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THE ENDOMORPHISM SPECTRUM OF A MONOUNARY ALGEBRA

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Dedicated to Professor Ján Jakubík on the occasion of his 90th birthday

(*Communicated by Jiří Rachůnek*)

ABSTRACT. The endomorphism spectrum $\operatorname{spec}\mathcal{A}$ of an algebra $\mathcal{A}$ is defined as the set of all positive integers, which are equal to the number of elements in an endomorphic image of $\mathcal{A}$, for all endomorphisms of $\mathcal{A}$. In this paper we study finite monounary algebras and their endomorphism spectrum. If a finite set S of positive integers is given, one can look for a monounary algebra $\mathcal{A}$ with $S = \operatorname{spec}\mathcal{A}$. We show that for countably many finite sets S, no such $\mathcal{A}$ exists. For some sets S, an appropriate $\mathcal{A}$ with $\operatorname{spec}\mathcal{A} = S$ are described.

For $n \in \mathbb{N}$ it is easy to find a monounary algebra $\mathcal{A}$ with $\{1, 2, \dots, n\} = \operatorname{spec}\mathcal{A}$. It will be proved that if $i \in \mathbb{N}$, then there exists a monounary algebra $\mathcal{A}$ such that $\operatorname{spec}\mathcal{A}$ skips i consecutive (consecutive eleven, consecutive odd, respectively) numbers.

Finally, for some types of finite monounary algebras (binary and at least binary trees) $\mathcal{A}$, their spectrum is shown to be complete.

1. Introduction

About universal algebra we can simply say that it deals with objects, their relationship and constructions, which either build something more complex from these objects, or simplify them (for example by classification of the objects according to some specific signs). The basic objects, the "building stones" of

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universal algebra, are algebraic structures and the constructions are creating, e.g., homomorphic images or direct product of the given algebraic structures.

The specific sign of an algebraic structure $\mathcal{A}$ we will be interested in, is the endomorphism spectrum $\operatorname{spec} \mathcal{A}$ of $\mathcal{A}$.

A monounary algebra is an algebra possessing a unique operation and the operation is unary (thus, it is a transformation of the corresponding set). Monounary algebras are transparent and they have the remarkable property that they can be visualized by drawing them as a planar oriented graph.

Among relational structures, the most investigated are structures with binary or unary relations. Natural properties of binary relations are reflexivity, symmetry, antisymmetry, transitivity. Combining them, e.g. graphs, oriented graphs, quasiordered sets, partially ordered sets are obtained. Several of such relations have also a transparent structure and in the finite case their drawing is often applied.

Partially ordered sets are in a close connection with monounary algebras. If $\mathcal{A} = (A, f)$ is a monounary algebra, we can assign to it a binary relation π defined as follows: if $a, b \in A$, then

$$a\pi b \iff (\exists n \geq 0)\big(f^n(a) = b\big)$$

($f^0(x) := x$, $f^n(x) := f(f^{n-1}(x)$ for $n \geq 1$, $x \in A$). Notice that the relation π is a quasiorder, in general. This relation is a partial order if and only if $\mathcal{A}$ contains no cycle of length at least two; such monounary algebras are said to be acyclic.

A significant role in graph theory plays their finiteness, since mainly only finite graphs are treated. In universal algebra usually the number of elements plays only a particular role. But there exist various assertions which are valid in the infinite case and fail to hold in the finite case, or, at least, in the infinite case have a nicer form. For example, if m is an infinite cardinal, then there exist 2^m non-isomorphic m-element monounary algebras ([2], 1969). For the finite case, not an exact, just an asymptotic formula was found much later in [6], 2011.

It appears now that finite algebraic structures are worth to be dealt with separately. Applying homomorphisms, endomorphisms and related notions, we would like to bring at least a partial contribution to a classification of finite monounary algebras.

One of the most important tools in studying universal algebra is the notion of endomorphism. A rather large series of further algebraic notions is based on endomorphisms. Some properties of monounary algebras connected with the notion of homomorphism were studied by several authors ([5], [10], [3], [11], [12]).

The endomomorphism spectrum of an algebraic structure is a typical example of this series. It is defined as follows:

Let $\mathcal{A}$ be an algebraic structure. The endomorphism spectrum of $\mathcal{A}$ (spec $\mathcal{A}$, briefly) is the set of all positive integers k such that there exists an endomorphism φ of $\mathcal{A}$ having the property that the image $\operatorname{Im}\varphi$ has k elements.

The notion of endomorphism spectrum of an algebraic structure was introduced by K. Grant, R. J. Nowakowski and I. Rival [4] for the case of partially ordered sets. Choosing from their results, they proved that for any positive integer k and any "enough large" partially ordered set P, $\operatorname{spec} P \supseteq \{1, 2, \dots, k\}$. An analogous assertion for monounary algebras fails to hold. It is easy to show that the endomorphism spectrum of the partially ordered set corresponding to an acyclic monounary algebra includes the whole spectrum of the monounary algebra. The converse implication is not valid in general.

All monounary algebras dealt with in the present paper are assumed to be finite.

It is easy to see that the spectrum of a monounary algebra consists of a single number if and only if the algebra is a cycle. Section 3 is devoted to the description of those monounary algebras which have a two-element spectrum.

In the second part of the paper we will work with root monounary algebras, which are (cf. B. Jónsson [9]) connected monounary algebras possessing a one element cycle (connected acyclic monounary algebras). In Section 4 it is proved that if A is a set with $(i+1)^2+2$ elements, then there exists a root algebra (A, f) such that $\operatorname{spec}(A, f)$ skips i consecutive numbers. Next it is shown that any root algebra with the spectrum skipping i consecutive numbers contains at least $(i+1)^2+2$ elements. Section 5 contains the results that for each $i \geq 1$ there exists a root algebra such that its spectrum skips i consecutive odd (even, respectively) numbers. In Section 6 it is proved that each at least binary (and specially each binary) tree has a complete spectrum.

2. Preliminaries

In what follows we will use several notions without quoting general definitions; we will describe the notions for the case of finite algebras. For an explanation of the basic terms in universal algebra and monounary algebras, we recommend [1], [7].

As usually, the symbol $\mathbb{N}$ is used for the set of all positive integers.

Let (A, f) be a monounary algebra. For $x \in A$ we denote $f^{-1}(x) = \{y \in A : f(y) = x\}$. Then $x \in A$ is said to be a leaf if $f^{-1}(x) = \emptyset$.

An element $a \in A$ is called cyclic if $f^n(a) = a$ for some $n \in \mathbb{N}$. Otherwise, a is said to be non-cyclic. The set of all cyclic elements of some connected component of (A, f) is called a cycle of (A, f). For a connected component $S \subseteq A$, $a \in S$, let $\mathrm{cn}(a)$, the cycle number of a, denote the length of the cycle in S. Since we consider only finite monounary algebras, for each $x \in A$ there exists the least $i \geq 0$ such that $f^i(x)$ is a cyclic element. Then i is called the height of x and we write $\mathrm{ht}(x) = i$. Further,

$$\mathrm{ht}(A) = \max\{\mathrm{ht}(a) : \ a \in A\}.$$

A monounary algebra (A, f) is said to be a root algebra if it is connected and acyclic. The unique cyclic element is called a top of (A, f).

Definition 2.1. Under a branch of an algebra (A, f) we understand a set B such that $(B, f \restriction B)$ is a connected partial subalgebra of (A, f), B contains at least one leaf and if $c \in B$ is the element with the least height (in A), then there exists $a \in A \setminus B$ such that $f(a) = c$.

Definition 2.2. Let (A, f_A) and (B, f_B) be monounary algebras. The mapping $\varphi\colon A \to B$ is called a homomorphism if $\varphi(f_A(a)) = f_B(\varphi(a))$ for each $a \in A$.

If φ is a homomorphism $(A, f) \to (A, f)$, then it is called an endomorphism. The set of all endomorphism of a monounary algebra (A, f) is denoted $\mathrm{End}(A, f)$.

Definition 2.3. The spectrum of a monounary algebra (A, f) (defined in Introduction) is said to be complete, if $\mathrm{spec}(A, f) = \{1, 2, \dots, |A|\}$.

Definition 2.4. The term 'spectrum of algebra (A, f) skips the set M', where $M \subseteq \{1, 2, \dots, |A|\}$ will mean that $\mathrm{spec}(A, f) = \{1, 2, \dots, |A|\} \setminus M$. If $M = \{m_1, \dots, m_k\}$, then we say also that the spectrum skips the numbers $m_1, \dots, m_k$.

The Figure 1 shows a root algebra (A, f) with the spectrum $\mathrm{spec}(A, f) = \{1, 2, 3, 4, 6\}$. This leads us to the following question.

Is there a root algebra (A, f) such that its spectrum skips a number $n < |A| \leq 5$? We can simply check all cases and verify, that the spectrum of each root algebra (A, f), $|A| < 6$, is $\{1, 2, \dots, |A|\}$, i.e., $\mathrm{spec}(A, f)$ is complete. The case $|A| \geq 6$ is contained in Section 4.

Remark 2.1. Let us remind some properties of homomorphisms. If φ is a homomorphism of (A, f) into (B, f), then

- $\varphi(f^n(x)) = f^n(\varphi(x))$ for each $x \in A$, $n \in \mathbb{N} \cup \{0\}$,

- if C is a cycle of (A, f), then
  $\varphi(C)$ is a cycle D of (B, f) and $|D|$ divides $|C|$,
- if $x \in A$ then $\mathrm{ht}(\varphi(x)) \leq \mathrm{ht}(x)$.

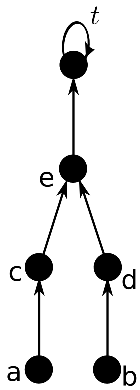

FIGURE 1. Algebra (A, f) with spectrum skipping number 5

If no confusion could occur we will use same symbols for algebras and the corresponding sets (A instead of (A, f)).

In [13], monounary algebras with a long pre-period were dealt with. A monounary algebra is said to be with a long pre-period if it is isomorphic to $(\{0, 1, \ldots, n\}, f)$ where $f(0) = 0$, $f(i) = i - 1$ for each $i \in \{1, \ldots, n\}$.

LEMMA 2.1. *The spectrum of a monounary algebra with a long pre-period is complete.*

Proof. Without loss of generality, $A = \{0, 1, \ldots, n\}$. Have a sequence of mappings $\varphi_i = f^i$, $i = 0, 1, \ldots, n-1$. Using induction we can prove that φ_i is an endomorphism. Further, $|\varphi_i(A)| = n-i$, which implies $\mathrm{spec}(A) = \{1, 2, \ldots, n\}$. □

LEMMA 2.2. *If (A, f) is a root algebra and $m = \mathrm{ht}(A)$, then the numbers $1, 2, \ldots, m+1$ are from the spectrum of (A, f).*

Proof. By the assumption there exists a leaf $c \in A$ with $\mathrm{ht}(c) = m$. We define the following sets: $V_0 = \{v\}$ is the one element cycle and by induction, $V_i = \big\{a \in A\colon f(a) \in V_{i-1}\big\}$, $i = 1, \ldots, m$. Clearly, $\bigcup\limits_{i=0}^{m} V_i = A$. Let us put $\varphi(v) = f^{m-i}(c)$ for $v \in V_i$, $i = 0, 1, \ldots, m-1$. It is easy to see that φ is an endomorphism. Then φ is an endomorphism of (A, f) onto the algebra $(\{c, f(c), \ldots, f^m(c)\}, f)$ with a long pre-period. The composition of two endomorphisms is an endomorphism, so by Lemma 2.1 it is true that $\{1, 2, \ldots, m+1\} \subseteq \mathrm{spec}(A)$. □

Definition 2.5. A connected partial monounary subalgebra $(B, f \restriction B)$ of an algebra (A, f) is called a tail, if it contains a leaf, $|f^{-1}(a)| = 1$ for each element $a \in B$ which is not a leaf and $|f^{-1}(f(b))| \geq 2$ for the element $b \in B$ of the least height (we note that such b exists and is uniquely determined). The element $f(b)$ is called a connection of the tail to the algebra (A, f). Next, $|B|$ is called the length of the tail B.

Notice that if there exists a tail in (A, f), then $|A| > 1$.

Lemma 2.3. *Let (A, f) be a root algebra and T be a tail of minimal length. Then there exists $\varphi \in \operatorname{End}(A)$ such that $\operatorname{Im}\varphi = A \setminus T$.*

Proof. Denote by t the top of (A, f). We analyse two cases: First let t be the connection of T. Since T is a tail of minimal length, $t \notin T$. Take a mapping $\varphi\colon A \to A$, where $\varphi(a) = t$ for each $a \in T$ and $\varphi(a) = a$ for each $a \in A \setminus T$. Obviously, $\varphi \in \operatorname{End}(A, f)$, $\operatorname{Im}\varphi = A \setminus T$.

Now let $c \neq t$ be the connection of T. Further, let d be the leaf belonging to T. Consider the set L of all leaves x such that there is $m(x) \in \mathbb{N}$ with $f^{m(x)}(x) = c$. Since c is a connection, L has at least two elements. From the minimality of $|T|$ it follows that $\operatorname{ht}(x) \geq \operatorname{ht}(d)$ for each $x \in L$. Denote $a = f^{m(x)-m(d)}(x)$, where $x \in L \setminus \{d\}$ is fixed. Now define

$$\varphi(y) = \begin{cases} f^k(a) & \text{if } y = f^k(d), \ \ k = 0, 1, \dots, m(d) - 1, \\ y & \text{otherwise.} \end{cases}$$

Then $\varphi \in \operatorname{End}(A, f)$ and $\operatorname{Im}\varphi = A \setminus T$. □

Lemma 2.4. *Let t be the top of (A, f) and let $f^{-1}(t)$ contain more than one element. If $B_1, B_2, \dots, B_n$ are branches, $B_i \cap B_j = \{t\}$ for each $i, j \in \{1, 2, \dots, n\}$, $i \neq j$, $\beta_1 \in \operatorname{spec}(B_1)$, $\beta_2 \in \operatorname{spec}(B_2), \dots, \beta_n \in \operatorname{spec}(B_n)$, then $\beta_1 + \beta_2 + \cdots + \beta_n - n + 1 \in \operatorname{spec}(A)$.*

Proof. Easy to show. □

3. Two-element spectrum

In this section we will characterize all two-element sets $S = \{m, k\}$ such that there exist monounary algebras (A, f) with $\operatorname{spec}(A, f) = S$.

It is easy to see that the following lemma is valid:

Lemma 3.1. *Let $S = \{1, k\}$, $k \in \mathbb{N}$, $k > 1$. There exists a monounary algebra (A, f) with $S = \operatorname{spec}(A)$.*

Proof. It suffices to take (A, f) consisting of a 1- and a $(k-1)$-element cycles. □

Lemma 3.2. *Let $S = \{2, k\}$, $k \in \mathbb{N}$, $k > 2$. A monounary algebra (A, f) with $S = \operatorname{spec}(A)$ exists if and only if k is even.*

Proof. Let $S = \operatorname{spec}(A)$ for a monounary algebra (A, f). Then $k = |A|$ and there are an endomorphism ψ and a 2-element subalgebra $\{a_1, a_2\}$ of (A, f) such that $\operatorname{Im}\psi = \{a_1, a_2\}$. If a_1 forms a one-element cycle, then a constant mapping $\varphi(x) = a_1$ for each $x \in A$ is an endomorphism, thus $1 \in \operatorname{spec}(A)$, a contradiction. Thus $\{a_1, a_2\}$ is a cycle. Hence each cycle of (A, f) must be even. Assume that not all elements of $\mathcal{A}$ are cyclic. Since $f \in \operatorname{End}(A)$, $2 < |\operatorname{Im} f| < k$, we arrived to a contradiction. This shows that (A, f) consists of even cycles, therefore k is even, too.

Conversely, let k be even. A monounary algebra (A, f) consisting of a 2- and $(k-2)$-element cycles has the property $\{2, k\} = \operatorname{spec}(A)$. □

Corollary 3.0.1. *There exists an infinite (countable) system of finite subsets S of $\mathbb{N}$ such that S fails to be an endomorphism spectrum of any monounary algebra (A, f).*

Proposition 3.1. *Let $S = \{m, k\}$, $m, k \in \mathbb{N}$, $k > m \geq 1$. A monounary algebra (A, f) with $S = \operatorname{spec}(A)$ exists if and only if either m divides k or there is a decomposition of m into a sum $m = t_1 + t_2 + \cdots + t_j$ of positive integers such that*

(i) *if t_{i_1} divides t_{i_2} for some $1 \leq i_1, i_2 \leq j$, then $i_1 = i_2$,*

(ii) *if $m \neq k-1$, then there is $1 \leq i_0 \leq j$ such that t_{i_0} divides $k - m$.*

Proof. Let $S = \operatorname{spec}(A)$ for a monounary algebra (A, f). As above, $k = |A|$.

First assume that not all elements of (A, f) are cyclic. Then $h = \operatorname{ht}(A) > 0$, $f^h \in \operatorname{End}(A)$, $|\operatorname{Im} f^h| < k$ imply $|\operatorname{Im} f^h| = m$. If $h > 1$, then $f^{h-1} \in \operatorname{End}(A)$, $|\operatorname{Im} f^{h-1}| < |\operatorname{Im} f^h| = m$, which is a contradiction. Thus $h = 1$ and there is $a \in A$ with $\operatorname{ht}(a) = 1$. The mapping

$$\psi(x) = \begin{cases} f^{\operatorname{cn}(a)}(a) & \text{if } x = a, \\ x & \text{otherwise} \end{cases}$$

is obviously an endomorphism of (A, f) and $|\operatorname{Im}\psi| = k-1$. From this it follows that $k - 1 = m$ and that (ii) holds. Notice that $\operatorname{Im} f^h$ is the set of all cyclic elements of (A, f) and $|\operatorname{Im} f^h| = m = k-1$. Therefore the set $A \setminus \{a\}$ consists of all cyclic elements of (A, f) (the cycles of length $t_1, t_2, \ldots, t_j$, whereas $m =$

$t_1 + t_2 + \cdots + t_j$) and the assumption yields that each endomorphism of this algebra must be surjective; hence (i) is valid.

Now suppose that (A, f) consists of cyclic elements only. Since $m \in \operatorname{spec}(A)$, there are an endomorphism φ and an m-element subalgebra B such that $\operatorname{Im}\varphi = B$. Then B consists of j cycles of length $t_1, t_2, \ldots, t_j$, $m = t_1+t_2+\cdots+t_j$. As above, (i) must be satisfied. Let $m \neq k-1$ and let C be any cycle of $A \setminus B$. Applying φ, we obtain that $\varphi(C)$ is one of the cycles in B, a cycle of length dividing $|C|$. To finish the proof of (ii), it suffices to show that $A = B \cup C$. Consider the mapping

$$\chi(x) = \begin{cases} \varphi(x) & \text{if } x \in B \cup C, \\ x & \text{otherwise.} \end{cases}$$

Then $\chi \in \operatorname{End}(A)$, $m + |A \setminus (B \cup C)| = |\operatorname{Im}\chi| < k$ imply $A \setminus (B \cup C) = \emptyset$, i.e., $A = B \cup C$.

Let us prove the converse implication. If $m \neq k-1$, take a monounary algebra (A, f) consisting of cycles with lengths $t_1, t_2, \ldots, t_j, k-m$. Let $\varphi \in \operatorname{End}(A)$. By (i), φ maps any of the cycles of length t_i $(1 \leq i \leq j)$ onto the same cycle. In view of (ii), the cycle of length $k-m$ can be mapped either onto the same cycle, or onto the i_0-element cycle. In the first case $|\operatorname{Im}\varphi| = k$, in the latter one $|\operatorname{Im}\varphi| = m$, which implies that $\operatorname{spec}(A) = \{m, k\}$.

On the other hand, if $m = k-1$, then consider a monounary algebra (A, f) consisting of cycles with lengths $t_1, t_2, \ldots, t_j$ and of one non-cyclic element. Similarly as above, $\operatorname{spec}(A) = \{k-1, k\}$. □

4. Spectrum skipping i consecutive numbers

Lemma 4.1. *If (A, f) is an algebra of minimal cardinality such that* $\operatorname{spec}(A)$ *skips some i consecutive numbers from the set $\{1, 2, \ldots, |A|\}$, $1 \leq i < |A|$, then (A, f) contains at least $i+1$ leaves.*

Proof. Let (A, f) be an algebra of minimal cardinality with the required property. By way of contradiction, assume that it has at most i leaves. Then $f \in \operatorname{End}(A, f)$ and for $B = \operatorname{Im} f$ we have $|B| = k \in \{|A|, |A|-1, \ldots |A|-i\}$, where $k \in \operatorname{spec}(A)$. Between the numbers k and $|A|$ there are less than i numbers, so our i consecutive numbers from the set $\{1, 2, \ldots, |A|\}$, $1 \leq i < |A|$ are smaller than k.

It is easy to show

$$\varphi \in \operatorname{End}(A) \implies \varphi \circ f = f \circ \varphi \in \operatorname{End}(B),$$
$$\psi \in \operatorname{End}(B) \implies (\exists \varphi \in \operatorname{End}(A))\,(\psi = \varphi \circ f).$$

Thereof we have found the monounary algebra (B, f), where $|B| < |A|$, but the above implications yield that its spectrum skips i consecutive numbers, which is a contradiction. □

Lemma 4.2. *If (A, f) is an algebra of minimal cardinality such that $\operatorname{spec}(A, f)$ skips some i consecutive numbers from the set $\{1, 2, \dots, |A|\}$, $1 \le i < |A|$, then (A, f) contains only tails of length at least $i + 1$.*

Proof. Let (A, f) be an algebra of minimal cardinality with the required property. Suppose that A contains a tail S of minimal length, $k \le i$. Using Lemma 2.3, there exists an endomorphism φ of A such that $\operatorname{Im} \varphi = A \setminus S$. Then $|A| - k \in \operatorname{spec}(A)$. Denote $B = \operatorname{Im} \varphi$. If we take the identity, also an endomorphism, we obtain $|A| \in \operatorname{spec}(A)$. Between $|A|$ and $|B| = |A| - k$ there are less than i numbers, therefore these i consecutive numbers from the set $\{1, 2, \dots, |A|\}$ are smaller than $|B|$. Further,

$$\alpha \in \operatorname{End}(A) \implies \varphi \circ \alpha \in \operatorname{End}(B),$$
$$\psi \in \operatorname{End}(B) \implies \varphi \circ \psi \in \operatorname{End}(A).$$

From this we deduce that $\operatorname{spec}(B)$ skips i consecutive numbers and $|B| < |A|$. We arrived to a contradiction. □

Theorem 4.1. *A root algebra (A, f) with the spectrum skipping some i consecutive numbers from the set $\{1, 2, \dots, |A|\}$, $1 \le i < |A|$, contains at least $(i+1)^2+2$ elements, and this boundary is the best possible.*

Proof. Suppose that (A, f) is an algebra of minimal cardinality skipping i consecutive numbers. Using Lemma 4.1 and Lemma 4.2, we deduce, that (A, f) has at least $i + 1$ leaves and the length of every tail is at least $i + 1$. The assumption $i \ge 1$ and Lemma 2.1 yield that A fails to be an algebra with a long pre-period. Thus the set L of all leaves of A has at least two elements. For each $x \in L$ there exists a unique tail $T(x)$ such that $x \in T(x)$. Notice that if $x, y \in L$, $x \ne y$, the $T(x) \cap T(y) = \emptyset$. The number of leaves is at least $i+1$, $|T(x)| \ge i+1$ for each $x \in L$, thus the number of elements in these tails is at least $(i + 1)^2$. Since the top t of A doest not belong to any tail, we obtain that A has at least $(i + 1)^2 + 1$ elements.

Let $|A| = (i + 1)^2 + 1$. From this assumption it follows that $i \ge 2$ and that t is a connection of any tail to A. If $x \in L$, then $T(x) \cup \{t\}$ is closed with respect to f and it is an algebra with a long pre-period. By Lemma 2.1 the spectrum of any algebra of this form is complete. These algebras are branches of A, their union equals A. Applying Lemma 2.4, spec(A) contains all numbers from $(i+1)-(i+1)+1=1$ to $|A|$, hence it is complete. This is a contradiction, therefore $|A| \geq (i+1)^2+2$.

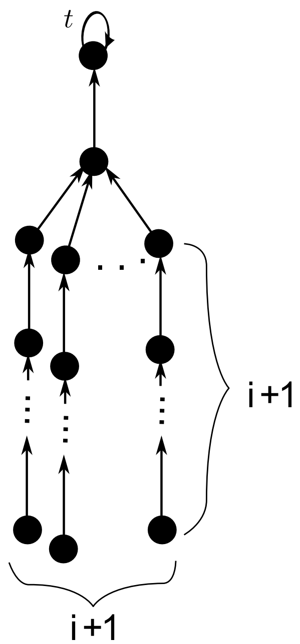

FIGURE 2. Algebra with spectrum skipping i consecutive numbers

Since we found the monounary algebra (see Fig. 4) with $(i+1)^2+2$ elements with our required attributes, we conclude that the boundary is the best possible. □

COROLLARY 4.1.1. *If (A, f) is a root algebra with the spectrum skipping one number from the set $\{1, 2, \ldots, |A|\}$, then A contains at least 6 elements and this boundary is the best possible.*

5. Spectrum skipping i consecutive even (odd) numbers

THEOREM 5.1. *For each $i \in \mathbb{N}$ there exists a root algebra (A, f) such that* spec(A) *skips i consecutive odd numbers.*

P r o o f. We are going to define a $6i$-element algebra (A, f). Let

$$A = \{a_1, a_2, \dots, a_{2i}\} \cup \{1, 2, \dots, 2i\} \cup \{1', 2', \dots, (2i)'\}.$$

Define the operation by setting $f(a_j) = a_{j+1}$ for $1 \le j < 2i$, $f(a_{2i}) = a_{2i}$, $f(j) = j-1$ and $f(j') = (j-1)'$ for $1 < j \le 2i$, $f(1) = f(1') = a_1$, c.f. Figure 3. Lemma 2.2 implies that $\{1, 2, \dots, 4i\} \subseteq \operatorname{spec}(A)$.

If $0 \le j < 2i$, $\psi \in \operatorname{End}(A)$, $\psi(a_1) = a_j$, then $f^{2i}(\psi(2i)) = \psi(f^{2i}(2i)) = \psi(a_1) = a_j$, which yields

$$\psi(2i) \in f^{-2i}(a_j) = \big\{2i - (j-1), (2i - (j-1))'\big\}. \tag{5.1}$$

For endomorphisms of (A, f) we will use the inclusion 5.1 and consider the following possibilities:

- $a_1 \mapsto a_1$: $2i \to \{2i, (2i)'\}$, hence $6i$, $4i \in \operatorname{spec}(A)$,
- $a_1 \mapsto a_2$: $2i \to \{2i-1, (2i-1)'\}$, hence $6i-2$, $4i-1 \in \operatorname{spec}(A)$,

  $\vdots$
- $a_1 \mapsto a_i$: $2i \to \{i+1, (i+1)'\}$, hence $6i - 2(i-1)$, $4i - (i-1) \in \operatorname{spec}(A)$,
- $a_1 \mapsto a_{i+1}$: $2i \to \{i, (i)'\}$, hence $6i-2((i+1)-1)$, $4i-((i+1)-1) \in \operatorname{spec}(A)$.

Notice that if $a_1 \mapsto a_k$, $k > i+1$, then the image of the corresponding endomorphism has less elements than $4i$.

We have proved that $\operatorname{spec}(A)$ skips exactly i odd numbers, namely $4i+1, 4i+3, \dots, 4i+(2i-1)$. □

Theorem 5.2. *For each $i \in \mathbb{N}$ there exists a root algebra (A, f) such that* $\operatorname{spec}(A)$ *skips i consecutive even numbers.*

P r o o f. We will proceed analogously as in Theorem 5.1. Put

$$A = \{a_0, a_1, a_2, \dots, a_{2i}\} \cup \{1, 2, \dots, 2i\} \cup \{1', 2', \dots, (2i)'\}$$

and let $f(a_j) = a_{j+1}$ for $0 \le j < 2i$, $f(a_{2i}) = a_{2i}$, $f(j) = j-1$ and $f(j') = (j-1)'$for $1 < j \le 2i$, $f(1) = f(1') = a_0$, c.f. Figure 4. Lemma 2.2 implies that $\{1, 2, \dots, 4i+1\} \subseteq \operatorname{spec}(A)$. Similarly as above if $0 \le j < 2i$, $\psi \in \operatorname{End}(A)$, $\psi(a_0) = a_j$, implies

$$\psi(2i) \in f^{-(2i)}(a_j) = \{2i - j, (2i-j)'\}.$$

Then we have the following possibilities:

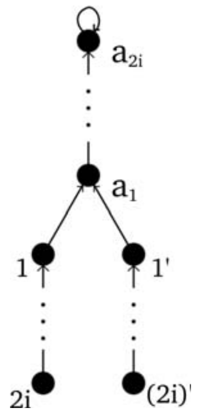

Figure 3. Algebra with spectrum skipping i consecutive odd numbers

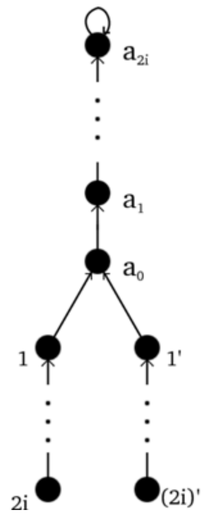

Figure 4. Algebra with spectrum skipping i consecutive even numbers

- $a_0 \mapsto a_0$: $2i \to \{2i, (2i)'\}$, hence $6i+1$, $4i+1 \in \operatorname{spec}(A)$,
- $a_0 \mapsto a_1$: $2i \to \{2i-1, (2i-1)'\}$, hence $(6i+1)-2$, $(4i+1)-1 \in \operatorname{spec}(A)$,

  $\vdots$
- $a_0 \mapsto a_{i-1}$: $2i \to \{i+1, (i+1)'\}$,
  hence $(6i+1)-2(i-1)$, $(4i+1)-(i-1) \in \operatorname{spec}(A)$
- $a_0 \mapsto a_i$: $2i \to \{i, (i)'\}$, hence $(6i+1)-2i$, $(4i+1)-i \in \operatorname{spec}(A)$.

Therefore $\operatorname{spec}(A)$ skips exactly i even numbers, namely, $4i+2, 4i+4, \dots$ $\dots, 4i+2i$. □

6. Binary and at least binary trees

In this chapter we are going to characterize the spectrum of specific monounary algebras, namely, binary and at least binary trees. Binary monounary algebras were dealt with, e.g., in [8]. Now a more general notion will be defined:

Definition 6.1. Let (A, f) be a root algebra. If $|f^{-1}(a)| \geq 2$ holds for any element which fails to be a leaf, then (A, f) is called an at least binary tree.

Let us remark that for a binary tree, if a is not a leaf then $|f^{-1}(a)| = 2$.

Our next aim is to prove that if (A, f) is an at least binary tree, then its spectrum is $\{1, 2, \dots, |A|\}$.

First we introduce some auxiliary notions.

Definition 6.2. Let (A, f) be an an at least binary tree, $b \in A$. If $B = \{a \in A : (\exists n \geq 0)(f^n(a) = b)\}$, then $(B, f \restriction B)$ is said to be a subtree of (A, f) (with the top b).

Note that if $B \neq A$, then B is a connected non-complete monounary algebra and it will be called a proper subtree. If $\mathrm{ht}(A) = k$, $\mathrm{ht}(b) = m$, then we say that $(B, f \restriction B)$ is of height $k - m$ ($\mathrm{ht}(B) = k - m$, briefly).

We remind some definitions for partial monounary algebras.

Definition 6.3. Let (A, f_A) and (B, f_B) be partial monounary algebras. The mapping $\varphi\colon A \to B$ is called a homomorphism, if

$$(\forall a \in \operatorname{dom} f_A)\big[\big(\varphi(a) \in \operatorname{dom} f_B\big) \wedge \big(\varphi(f_A(a)) = f_B(\varphi(a))\big)\big].$$

The notions of endomorphism and endomorphism spectrum are defined similarly as for complete algebras.

Lemma 6.1. *Let (A, f) be a connected partial monounary algebra with $\operatorname{dom} f = A \setminus \{c\}$, $\mathrm{ht}(A) = k$. If $\varphi \in \mathrm{End}(A, f)$, then $\varphi(c) = c$.*

Proof. By way of contradiction, suppose that $\varphi(c) = a \neq c$. Then $a \in \operatorname{dom} f$ which yields $\varphi(a) \neq c$. There exists $1 \leq m \leq k$ such that $f^m(a) = c$. By definition of $\mathrm{ht}(A)$, there is $b \in A$ with $f^k(b) = c$. We obtain

$$c = f^m(a) = f^m(\varphi(c)) = f^m(\varphi(f^k(b))) = f^m(f^k(\varphi(b))) = f^{k+m}(\varphi(b)).$$

This implies $\mathrm{ht}(A) \geq k + m$, a contradiction. □

Lemma 6.2. *Let (A, f) be an at least binary tree. If $\mathrm{ht}(A) = k$ then $|A| \geq 2^{k+1} - 1$.*

Proof. For $|A|$ we have $|A| \geq 1 + 2 + 2^2 + \dots + 2^k = 2^{k+1} - 1$. □

Lemma 6.3. *If (B, f) is a proper subtree of an at least binary tree (A, f) such that $\mathrm{ht}(B) = k$, then $\mathrm{spec}(B) = \{k + 1, k + 2, \dots, |B|\}$.*

Proof. Denote by b the top of B. We proceed by induction according to k.

$k = 1$: Then $f^{-1}(b) = \{b_1, b_2, \dots, b_m\}$ for some $m \geq 2$. For $0 \leq i \leq m - 2$ we define a mapping $\varphi_i\colon B \mapsto B$ as follows:

$$\varphi_i(x) = \begin{cases} b_1 & \text{if } x = b_j,\ \ i + 1 \leq j \leq m, \\ x & \text{otherwise.} \end{cases}$$

Obviously, $\varphi_i \in \mathrm{End}(B)$ for each $1 \leq i \leq m$ and $|\operatorname{Im}\varphi_i| = i + 1$.

Suppose that $k \geq 1$ and each proper subtree (C, f) of height $k' \leq k$ has the spectrum $\{k'+1, k'+2, \dots, |C|\}$.

Now take a proper subtree B of height $k+1$, with the top b. Then $f^{-1}(b) = \{b_1, b_2, \dots, b_m\}$, $m \geq 2$. Denote by $B_1, B_2, \dots, B_m$ the subtrees with the tops $b_1, b_2, \dots, b_m$. Without loss of generality we can suppose that $\mathrm{ht}(B_1) = k$. Let $i \in \{1, \dots, m\}$. We have $\mathrm{ht}(B_i) = n_i \leq k$; $k = n_1 \geq n_2 \geq \dots \geq n_m$. By induction assumption,

$$\mathrm{spec}(B_i) = \big\{n_i + 1, n_i + 2, \dots, |B_i| = p_i\big\}.$$

Let $\varphi \in \mathrm{End}(B, f \restriction B)$. According to Lemma 6.1, $\varphi(b) = b$. By definition of a homomorphism, $\varphi(b_i) \in \big\{b_j : \ j \in \{1, \dots, m\}\big\}$. Therefore $\mathrm{spec}(B) \subseteq \{k+2, k+3, \dots, |B|\}$.

First we will verify

$$|B_i| - n_{i+1} \geq n_i + 1.$$

According to Lemma 6.2 and $2(n_i + 1) \leq 2 \cdot 2^{n_i}$ we obtain

$$|B_i| - n_{i+1} \geq 2^{n_i+1} - 1 - n_{i+1} \geq 2^{n_i+1} - 1 - n_i \geq n_i + 1.$$

Hence

$$\mathrm{spec}(B_1) = \{n_1 + 1, n_1 + 2, \dots, p_1 - n_2, \dots, p_1 - 1, p_1\},$$
$$\mathrm{spec}(B_2) = \{n_2 + 1, n_2 + 2, \dots, p_2 - n_3, \dots, p_2 - 1, p_2\}.$$

Since $n_1 \geq n_2$ there exists a homomorphism of B_2 into B_1, which yields $\mathrm{spec}(B) \supseteq \big\{\beta + 1 : \ \beta \in \mathrm{spec}(B_1)\big\}$. Further, if $\beta_1 \in \mathrm{spec}(B_1), \beta_2 \in \mathrm{spec}(B_2)$, then $1 + \beta_1 + \beta_2 \in \mathrm{spec}(B)$. Consider the set

$$\begin{aligned} S = &\{n_1 + 1, n_1 + 2, \dots, p_1 - n_2, \dots, p_1 - 1, p_1\} \\ &\cup \{(p_1 - n_2) + (n_2 + 1), (p_1 - n_2) + (n_2 + 2), \dots, (p_1 - n_2) + p_2\} \\ &\cup \{p_2 + p_1, p_2 + (p_1 - 1), \dots, p_2 + (p_1 - n_2 + 1)\}. \end{aligned}$$

If $\beta \in S$ then $\beta + 1 \in \mathrm{spec}(B)$. Therefore $\{n_1 + 2, n_1 + 3, \dots p_1 + p_2 + 1\} \subseteq \mathrm{spec}(B)$. If $m = 2$ we are done. The required assertion can be proved by induction with respect to m. This is just a mechanical matter. $\square$

Theorem 6.1. *The spectrum of any at least binary tree is complete.*

Proof. Let (B, f) be an at least binary tree with a top b and $\mathrm{ht}(B) = k$. Lemma 2.2 implies that $\{1, 2, \dots, k+1\} \subseteq \mathrm{spec}(B)$.

First assume that $|f^{-1}(b)| \geq 3$. We define a monounary algebra (D, g) of height $k+1$ by putting $D = B \cup \{c, d\}$ and

$$g(x) = \begin{cases} b & \text{if } x = c, \\ d & \text{if } x \in \{b, d\}, \\ f(x) & \text{otherwise.} \end{cases}$$

Then $(B, g \restriction B)$ is a proper subtree of the at least binary tree (D, g). According to Lemma 6.3, $\operatorname{spec}(B, g \restriction B) = \{k+1, k+2, \ldots, |B|\}$. Notice that $\operatorname{spec}(B, g \restriction B) \subseteq \operatorname{spec}(B, f)$. This yields

$$\operatorname{spec}(B, f) = \big\{1, 2, \ldots, |B|\big\}.$$

Now assume that $f^{-1}(b) = \{b, c\}$, $c \neq b$. Denote $C = B \backslash \{b\}$. Then $(C, f \restriction C)$ is a proper subtree of the at least binary tree (B, f) and $\operatorname{ht}(C) = k - 1$. By Lemma 6.3 we get that $\operatorname{spec}(C, f \restriction C) = \{k, k+1, \ldots, |C|\}$. For each $\varphi \in \operatorname{End}(C)$, the extension $\varphi' \colon B \to B$ of φ such that $\varphi(c) = c$, belongs to $\operatorname{End}(B)$. Since $\operatorname{Im} \varphi' = \operatorname{Im} \varphi \cup \{b\}$, we obtain

$$\begin{aligned} \operatorname{spec}(B, f) &\supseteq \big\{i + 1 : \ i \in \operatorname{spec}(C, f \restriction C)\big\} \\ &= \big\{k+1, k+2, \ldots, |C| + 1\big\} \\ &= \big\{k+1, k+2, \ldots, |B|\big\}. \end{aligned}$$

Therefore $\operatorname{spec}(B, f) = \big\{1, 2, \ldots, |B|\big\}$. □

Corollary 6.1.1. *The spectrum of any binary tree is complete.*

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