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LEXICO GROUPS AND RELATED RADICAL CLASSES

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Dedicated to Professor Ján Jakubík on the occasion of his 90th birthday

(*Communicated by Jiří Rachůnek*)

ABSTRACT. There are defined and studied radical classes of lattice ordered groups, in some way related to the notion of lexico extensions. It is shown that such radical classes form a proper class and they fail to be torsion classes.

1. Introduction

For lattice ordered groups we apply the terminology and the notation as in [1]. Thus the group operation in a lattice ordered group (ℓ-group) is written additively, although we do not assume the validity of the commutative law.

The notion of lexico extensions of ℓ-groups has been applied in several papers (cf., e.g., [2], [3], [4], [7], [8]).

For the notion of radical classes and torsion classes of ℓ-groups, see, e.g. [5], [6], [9].

Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be the collections of all torsion classes and radical classes, respectively, and let $\mathbf{X}_3 = \mathbf{X}_2 \setminus \mathbf{X}_1$. The very useful Conrads article [5] is a survey paper on torsion classes. It characterizes all torsion classes which have been previously described in the literature. On the other hand, published material concerning the collection $\mathbf{X}_3$ is rather modest. Therefore, when we are interested in $\mathbf{X}_3$, we have to begin by searching radical classes which fail to be torsion classes. This paper presents a step in this direction. In fact, we do not deal with an isolated radical class, but with a collection of classes in $\mathbf{X}_3$ which have mutually similar basic ideas.

We recall the corresponding terminology.

Let G be an ℓ-group and $x \in G$, $\emptyset \neq Y \subseteq G$. We will write $x > Y$ if $x > y$ for each $y \in Y$. The meaning of $x < Y$ is analogous.

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An ℓ-group A will be said to be a *lexico extension* of its subgroup A_0, if $A_0 \neq A$ and for each $x \in A \setminus A_0$ we have either $x > A_0$ or $x < A_0$. In such case we also say that A is a *lexico group*.

If G is an ℓ-group, then the symbol $c(G)$ will be used for the system of all convex ℓ-subgroups of G.

It is easy to see that if an ℓ-group A is a lexico extension of its subgroup A_0, then $A_0 \in c(A)$. If $A \in c(G)$ and A is a lexico group, then it is said to be a *lexico subgroup* (*lex-subgroup*) of G.

Let us remark that the definitions of lexico extensions introduced by Conrad in [3] and by Jakubík in [8], we are working with, are identical. Namely, if A is a lexico extension of A_0 in our sense, then the lattice A/A_0 of right cosets is linearly ordered (cf. [8], Lemma 2.1), i.e., A_0 is a prime convex ℓ-subgroup of A.

We give some simple examples of lexico extensions.

Example 1. A linearly ordered group is a lexico extension of $\{0\}$, but also of any non-zero proper convex subgroup.

Example 2. The lexicographic product $L \circ G$ of any linearly ordered group L and ℓ-group G is a lexico extension of G.

Example 3. The wreath product $\mathbb{R}\ Wr\ \mathbb{Z}$ is a lexico extension of $\mathbb{R}^{\mathbb{Z}}$, which fails to be a lexicographic product.

In what follows, $\mathcal{G}$ will denote the class of all ℓ-groups.

Consider the following conditions for a non-empty subclass $\mathcal{T}$ of $\mathcal{G}$:

(i) $\mathcal{T}$ is closed with respect to isomorphisms;

(ii) if $G \in \mathcal{T}$, then $c(G) \subseteq \mathcal{T}$;

(iii) if $G \in \mathcal{G}$ and $\{H_i\}_{i \in I} \subseteq c(G) \cap \mathcal{T}$, then $\bigvee\limits_{i \in I} H_i \in \mathcal{T}$;

(iv) $\mathcal{T}$ is closed with respect to homomorphic images.

A class $\mathcal{T}$ of ℓ-groups fulfilling conditioms (i)–(iii) is said to be a *radical class*. If, moreover, $\mathcal{T}$ fulfills (iv), $\mathcal{T}$ is called a *torsion class*.

We will define and study certain radical classes, related to the notion of lexico extension (for the definition see Section 2). Besides, we will focus on some properties of the ordered system of such radical classes.

2. Radical classes of the type $\mathcal{R}_{\mathcal{M}}$

Let $\mathbf{M}$ be the system of all $\emptyset \neq \mathcal{M} \subseteq \mathcal{G}$ fulfilling conditions (i) and (ii).

For an ℓ-group G and a system $\mathcal{M} \in \mathbf{M}$, consider the following condition:

($*$) if $H_1, H_2 \in c(G)$ and H_2 is a lexico extension of H_1, then $H_1 \notin \mathcal{M}$.

Let $\mathcal{R}_{\mathcal{M}}$ be the system of all $G \in \mathcal{G}$ satisfying the condition ($*$).

In other words, $G \notin \mathcal{R}_{\mathcal{M}}$ if and only if there exist $H_1, H_2 \in c(G)$, H_2 being a lexico extension of H_1, with $H_1 \in \mathcal{M}$.

Theorem 2.1. *For each $\mathcal{M} \in \mathbf{M}$, $\mathcal{R}_{\mathcal{M}}$ is a radical class.*

Proof. Each $\mathcal{R}_{\mathcal{M}}$ is evidently closed with respect to isomorphisms and convex ℓ-subgroups. To show the validity of (iii), take $G \in \mathcal{G}$, $\{G_i\}_{i\in I} \in c(G) \cap \mathcal{R}_{\mathcal{M}}$ and put $G' = \bigvee\limits_{i\in I} G_i$ (the operation $\bigvee$ means the operation sup in the complete lattice $c(G)$).

We want to show that $G' \in \mathcal{R}_{\mathcal{M}}$. By way of contradiction, let us suppose that $G' \notin \mathcal{R}_{\mathcal{M}}$. Then there exist $H_1, H_2 \in c(G')$, H_2 being a lexico extension of H_1, with $H_1 \in \mathcal{M}$.

Let us suppose that $H_1 \cap G_i = H_2 \cap G_i$ for all $i \in I$.

Then we have

$$H_1 = H_1 \cap G' = H_1 \cap \bigvee_{i\in I} G_i = \bigvee_{i\in I}(H_1 \cap G_i) = \bigvee_{i\in I}(H_2 \cap G_i) = H_2 \cap \bigvee_{i\in I} G_i = H_2 \cap G' = H_2\,,$$

a contradiction.

Thus there exists $i_0 \in I$ such that $H_1 \cap G_{i_0} \subsetneqq H_2 \cap G_{i_0}$. Evidently $H_2 \cap G_{i_0}$ is a lexico extension of $H_1 \cap G_{i_0}$ in G_{i_0}. As $G_{i_0} \in \mathcal{R}_{\mathcal{M}}$, it is $H_1 \cap G_{i_0} \notin \mathcal{M}$, which implies $H_1 \notin \mathcal{M}$, again a contradiction. □

Let us remark that $\mathcal{R}_{\mathcal{M}}$ for $\mathcal{M}$ containing only one-element ℓ-groups is the system of all ℓ-groups without non-zero convex linearly ordered subgroups. On the other hand, $\mathcal{R}_{\mathcal{G}}$ is the system of all ℓ-groups which have no lex-subgroups.

A natural question is whether $\mathcal{R}_{\mathcal{M}_1} = \mathcal{R}_{\mathcal{M}_2}$ can occur for $\mathcal{M}_1, \mathcal{M}_2 \in \mathbf{M}$, $\mathcal{M}_1 \neq \mathcal{M}_2$. We are going to show that the answer to this question is positive.

The following assertion is evident.

Lemma 2.2. *If $\mathcal{M}_1, \mathcal{M}_2 \in \mathbf{M}$, $\mathcal{M}_1 \subseteq \mathcal{M}_2$, then $\mathcal{R}_{\mathcal{M}_2} \subseteq \mathcal{R}_{\mathcal{M}_1}$.*

First, we want to describe the least radical class of the type $\mathcal{R}_{\mathcal{M}}$ containing an arbitrary non-empty subclass $\mathcal{L} \subseteq \mathcal{G}$.

Let us introduce the following terminology and notation.

If an ℓ-group A is a lexico extension of A_0, we will say that A_0 is a *heel* of A. Heels of an ℓ-group G will be the heels of all lex-subgroups of G. The set of all heels of G will be denoted by $\mathbf{h}(G)$. Further, let us introduce the following notation for any $\emptyset \neq \mathcal{X} \subseteq \mathcal{G}$:

$$\mathbf{h}(\mathcal{X}) = \bigcup_{G\in\mathcal{X}} \mathbf{h}(G)\,,$$

$$\mathbf{e}(\mathcal{X}) = \big\{G \in \mathcal{G} : \text{ there exists } X \in \mathcal{X} \text{ with } X \in c(G)\big\}\,.$$

Both $\mathbf{h}(\mathcal{X})$ and $\mathbf{e}(\mathcal{X})$ are assumed to be closed with respect to isomorphisms.

Let $\mathbf{R}$ be the system of all $\mathcal{R}_{\mathcal{M}}$ with $\mathcal{M} \in \mathbf{M}$.

Theorem 2.3. *Let $\emptyset \neq \mathcal{L} \subseteq \mathcal{G}$, $\mathcal{L} \nsubseteq \mathcal{R}_{\mathcal{G}}$. Let $\mathcal{M} = \mathcal{G} \setminus \mathbf{e}(\mathbf{h}(\mathcal{L}))$. Then*

1) *$\mathcal{M} \in \mathbf{M}$ and $\mathcal{L} \subseteq \mathcal{R}_{\mathcal{M}}$;*
2) *if $\mathcal{R}_{\mathcal{M}'} \supseteq \mathcal{L}$ for some $\mathcal{M}' \in \mathbf{M}$, then $\mathcal{M} \supseteq \mathcal{M}'$ and hence $\mathcal{R}_{\mathcal{M}} \subseteq \mathcal{R}_{\mathcal{M}'}$.*

Proof.

1) Evidently $\mathcal{M} \in \mathbf{M}$. To show that $\mathcal{L} \subseteq \mathcal{R}_{\mathcal{M}}$, let $G \in \mathcal{L}$, $H_1, H_2 \in c(G)$, H_2 be a lexico extension of H_1. We want to show that $H_1 \notin \mathcal{M}$. As $H_1 \in \mathbf{h}(\mathcal{L})$, we have also $H_1 \in \mathbf{e}(\mathbf{h}(\mathcal{L}))$, which yields $H_1 \notin \mathcal{M}$.

2) Let $\mathcal{R}_{\mathcal{M}'} \supseteq \mathcal{L}$ for some $\mathcal{M}' \in \mathbf{M}$. Suppose that $G \notin \mathcal{M}$. Then $G \in \mathbf{e}(\mathbf{h}(\mathcal{L}))$, so there exist $L \in \mathcal{L}$ and $H_1 \in \mathbf{h}(L)$ such that $H_1 \in c(G)$. Since $L \in \mathcal{L} \subseteq \mathcal{R}_{\mathcal{M}'}$, we have $H_1 \notin \mathcal{M}'$, which implies $G \notin \mathcal{M}'$. □

Lemma 2.4. *For any $\mathcal{M} \in \mathbf{M}$, the following conditions are equivalent for $G \in \mathcal{G}$:*

(1) *$G \in \mathbf{h}(\mathcal{R}_{\mathcal{M}})$;*
(2) *$G \in \mathcal{R}_{\mathcal{M}} \setminus \mathcal{M}$;*
(3) *$G \notin \mathcal{M}$ & $\mathbf{h}(G) \cap \mathcal{M} = \emptyset$.*

Proof. The implications (1) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (3) are evident. To prove the implication (3) $\Longrightarrow$ (1), let $G \notin \mathcal{M}$ and $\mathbf{h}(G) \cap \mathcal{M} = \emptyset$. We have to show that there exists $G_1 \in \mathcal{R}_{\mathcal{M}}$ with $G \in \mathbf{h}(G_1)$. It is sufficient to take $G_1 = \mathbb{Z} \circ G$. □

Now let us apply Theorem 2.3 to $\mathcal{L} = \mathcal{R}_{\mathcal{M}}$ for any fixed $\mathcal{M} \in \mathbf{M}$, $\mathcal{M} \neq \mathcal{G}$. As both $\mathcal{R}_{\mathcal{M}}$ and $\mathcal{R}_{\mathcal{G} \setminus \mathbf{e}(\mathbf{h}(\mathcal{R}_{\mathcal{M}}))}$ are the least elements of $\mathbf{R}$ containing $\mathcal{R}_{\mathcal{M}}$, we have $\mathcal{R}_{\mathcal{G} \setminus \mathbf{e}(\mathbf{h}(\mathcal{R}_{\mathcal{M}}))} = \mathcal{R}_{\mathcal{M}}$ and, moreover, if $\mathcal{R}_{\mathcal{M}'} \supseteq \mathcal{R}_{\mathcal{M}}$ for some $\mathcal{M}' \in \mathbf{M}$, then $\mathcal{G} \setminus \mathbf{e}(\mathbf{h}(\mathcal{R}_{\mathcal{M}})) \supseteq \mathcal{M}'$. Set $\overline{\mathcal{M}} = \mathcal{G} \setminus \mathbf{e}(\mathbf{h}(\mathcal{R}_{\mathcal{M}}))$ for $\mathcal{M} \neq \mathcal{G}$, and $\overline{\mathcal{G}} = \mathcal{G}$. Then we have:

Corollary 2.5. *For each $\mathcal{M} \in \mathbf{M}$, $\mathcal{R}_{\mathcal{M}} = \mathcal{R}_{\overline{\mathcal{M}}}$ and if $\mathcal{R}_{\mathcal{M}'} \supseteq \mathcal{R}_{\mathcal{M}}$, then $\overline{\mathcal{M}} \supseteq \mathcal{M}'$.*

Thus, $\overline{\mathcal{M}}$ is the greatest element of the system $\{\mathcal{M}' \in \mathbf{M} : \mathcal{R}_{\mathcal{M}'} = \mathcal{R}_{\mathcal{M}}\}$.

Let us observe that for any $\mathcal{M} \in \mathbf{M}$, it is $G \in \overline{\mathcal{M}}$ if and only if each $H \in c(G) \setminus \mathcal{M}$ has a heel in $\mathcal{M}$.

It is easy to prove the properties of the mapping $\mathcal{M}(\in \mathbf{M}) \mapsto \overline{\mathcal{M}}$, which are summarized in the following theorem.

Theorem 2.6. *For all $\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2 \in \mathbf{M}$, it is*

(1) *$\mathcal{M} \subseteq \overline{\mathcal{M}}$;*
(2) *$\mathcal{M}_1 \subseteq \mathcal{M}_2 \implies \overline{\mathcal{M}}_1 \subseteq \overline{\mathcal{M}}_2$;*
(3) *$\overline{\overline{\mathcal{M}}} = \overline{\mathcal{M}}$;*
(4) *$\mathcal{M}_1 \subseteq \overline{\mathcal{M}}_2 \implies \overline{\mathcal{M}}_1 \subseteq \overline{\mathcal{M}}_2$;*
(5) *$\mathcal{R}_{\mathcal{M}_1} \subseteq \mathcal{R}_{\mathcal{M}_2} \iff \overline{\mathcal{M}_2} \subseteq \overline{\mathcal{M}}_1$;*
(6) *$\mathcal{R}_{\mathcal{M}_1} = \mathcal{R}_{\mathcal{M}_2} \iff \overline{\mathcal{M}_1} = \overline{\mathcal{M}}_2$.*

The elements $\mathcal{M}(\in \mathbf{M})$ with $\mathcal{M} = \overline{\mathcal{M}}$ are called closed.

Corollary 2.7. *The mapping $\varphi\colon \mathcal{R}_{\mathcal{M}} \mapsto \overline{\mathcal{M}}$ is a dual isomorphism of the system $(\mathbf{R}, \subseteq)$ onto the ordered system of closed elements of $\mathbf{M}$.*

Let us remark that for each $\mathcal{M} \in \mathbf{M}$, it is $\mathcal{R}_{\mathcal{G}} \subseteq \mathcal{R}_{\mathcal{M}} \subseteq \mathcal{R}_{\{\{0\}\}}$. Consequently, $\mathcal{R}_{\mathcal{M}}$ contains all ℓ-groups without lex-subroups. If $\mathcal{R}_{\mathcal{M}} \neq \mathcal{R}_{\mathcal{G}}$, then $\mathcal{R}_{\mathcal{M}}$ contains also ℓ-groups having lex-subgroups, but these lex-subgroups are different from linear ones. Obviously, $G \in \mathcal{R}_{\mathcal{M}} \setminus \mathcal{R}_{\mathcal{G}}$ implies $G \in \mathcal{R}_{\mathcal{M}} \setminus \mathcal{M}$.

Evidently, $\mathcal{R}_{\mathcal{G}}$ contains, e.g., all archimedean ℓ-groups without non-zero convex linearly ordered subgroups. Since any direct product of such ℓ-groups is again an archimedean ℓ-group without non-zero convex linearly ordered subgroups, $\mathcal{R}_{\mathcal{G}}$ is a proper class. But then the same holds for all $\mathcal{R}_{\mathcal{M}} \in \mathbf{R}$.

Finally, we are going to show that if $\mathcal{R}_{\mathcal{M}} \neq \mathcal{R}_{\mathcal{G}}$, then $\mathcal{R}_{\mathcal{M}}$ fails to be a torsion class.

So let $\mathcal{R}_{\mathcal{M}} \neq \mathcal{R}_{\mathcal{G}}$, $G \in \mathcal{R}_{\mathcal{M}} \setminus \mathcal{R}_{\mathcal{G}}$. As we have remarked, $G \in \mathcal{R}_{\mathcal{M}} \setminus \mathcal{M}$. Take, as in the proof of Lemma 2.4, $G_1 = \mathbb{Z} \circ G$. Then $G_1 \in \mathcal{R}_{\mathcal{M}}$, G is an ℓ-ideal of G_1, but $G_1/G \notin \mathcal{R}_{\mathcal{M}}$, because $G_1/G \cong \mathbb{Z}$.

We have proved:

Theorem 2.8. *If $\mathcal{R}_{\mathcal{M}} \neq \mathcal{R}_{\mathcal{G}}$, then $\mathcal{R}_{\mathcal{M}}$ fails to be a torsion class.*

3. The lattice R

Theorem 2.6 says that the mapping $\mathcal{M} \mapsto \overline{\mathcal{M}}$ is a closure operator (we ignore the fact that $\mathbf{M}$ as well as some $\mathcal{M} \in \mathbf{M}$ are proper classes, as it is frequently done). It is well known that $\left(\{\overline{\mathcal{M}} : \ \mathcal{M} \in \mathbf{M}\}, \subseteq\right)$ is a complete lattice. If $\left\{\overline{\mathcal{M}}_i : \ i \in I\right\} \subseteq \mathbf{M}$, then

$$\bigwedge_{i\in I} \overline{\mathcal{M}}_i = \bigcap_{i\in I} \overline{\mathcal{M}}_i\,, \quad \bigvee_{i\in I} \overline{\mathcal{M}}_i = \overline{\bigcup_{i\in I} \overline{\mathcal{M}}_i}\,.$$

Thus, in view of Corollary 2.7, we have:

Theorem 3.1. *$(\mathbf{R}, \subseteq)$ is a complete lattice. Its least element is $\mathcal{R}_{\mathcal{G}}$, the greatest one is $\mathcal{R}_{\{\{0\}\}}$. Further, if $\left\{\overline{\mathcal{M}}_i : \ i \in I\right\} \subseteq \mathbf{M}$, then*

$$\bigwedge_{i\in I} \mathcal{R}_{\overline{\mathcal{M}}_i} = \mathcal{R}_{\bigvee_{i\in I} \overline{\mathcal{M}}_i} = \mathcal{R}_{\overline{\bigcup_{i\in I} \overline{\mathcal{M}}_i}}\,,$$

$$\bigvee_{i\in I} \mathcal{R}_{\overline{\mathcal{M}}_i} = \mathcal{R}_{\bigwedge_{i\in I} \overline{\mathcal{M}}_i} = \mathcal{R}_{\bigcap_{i\in I} \overline{\mathcal{M}}_i}\,.$$

In fact, for any $\{\mathcal{M}_i : \ i \in I\} \subseteq \mathbf{M}$, $\mathcal{M}_i$ not necessarily closed, it is

$$\bigwedge_{i\in I} \mathcal{R}_{\mathcal{M}_i} = \mathcal{R}_{\bigcup_{i\in I} \mathcal{M}_i}\,,$$

$$\bigvee_{i\in I} \mathcal{R}_{\mathcal{M}_i} = \mathcal{R}_{\bigcap_{i\in I} \overline{\mathcal{M}}_i}\,.$$

We are going to show that $\mathbf{R}$ is a proper class.

Take any archimedean ℓ-group without non-zero convex linearly ordered subgroups.

For any infinite cardinal number α, we have $c(A) \subseteq c(A^\alpha)$, so that $\mathcal{R}_{c(A^\alpha)} \subseteq \mathcal{R}_{c(A)}$. As evidently $G = \mathbb{Z} \circ A^\alpha \in \mathcal{R}_{c(A)} \setminus \mathcal{R}_{c(A^\alpha)}$, we have $\mathcal{R}_{c(A^\alpha)} \neq \mathcal{R}_{c(A)}$. Further, if α, β are different infinite cardinal numbers, e.g., $\alpha < \beta$, then $\mathcal{R}_{c(A^\beta)} \subseteq \mathcal{R}_{c(A^\alpha)}$. But again, $G = \mathbb{Z} \circ A^\beta \in \mathcal{R}_{c(A^\alpha)} \setminus \mathcal{R}_{c(A^\beta)}$, so that $\mathcal{R}_{c(A^\beta)} \neq \mathcal{R}_{c(A^\alpha)}$. Taking A^α for each infinite cardinal number α, we obtain a descending chain of radical classes $\mathcal{R}_{c(A^\alpha)}$, which is a proper class.

If $\{A_i : i \in I\}$ is a system of non-comparable archimedean ℓ-groups without non-zero convex linearly ordered subgroups (the non-comparability means that neither $A_i \in c(A_j)$, nor $A_j \in c(A_i)$ for $i, j \in I$, $i \neq j$), it is easy to see that $\left\{\mathcal{R}_{c(A_i)} : i \in I\right\}$ is an antichain.

Since $\mathbf{R}$ is not a sublattice of the lattice of all radical classes of ℓ-groups, the lattice $\mathbf{R}$ can have properties different from the properties of the lattice of all radical classes of ℓ-groups. Questions such as whether $\mathbf{R}$ is distributive or not, whether $\mathbf{R}$ has atoms or dual atoms, and so on, could be objects of a further investigation.

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