 0 & \text{if } \omega_\lambda \leq \kappa, \\ \sup\big\{g_\xi(\lambda):\ \xi < \kappa\big\} & \text{if } \omega_\lambda > \kappa. \end{cases}$$

Since each ω_λ is a regular cardinal, it follows that $\sup\big\{g_\xi(\lambda):\ \xi < \kappa\big\} < \omega_\lambda$ for all $\lambda \in I$ with $\omega_\lambda > \kappa$. As a consequence we obtain $g \in \prod\limits_{\lambda\in I}\omega_\lambda$.

According to Lemma 2.4 we have

$$|I \setminus \{\lambda:\ \omega_\lambda > \kappa\}| = |\{\lambda:\ \omega_\lambda \leq \kappa\}| < |I|\,.$$

Since the ultrafilter U is uniform over $I = \mathrm{cf}(\varkappa)$, we obtain

$$\big\{\lambda \in I:\ g(\lambda) \geq g_\xi(\lambda)\big\} \supseteq \big\{\lambda \in I:\ \omega_\lambda > \kappa\big\} \in U$$

for all $\xi < \kappa$.

Hence $[g]/U$ is the upper bound of the set Y in $\prod\limits_{\lambda\in I}\omega_\lambda/U$. This yields that $\mathrm{cf}\Big(\prod\limits_{\lambda\in I}\omega_\lambda/U\Big) \geq \varkappa$ since any subset of $\prod\limits_{\lambda\in I}\omega_\lambda/U$ with lower cardinality that $\varkappa$ is not cofinal. If ω_α is the successor cardinal of singular $\varkappa$, than also $\mathrm{cf}\Big(\prod\limits_{\lambda\in I}\omega_\lambda/U\Big) \geq \varkappa^+ = \omega_\alpha$, since cofinality is always a regular cardinal. Similarly as in the previous case, using Lemma 2.5 we obtain $\omega_\alpha \in \mathbf{HCP}\big(\{\omega_\lambda :\ \lambda < \mathrm{cf}(\varkappa)\}\big) \subseteq \mathbf{HCP}\big(\{\omega_0\}\big)$. □

Next, we provide an another proof of Theorem 3.1. We will use some notions and results of model theory. First we recall the notion of elementary equivalence of two models.

Let $\mathcal{L}$ be a signature. Two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$ are elementarily equivalent if for all $\mathcal{L}$-sentences ϕ, $\mathcal{M} \models \phi$ if and only if $\mathcal{N} \models \phi$. Hence, interpretations (models) for a first-order language $\mathcal{L}$ are elementarily equivalent in $\mathcal{L}$, provided that they make exactly the same sentences from $\mathcal{L}$ true.

The next theorem is the well-known Downward Löwenheim-Skolem Theorem, cf. [14].

Theorem 3.2. *Suppose that $\mathcal{M}$ is an $\mathcal{L}$–structure and $X \subseteq \mathcal{M}$. Then there is an elementary submodel $\mathcal{N}$ of $\mathcal{M}$ such that $X \subseteq \mathcal{N}$ and $|\mathcal{N}| \leq |X| + |\mathcal{L}| + \aleph_0$.*

Next we recall so-called Keisler-Shelah Isomorphism Theorem, which provides a characterization of elementary equivalence. This theorem was formulated and proved by H. J. Keisler with the assumption of the generalized continuum hypothesis. Later, S. Shelah proved this theorem in its full generality, i.e., avoiding GCH (see [16]).

Theorem 3.3. *Let $\mathcal{L}$ be a signature and let $\mathcal{A}, \mathcal{B}$ be two $\mathcal{L}$–structures. Then $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent if and only if there are a set I and an ultrafilter U over I such that $\mathcal{A}^I/U \cong \mathcal{B}^I/U$.*

In order to give an alternative proof of Theorem 3.1 we will need a result concerning convexity generated by real numbers, cf. [13].

Theorem 3.4. $\mathbf{HCP}(\{\omega_0\}) = \mathbf{HCP}(\{\mathbb{Z}\}) = \mathbf{HCP}(\{\mathbb{R}\})$.

As an interesting fact, let us remark that in [6] it was shown that the convexity of ℓ-groups generated by $\mathbb{R}$ is the proper subclass of the convexity of ℓ-groups generated by the set of all integers $\mathbb{Z}$.

Now we can provide another proof of Theorem 3.1

Proof. Let $\beta \in \mathit{Ord}$ be any infinite ordinal. According to Theorem 3.4 it is sufficient to show that $\beta \in \mathbf{HCP}(\{\mathbb{R}\})$.

Denote by α a countable ordinal such that β and α are elementarily equivalent. The existence of the countable ordinal α is given by Downward Löwenheim–Skolem Theorem 3.2. It is a well-known fact that any countable linearly ordered set can be embedded into $\mathbb{R}$. Let $\varphi\colon \alpha \to \mathbb{R}$ be an order embedding and $C(\varphi(\alpha))$ be the smallest convex sublattice of $\mathbb{R}$ containing $\varphi(\alpha)$, i.e., the convex hull of $\varphi(\alpha)$. Evidently $\varphi(\alpha)$ is cofinal in $C(\varphi(\alpha))$. Hence, due to Lemma 2.5, the ordinal α is a homomorphic image of the convex hull of it's own copy in the chain of real numbers and $\alpha \in \mathbf{HCP}(\{\mathbb{R}\})$.

Since α and β are elementarily equivalent, according to Theorem 3.3, there are a set I and an ultrafilter U over I such that $\alpha^I/U \cong \beta^I/U$. Obviously, β^I/U contains an isomorphic copy of β, as a sublattice. Consequently, the same is true for α^I/U. Taking convex hull $C(\beta)$ of β in α^I/U and using Lemma 2.5, we obtain $\beta \in \mathbf{HCP}(\{\alpha\}) \subseteq \mathbf{HCP}(\{\mathbb{R}\})$. □

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