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SYMMETRIES IN SYNAPTIC ALGEBRAS

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Dedicated to Professor Ján Jakubík on the occasion of his 90th birthday

(*Communicated by Anatolij Dvurečenskij*)

ABSTRACT. A synaptic algebra is a generalization of the Jordan algebra of self-adjoint elements of a von Neumann algebra. We study symmetries in synaptic algebras, i.e., elements whose square is the unit element, and we investigate the equivalence relation on the projection lattice of the algebra induced by finite sequences of symmetries. In case the projection lattice is complete, or even centrally orthocomplete, this equivalence relation is shown to possess many of the properties of a dimension equivalence relation on an orthomodular lattice.

1. Introduction

Synaptic algebras, which were introduced in [6] and further studied in [10,25] tie together the notions of an order-unit normed space [1: p. 69], a special Jordan algebra [22], a convex effect algebra [14], and an orthomodular lattice [2,18]. The self-adjoint part of a von Neumann algebra is an example of a synaptic algebra; see [6,10,25] for numerous additional examples.

Our purpose in this article is to study symmetries s in a synaptic algebra A and the equivalence relation $\sim$ induced by finite sequences of symmetries on the orthomodular lattice P of all projections p in A. For a symmetry s, we have $s^2 = 1$ (the unit element in A), and $p^2 = p$ for a projection p. If P is a complete lattice, or even centrally orthocomplete, i.e., every family of projections that is dominated by an orthogonal family of central projections has a supremum, then we show that $\sim$ acquires many of the properties of a dimension equivalence relation on an orthomodular lattice [21].

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In Section 2 we review the definition and basic properties of a synaptic algebra A. Since the projections in A form an orthomodular lattice (OML) P, we sketch in Section 3 a portion of the theory of OMLs that will be needed for our subsequent work. In Section 4 we focus on the special properties of the OML P that are acquired due to the fact that $P \subseteq A$. In Section 5 we introduce the notion of a symmetry s in A, study exchangeability of projections by a symmetry, and relate symmetries to the notion of perspectivity of projections. The condition of central orthocompleteness is defined in Section 6, and it is observed that, if P is centrally orthocomplete, then the center of A is a complete boolean algebra and A hosts a central cover mapping. From Section 6 onward, it is assumed that P is, at least, centrally orthocomplete. The equivalence relation $\sim$ on P induced by finite sequences of symmetries is introduced in Section 7 where we investigate the extent to which $\sim$ is a dimension equivalence relation. Finally, in Section 8 we cover some of the features of the relation of exchangeability of projections by symmetries that require completeness of the OML P.

2. Basic properties of a synaptic algebra

In this section, we review the definition of a synaptic algebra and sketch some basic facts about synaptic algebras. For more details, see [6, 10, 25]. We use the notation := for "equals by definition" and "iff" abbreviates "if and only if."

2.1. Definition. ([6: Definition 1.1]) Let R be a linear associative algebra with unity element 1 over the real numbers $\mathbb{R}$, and let A be a real vector subspace of R. Let $a, b \in A$. We understand that the product ab is calculated in R, and that it may or may not belong to A. We write aCb iff a and b commute (i.e. $ab = ba$) and we define $C(a) := \{b \in A : \ aCb\}$. If $B \subseteq A$, then

$$C(B) := \bigcap_{b \in B} C(b), \quad CC(B) := C(C(B)), \quad \text{and} \quad CC(b) := C(C(b)).$$

The vector space A is a *synaptic algebra* with *enveloping algebra* R iff the following conditions are satisfied:

SA1. A is a partially ordered archimedean real vector space with positive cone $A^+ = \{a \in A : \ 0 \le a\}$, $1 \in A^+$ is an order unit in A, and $\|\cdot\|$ is the corresponding order unit norm on A.

SA2. If $a \in A$ then $a^2 \in A^+$.

SA3. If $a, b \in A^+$, then $aba \in A^+$.

SA4. If $a \in A$ and $b \in A^+$, then $aba = 0 \implies ab = ba = 0$.

SA5. If $a \in A^+$, there exists $b \in A^+ \cap CC(a)$ such that $b^2 = a$.

SA6. If $a \in A$, there exists $p \in A$ such that $p = p^2$ and, for all $b \in A$, $ab = 0 \iff pb = 0$.

SA7. If $1 \leq a \in A$, there exists $b \in A$ such that $ab = ba = 1$.

SA8. If $a, b \in A$, $a_1 \leq a_2 \leq a_3 \leq \dots$ is an ascending sequence of pairwise commuting elements of $C(b)$ and $\lim_{n\to\infty} \|a - a_n\| = 0$, then $a \in C(b)$.

We define $P := \{p \in A : \ p = p^2\}$ and we refer to elements $p \in P$ as *projections*. Elements e in the "unit interval" $E := \{e \in A : \ 0 \leq e \leq 1\}$ are called *effects*. The set $C(A)$ is called the *center* of A. We understand that subsets of A such as P, E, and $C(A)$ are partially ordered by the respective restrictions of the partial order $\leq$ on A. If $p, q \in P$ and $p \leq q$, we say that p is a *subprojection* of q, or equivalently, that q *dominates* p.

2.2. Standing Assumptions. For the remainder of this article, A is a synaptic algebra with unit 1, with enveloping algebra R, with E as its unit interval, and with P as its set of projections. To avoid trivialities, we shall assume that A is "non-degenerate", i.e., $0 \neq 1$. Also, we shall follow the usual convention of identifying each real number $\lambda \in \mathbb{R}$ with the element $\lambda 1 \in A$, so that $\mathbb{R} \subseteq C(A)$.

As A is an order unit space with order unit 1, the *order-unit norm* $\|\cdot\|$ is defined on A by $\|a\| := \inf\{0 < \lambda \in \mathbb{R} : \ -\lambda \leq a \leq \lambda\}$. If $a \in A$, then by [6: Theorem 8.11], $C(a)$ is norm closed in A. In fact, it can be shown that, in the presence of axioms SA1–SA7, axiom SA8 is equivalent to the condition that $C(a)$ is norm closed in A for all $a \in A$.

Since A is closed under squaring, it is a *special Jordan algebra* under the Jordan product

$$a \circ b := \frac{1}{2}((a+b)^2 - a^2 - b^2) = \frac{1}{2}(ab + ba) \in A \qquad \text{for all} \quad a, b \in A.$$

If $a, b \in A$, then $ab + ba = 2(a \circ b) \in A$ and $aCb \implies ab = ba = a \circ b = b \circ a \in A$. Also, $aba = 2a \circ (a \circ b) - a^2 \circ b \in A$.

2.3. Definition. ([6: Definition 4.1]) If $a \in A$, the mapping $J_a \colon A \to A$ defined for $b \in A$ by $J_a(b) := aba$ is called the *quadratic mapping* determined by a. If $p \in P$, then the quadratic mapping J_p is called the *compression* determined by p [5].

If $a \in A$, then by [6: Theorem 4.2, Lemma 4.4] the quadratic mapping $J_a \colon A \to A$ is linear, order preserving, and norm-continuous. In particular, if $0 \leq b \in A$, then $0 \leq J_a(b) = aba$, which is a stronger version of axiom SA3.

If $a, b, c \in A$, then abc belongs to R, but not necessarily to A. However, we have the following.

2.4. Lemma. *If $a, b, c \in A$, then $abc + cba \in A$.*

Proof. $abc+cba = (a+c)b(a+c) - aba - cbc = J_{a+c}(b) - J_a(b) - J_c(b) \in A$. □

By [6: Theorem 2.2], each element $a \in A^+$ has a uniquely determined *square root* $a^{1/2} \in A^+$ such that $(a^{1/2})^2 = a$; moreover, $a^{1/2} \in CC(a)$. If $a \in A$, then $a^2 \in A^+$, whence a has an *absolute value* $|a| := (a^2)^{1/2} \in CC(a^2) \subseteq CC(a)$ which is uniquely determined by the properties $|a| \in A^+$ and $|a|^2 = a^2$.

By [6: Lemma 7.1, Theorem 7.2], an element $a \in A$ has an *inverse* $a^{-1} \in A$ such that $aa^{-1} = a^{-1}a = 1$ iff there exists $0 < \varepsilon \in \mathbb{R}$ such that $\varepsilon \leq |a|$; moreover, if a is *invertible* (i.e., a^{-1} exists in A), then $a^{-1} \in CC(a)$.

If $a \in A$, then by [6: Theorem 3.3],

$$a^+ := \frac{1}{2}(|a| + a) \in A^+ \cap CC(a) \qquad \text{and} \qquad a^- := \frac{1}{2}(|a| - a) \in A^+ \cap CC(a).$$

Moreover, we have $a = a^+ - a^-$, $|a| = a^+ + a^-$, and $a^+a^- = a^-a^+ = 0$.

Clearly, $P \subseteq E \subseteq A$. An effect $e \in E$ is said to be *sharp* iff the only effect $f \in E$ such that $f \leq e$ and $f \leq 1 - e$ is $f = 0$. Obviously, the unit interval E is convex — in fact, E forms a *convex effect algebra* [14] under the partial binary operation obtained by restriction to E of the addition operation on A. By [6: Theorem 2.6], P is the set of all sharp effects, and it is also the set of all extreme points of the convex set E.

The *generalized Hermitian algebras*, introduced and studied in [9, 12], are special cases of synaptic algebras; in fact, the synaptic algebra A is a generalized Hermitian algebra iff it satisfies the condition that every bounded ascending sequence $a_1 \leq a_2 \leq \dots$ of pairwise commuting elements in A has a supremum a in A and $a \in CC(\{a_n : n \in \mathbb{N}\})$ [9: Section 6].

If $(A_i : i \in I)$ is a nonempty family of synaptic algebras and R_i is the enveloping algebra of A_i for each $i \in I$, then with coordinatewise operations and partial order, the cartesian product $\bigtimes_{i\in I} A_i$ is again a synaptic algebra with $\bigtimes_{i\in I} R_i$ as its enveloping algebra.

3. Review of orthomodular lattices

As we have mentioned, it turns out that the set P of projections in the synaptic algebra A forms an orthomodular lattice (OML) [2,18]; hence we devote this section to a brief review of some of the theory of OMLs that we shall require in what follows.

Let L be a nonempty set partially ordered by $\leq$. If there is a smallest element, often denoted by 0, and a largest element, often denoted by 1, in L, then we say that L is *bounded*. If, for every $p, q \in L$, the *meet* $p \wedge q$ (i.e., the greatest lower bound, or infimum) and the *join* $p \vee q$ (i.e., the least upper bound, or supremum) of p and q exist in L, then L is called a *lattice*. If L is a bounded lattice, then elements $p, q \in L$ are said to be *complements* of each other iff $p \wedge q = 0$ and $p \vee q = 1$.

If every subset of L has an infimum and a supremum, then L is called a *complete* lattice. A subset S of L is said to be *sup/inf-closed* in L iff whenever a nonempty subset Q of S has a supremum $s := \bigvee Q$ (respectively, an infimum $t := \bigvee Q$) in P, then $s \in S$, whence s is the supremum of Q as calculated in S (respectively, $t \in S$, whence t is the infimum of Q as calculated in S).

Let L be a bounded lattice. A mapping $p \mapsto p^{\perp}$ on L is called an *orthocomplementation* iff, for all $p, q \in L$,

(i) $p^{\perp}$ is a complement of p in L,

(ii) $(p^{\perp})^{\perp} = p$, and $p \leq q \implies q^{\perp} \leq p^{\perp}$.

We say that L is an *orthomodular lattice* (OML) iff it is equipped with an orthocomplementation $p \mapsto p^{\perp}$ that satisfies the *orthomodular law*: $p \leq q \implies q = p \vee (q \wedge p^{\perp})$ for all $p, q \in L$. If L is an OML, then elements $p, q \in L$ are *orthogonal*, in symbols $p \perp q$, iff $p \leq q^{\perp}$. *For the remainder of this section, we assume that L is an OML.*

The following *De Morgan duality* holds in L: If $Q \subseteq L$ and the supremum $\bigvee Q$ (respectively, the infimum $\bigwedge Q$) exists in L, then $(\bigvee Q)^{\perp} = \bigwedge\{q^{\perp} : q \in Q\}$ (respectively, $(\bigwedge Q)^{\perp} = \bigvee\{q^{\perp} : q \in Q\}$).

The elements $p, q \in L$ are said to be (*Mackey*) *compatible* in L iff there are pairwise orthogonal elements $p_1, q_1, d \in L$ such that $p = p_1 \vee d$ and $q = q_1 \vee d$. For instance, if $p \leq q$, or if $p \perp q$, then p and q are compatible; also, if p and q are compatible, then so are p and $q^{\perp}$. As is well-known (e.g., see [18: Proposition 4, p. 24] or [24: Proposition 1.3.8]), compatibility is preserved under the formation of arbitrary existing suprema or infima in L. Computations in L are facilitated by the following result: If $p, q, r \in L$ and one of the elements p, q, or r is compatible with the other two, then the distributive relations $(p \vee q) \wedge r = (p \wedge r) \vee (q \wedge r)$ and $(p \wedge q) \vee r = (p \vee r) \wedge (q \vee r)$ hold [4].

The subset of L consisting of all elements of L that are compatible with every element of L is called the *center* of L. As is well-known [18: p, 26], the center of L forms a boolean algebra, i.e., a bounded, complemented, distributive lattice [26], and it is sup/inf-closed in L.

For each $p \in L$, the mapping $\phi_p \colon P \to P$ defined for $q \in L$ by $\phi_p q := p \wedge (p^{\perp} \vee q)$ is called the *Sasaki projection* corresponding to p. The Sasaki projection has the following properties for all $p, q, r \in L$:

(i) $\phi_p q \perp r \iff q \perp \phi_p r$.

(ii) $\phi_p \colon P \to P$ is order preserving.

(iii) $\phi_p(\phi_p q) = \phi_p q$.

(iv) p and q are compatible iff $\phi_p q = p \wedge q$ iff $\phi_p q \leq q$.

(v) $p \perp q$ iff $\phi_p q = 0$.

(vi) ϕ_p preserves arbitrary existing suprema in L.

If $p \in L$, the *p-interval*, defined and denoted by $L[0,p] := \{q \in L : 0 \leq q \leq p\}$, is a sublattice of L with greatest element p and it forms an OML in its own right with $q \mapsto q^{\perp_p} = q^{\perp} \wedge p$ as the orthocomplementation. If c belongs to the center of L, it is easy to see that $c \wedge p$ belongs to the center of $L[0,p]$. If, conversely, for every $p \in L$, every element d of the center of $L[0,p]$ has the form $d = c \wedge p$ for some c in the center of L, then L is said to have the *relative center property* [3].

3.1. Lemma. *Let $p \in L$, let $q, r \in L[0,p]$, and let $\phi_q^p \colon L[0,p] \to L[0,p]$ be the Sasaki projection determined by q on the OML $L[0,p]$. Then:*

(i) *$\phi_q^p r = \phi_q r$, i.e., ϕ_q^p is the restriction to $L[0,p]$ of the Sasaki projection ϕ_q on L.*

(ii) *$\phi_q^p(r^{\perp_p}) = \phi_q(r^{\perp})$.*

Proof.

(i) Since $r = r \wedge p$, $q = q \wedge p$, and p is compatible with both $q^{\perp}$ and r, we have $\phi_q^p r = q \wedge (q^{\perp_p} \vee r) = q \wedge ((q^{\perp} \wedge p) \vee (r \wedge p)) = q \wedge p \wedge (q^{\perp} \vee r) = q \wedge (q^{\perp} \vee r) = \phi_q(r)$.

(ii) Similarly, $\phi_q^p(r^{\perp_p}) = q \wedge (q^{\perp_p} \vee r^{\perp_p}) = q \wedge ((q^{\perp} \wedge p) \vee (r^{\perp} \wedge p)) = q \wedge p \wedge (q^{\perp} \vee r^{\perp}) = q \wedge (q^{\perp} \vee r^{\perp}) = \phi_q(r^{\perp})$. □

If p and q share a common complement in L, they are said to be *perspective* and if p and q are perspective in $L[0, p \vee q]$, they are said to be *strongly perspective*. Strongly perspective elements are perspective, but in general, not conversely. In fact, L is modular (i.e., for all $p, q, r \in L$, $p \leq r \implies p \wedge (q \vee r) = (p \wedge q) \vee r$) iff perspective elements in L are always strongly perspective [15: Theorem 2]. The transitive closure of the relation of perspectivity is an equivalence relation on L called *projectivity*. If L is modular and complete as a lattice, then by classic results of von Neumann [23] and Kaplansky [20], perspectivity is transitive on L, and therefore it coincides with projectivity.

Proof of the following lemma is a straightforward OML-calculation.

3.2. Lemma. *If $e, f, p \in L$ and if e and f are perspective in $L[0,p]$, then e and f are perspective in L. In fact, if $q \in L[0,p]$ is a common complement of e and f in $L[0,p]$, then $q \vee p^{\perp}$ is a common complement of e and f in L.*

If $p, q \in L$, we have the *parallelogram law* asserting that $(p \vee q) \wedge p^{\perp} = \phi_{p^{\perp}}(q)$ is strongly perspective to $q \wedge (p \wedge q)^{\perp} = \phi_q(p^{\perp})$ (see the proof of [15: Corollary 1]). Replacing p by $p^{\perp}$, we obtain an alternative version of the parallelogram law asserting that $\phi_p(q)$ is strongly perspective to $\phi_q(p)$. (Another version of the parallelogram law is given in Theorem 5.9(ii) below.)

The following theorem provides an analogue for strong perspectivity of [21: Lemma 43] for a dimension equivalence relation on an OML.

3.3. Theorem. *Suppose $p, q, e, f \in P$ with $p \perp q$, $e \perp f$ and $p \vee q = e \vee f$. Put $p_1 := p \wedge (p \wedge f)^{\perp}$, $p_2 := p \wedge f$, $q_1 := q \wedge e$, $q_2 := q \wedge (q \wedge e)^{\perp}$, $e_1 := e \wedge (e \wedge q)^{\perp}$, and $f_2 := f \wedge (f \wedge p)^{\perp}$. Then:*

(i) *p_1 and e_1 are strongly perspective.*

(ii) *q_2 and f_2 are strongly perspective.*

(iii) *$p_1 \perp p_2$ with $p_1 \vee p_2 = p$, $q_1 \perp e_1$ with $q_1 \vee e_1 = e$, $p_2 \perp f_2$ with $p_2 \vee f_2 = f$, and $q_1 \perp q_2$ with $q_1 \vee q_2 = q$.*

(iv) *$p_1 \perp q_1$ and $p_1 \vee q_1$ is strongly perspective to e.*

(v) *$p_2 \perp q_2$ and $p_2 \vee q_2$ is strongly perspective to f.*

Proof.

(i) Let $k := p \vee q = e \vee f$. By the parallelogram law in the OML $L[0, k]$, and with the notation and results of Lemma 3.1, we find that $\phi_p^k(e) = \phi_p^k(f^{\perp_k}) = \phi_p(f^{\perp}) = p \wedge (p^{\perp} \vee f^{\perp}) = p \wedge (p \wedge f)^{\perp} = p_1$ and $\phi_e^k(p) = \phi_e^k(q^{\perp_k}) = \phi_e(q^{\perp}) = e \wedge (e^{\perp} \vee q^{\perp}) = e \wedge (e \wedge q)^{\perp} = e_1$ have a common complement in $(L[0, k])[0, p_1 \vee e_1] = L[0, p_1 \vee e_1]$, proving (i).

(ii) Since $k = f \vee e = q \vee p$, (ii) follows from (i) by symmetry.

(iii) The assertions in (iii) are obvious.

(iv) We have $p_1 \leq p \leq q^{\perp} \leq q^{\perp} \vee e^{\perp} = {q_1}^{\perp}$, so $p_1 \perp q_1$. By (i) there exists $v_1 \in L[0, p_1 \vee e_1]$ such that $p_1 \vee v_1 = e_1 \vee v_1 = p_1 \vee e_1$ and $p_1 \wedge v_1 = e_1 \wedge v_1 = 0$. We claim that v_1 is also a common complement of $p_1 \vee q_1$ and e in $L[0, (p_1 \vee q_1) \vee e]$. We note that $(p_1 \vee q_1) \vee e = p_1 \vee q_1 \vee e_1 \vee q_1 = p_1 \vee q_1 \vee e_1 = p_1 \vee e$. Also, $(p_1 \vee q_1) \vee v_1 = (p_1 \vee v_1) \vee q_1 = p_1 \vee e_1 \vee q_1 = p_1 \vee e$ and $e \vee v_1 = q_1 \vee e_1 \vee v_1 = q_1 \vee p_1 \vee e_1 = p_1 \vee e$. Moreover, $p_1 \leq {q_1}^{\perp}$ and $e_1 = e \wedge (e \wedge q)^{\perp} \leq (e \wedge q)^{\perp} = {q_1}^{\perp}$, whence $v_1 \leq p_1 \vee e_1 \leq {q_1}^{\perp}$, and we have $v_1 \perp q_1$. Thus, since q_1 is orthogonal to, hence compatible with, both p_1 and v_1, it follows that $(p_1 \vee q_1) \wedge v_1 = (p_1 \wedge v_1) \vee (q_1 \wedge v_1) = 0$. Similarly, as q_1 is orthogonal to both e_1 and v_1, we have $(e_1 \vee q_1) \wedge v_1 = (e_1 \wedge v_1) \vee (q_1 \wedge v_1) = 0$, completing the proof of our claim.

(v) Since $k = q \vee p = f \vee e$, (v) follows from (iv) by symmetry. □

3.4. Remark. We recall that the OML L is organized into an effect algebra [11: p. 284] in which every element is principal [11: p. 286] by defining the orthosum $p \oplus q := p \vee q$ of p and q in L iff $p \perp q$. Then the effect-algebra partial order coincides with the partial order on L and the effect-algebra orthosupplementation is the orthocomplementation on L. Thus the theory of effect algebras is applicable to OMLs.

As is easily seen, if the OML L is regarded as an effect algebra, then a family of elements in L is *orthogonal* iff it is pairwise orthogonal, such an orthogonal family is orthosummable iff it has a supremum, and if the family is orthosummable, then its supremum is its orthosum [11: p. 286]. If every orthogonal family in an effect algebra is orthosummable, then the effect algebra is called *orthocomplete* [17]. If the OML L is regarded as an effect algebra, then L is orthocomplete iff it is complete as a lattice [16].

4. The orthomodular lattice of projections

By [6: Theorem 5.6], under the partial order inherited from A, the set P of projections forms an orthomodular lattice with $p \mapsto p^{\perp} := 1 - p$ as the orthocomplementation. As $P \subseteq A$, the OML P acquires several special properties not enjoyed by OMLs in general.

Let $p, q \in P$. By [6: Theorem 2.4, Lemma 5.3], $p \leq q \iff p = pq \iff p = qp \iff p = qpq = J_q(p)$ and $p \leq q \implies q - p = q \wedge p^{\perp}$. Moreover, $pCq \implies pq = qp = p \wedge q$. Evidently, $p \perp q$ iff $p + q \leq 1$. Also by [6: Lemma 5.3], $p \perp q \iff pq = qp = 0$ and $p \perp q \implies p \vee q = p + q$. We refer to $p + q$ as the *orthogonal sum* of p and q iff $p \perp q$. A simple argument yields the important result that p and q are compatible iff pCq [7: Theorem 3.11].

By [6: Theorems 2.7, 2.10], each element $a \in A$ has a *carrier projection* $a^{\mathrm{o}} \in P$ such that, for all $b \in A$, $ab = 0 \iff a^{\mathrm{o}}b = 0$; moreover, $a^{\mathrm{o}} \in CC(a)$, $a = aa^{\mathrm{o}} = a^{\mathrm{o}}a$, $a^{\mathrm{o}} = |a|^{\mathrm{o}}$, and for all $b \in A$, $ab = 0 \iff a^{\mathrm{o}}b^{\mathrm{o}} = 0 \iff b^{\mathrm{o}}a^{\mathrm{o}} = 0 \iff ba = 0$. Furthermore, if $q \in P$, then $aq = a \iff qa = a \iff a^{\mathrm{o}} \leq q$. The carrier projection a^{o} is uniquely characterized by the property $ap = 0 \iff a^{\mathrm{o}}p = 0$ for all $p \in P$, or equivalently, by the property that a^{o} is the smallest projection $q \in P$ such that $a = aq$.

4.1. Lemma. *If $a_1, a_2, \ldots, a_n \in A^+$, then* $\left(\sum\limits_{i=1}^{n} a_i\right)^{\mathrm{o}} = \bigvee\limits_{i=1}^{n} (a_i)^{\mathrm{o}}$.

Proof. By [25: Lemma 3.1], the lemma holds for the case $n = 2$, and the general case then follows by mathematical induction. □

If $a \in A$, then $\operatorname{sgn}(a) := (a^+)^{\mathrm{o}} - (a^-)^{\mathrm{o}}$ is called the *signum* of a, and by [6: Theorem 3.6], $\operatorname{sgn}(a) \in CC(a)$, $(\operatorname{sgn}(a))^2 = a^{\mathrm{o}}$, and $a = \operatorname{sgn}(a)|a| = |a| \operatorname{sgn}(a)$, the latter formula being called the *polar decomposition* of a.

Each element $a \in A$ has a *spectral resolution* [6: Section 8], [8] that both determines and is determined by a, namely the right continuous ascending family $(p_{a,\lambda} : \lambda \in \mathbb{R})$ of projections in $CC(a)$ given by

$$p_{a,\lambda} := 1 - ((a - \lambda)^+)^{\mathrm{o}} = (((a - \lambda)^+)^{\mathrm{o}})^{\perp} \qquad \text{for all} \quad \lambda \in \mathbb{R}.$$

By [6: Theorems 8.4, 8.5], $L := \sup\{\lambda \in \mathbb{R} : \ p_{a,\lambda} = 0\} \in \mathbb{R}$, $U := \inf\{\lambda \in \mathbb{R} : p_{a,\lambda} = 1\} \in \mathbb{R}$, and $a = \int\limits_{L-0}^{U} \lambda \, dp_{a,\lambda}$, where the Riemann-Stieltjes type integral converges in norm.

By [10: Theorem 8.3], any one of the following conditions is sufficient to guarantee modularity of the projection lattice P:

(i) If $p, q \in P$, there exists $0 < \varepsilon \in \mathbb{R}$ such that $\varepsilon(pqp)^{\mathrm{o}} \leq pqp$.

(ii) If $p, q \in P$, then pqp is an algebraic element of A;

(iii) A is finite dimensional over $\mathbb{R}$;

(iv) P satisfies the ascending chain condition.

Let $p \in P$. Then, according to [6: Theorem 4.9],

$$pAp := \{pap : \ a \in A\} = \{a \in A : \ pa = ap = a\} = \{a \in A : \ a^{\mathrm{o}} \leq p\} = J_p(A)$$

is norm closed in A, and with the partial order inherited from A, it is a synaptic algebra (degenerate if $p = 0$) with p as its order unit, pRp as its enveloping algebra, and the order unit norm on pAp is the restriction to pAp of the order unit norm on A. Moreover, if $a, b \in pAp$, then $a \circ b$, a^{o}, $|a|$, a^+, $a^- \in pAp$, and if $a \in A^+$, then $a^{1/2} \in pAp$. Consequently, if $a \in pAp$, then $pp_{a,\lambda} = p_{a,\lambda}p$ $= p \wedge p_{a,\lambda}$ for all $\lambda \in \mathbb{R}$, and the spectral resolution of a as calculated in pap is $(pp_{a,\lambda} : \ \lambda \in \mathbb{R})$. Clearly, the OML of projections in the synaptic algebra pAp is the p-interval $P[0,p]$ in P, and the orthocomplementation on $P[0,p]$ is given by

$$q \mapsto q^{\perp_p} = p - q = J_p(q^\perp) = pq^\perp = q^\perp p = p \wedge q^\perp \qquad \text{for} \quad q \in P[0,p].$$

Suppose that $B \subseteq A$. Then $C(B) = \bigcap\limits_{b \in B} C(b)$ is norm closed in A, and with the partial order inherited from A, $C(B)$ is a synaptic algebra with order unit 1 and enveloping algebra R. Moreover, if $a, c \in C(B)$, then $a \circ c$, a^{o}, $|a|$, a^+, a^- $\in C(B)$ and $0 \leq a \implies a^{1/2} \in C(B)$. Consequently, if $a \in C(B)$, then the spectral resolution of a is the same whether calculated in A or in $C(B)$. Also it is clear that the OML of projections in the synaptic algebra $C(B)$ is just $P \cap C(B)$, and we have the following result.

4.2. Lemma. *If $B \subseteq A$, then $P \cap C(B)$ is sup/inf-closed in P.*

Proof. As $C(B) = \bigcap\limits_{b \in B} C(b)$, it will be sufficient to prove the lemma for the special case $B = \{b\}$. Thus, assume that $Q \subseteq P \cap C(b)$ and that $h = \bigvee Q$ exists in P. For the projections in the spectral resolution of b, we have $C(b) \subseteq C(p_{b,\lambda})$, whence $Q \subseteq P \cap C(p_{b,\lambda})$ for every $\lambda \in \mathbb{R}$. We recall that compatibility is preserved under the formation of arbitrary existing suprema and infima. Thus, $h \in C(p_{b,\lambda})$ for all $\lambda \in \mathbb{R}$, and it follows from [6: Theorem 8.10] that $h \in C(b)$. A similar argument applies to the infimum k, if it exists in P. $\square$

We shall make extensive use of the next theorem, often without explicit attribution.

4.3. Theorem. *The center of P is $P \cap C(P) = P \cap C(A)$.*

Proof. As two projections in P are compatible iff they commute, the center of P is $P \cap C(P)$. Clearly, $P \cap C(A) \subseteq P \cap C(P)$. Conversely, by [6: Theorem 8.10], $P \cap C(P) \subseteq P \cap C(A)$, so $P \cap C(A) = P \cap C(P)$. □

In view of Theorem 4.3, if we say that c is a *central projection* in A, we mean that $c \in P \cap C(A)$, or what is the same thing, that c belongs to the center $P \cap C(P)$ of the OML P. As is easily seen, if P is regarded as an effect algebra, then the center $P \cap C(P)$ of P coincides with the effect-algebra center of P [11: p. 287].

4.4. Remarks. Suppose that c is a central projection in A. Then $c^\perp$ is also a central projection and A is the (internal) *direct sum* of the synaptic algebras $cAc = cA = Ac$ and $c^\perp A c^\perp = c^\perp A = A c^\perp$ in the sense that:

(1) As a vector space, A is the (internal) direct sum of the vector subspaces cA and $c^\perp A$.

(2) If $a = x + y$ with $x \in cA$ and $y \in c^\perp A$, then $0 \le a \iff 0 \le x$ and $0 \le y$.

(3) If $a_i = x_i + y_i$ with $x_i \in cA$ and $y_i \in c^\perp A$ for $i = 1, 2$, then $a_1 a_2 = x_1 x_2 + y_1 y_2$ (the products being calculated in R).

With coordinatewise operations and partial order, the cartesian product $cA \times c^\perp A$ is a synaptic algebra with order unit $(c, c^\perp)$ and enveloping algebra $cRc \times c^\perp R c^\perp$, and A is order, linear, and Jordan isomorphic to $cA \times c^\perp A$ under the mapping $a \mapsto (ca, c^\perp a)$.

Naturally, A is called a *commutative* synaptic algebra iff aCb for all $a, b \in A$, i.e., iff $A = C(A)$. Thus, a commutative synaptic algebra is a commutative associative archimedean partially ordered linear algebra with a unity element. In [12: Section 4], the following result is stated without proof.

4.5. Theorem. *The synaptic algebra A is commutative iff P is a boolean algebra i.e., iff the lattice P is distributive.*

Proof. If A is commutative, then $P \cap C(A) = P \cap A = P$, i.e., P is its own center, whence P is a boolean algebra. Conversely, if P is a boolean algebra, then any two projections $p, q \in P$ are compatible, and therefore $p, q \in P \implies pCq$. Let $a, b \in A$. Then, for all $\lambda, \mu \in \mathbb{R}$, $p_{a,\lambda} C p_{b,\mu}$, whence $aCp_{b,\mu}$ by [6: Theorem 8.10]. Therefore, aCb by a second application of [6: Theorem 8.10], and it follows that A is commutative. □

It can be shown that every boolean algebra can be realized as the lattice of projections in a commutative synaptic algebra. The center $C(A)$ is a commutative synaptic algebra, and if B is a subset of A consisting of pairwise commuting elements, then $CC(B)$ is a commutative synaptic algebra. In particular, $CC(a)$ is a commutative synaptic algebra for any choice of $a \in A$.

5. Symmetries and perspectivities

Although there is some overlap between this section and [25: Section 3], the material here is arranged a little differently, so for the reader's convenience, we give proofs of most of our results.

5.1. Definition. An element $s \in A$ is called a *symmetry* iff $s^2 = 1$. An element $t \in A$ is called a *partial symmetry* iff $t^2 \in P$.

Proofs of the following statements are straightforward.

(i) If $t \in A$ is a partial symmetry with $p := t^2$, then t is a symmetry in the synaptic algebra pAp.

(ii) If $a \in A$, then the element $t := \operatorname{sgn}(a)$ in the polar decomposition $a = t|a|$ is a partial symmetry with $t^2 = a^{\circ}$.

(iii) If s is a symmetry, then $-s$ is a symmetry as well.

(iv) There is a bijective correspondence $p \leftrightarrow s$ between symmetries in A and projections in P given by $s = 2p - 1$ and $p = \frac{1}{2}(1 + s)$.

(v) If s is a symmetry, then $\|s\|^2 = \|s^2\| = \|1\| = 1$, so $\|s\| = 1$.

(vi) If s is a symmetry, then as $0 \leq \frac{1}{2}(1 \pm s) \in P$, it follows that $-1 \leq s \leq 1$.

If $p \in P$ and $s := 2p - 1 = p - (1 - p)$ is the corresponding symmetry, then s is the difference of the orthogonal projections p and $1 - p$. More generally, by the following theorem, a difference of two orthogonal projections p and q is a partial symmetry and *vice versa*.

5.2. Theorem. *If p and q are orthogonal projections, then $t := p - q$ is a partial symmetry with $t^2 = p + q = p \vee q \in P$. Conversely, if t is a partial symmetry, then both $p := t^+$ and $q := t^-$ are projections, $pq = 0$, $t = p - q$, $t^2 = p + q = |t| \in P$, and $s := t + (1 - t^2) = t + (1 - |t|) = 1 - 2q$ is a symmetry.*

Proof. The first statement of the lemma is obvious. So assume that $u := t^2 \in P$, $p := t^+$ and $q := t^-$. Then $t = p - q$ and $pq = 0$. Also, as $0 \leq u$ and $u^2 = u$, we have $p + q = |t| = (t^2)^{\frac{1}{2}} = u^{\frac{1}{2}} = u$. Moreover, as $0 \leq t^+ = p$, $0 \leq t^- = q$ and $u \in P$, we have $0 \leq p \leq p + q = u \leq 1$, so $p \in E$, and it follows from [6: Theorem 2.4] that $p = pu = p(p + q) = p^2 + pq = p^2$, so $p \in P$ and likewise, $q \in P$. Then $s := t + (1 - t^2) = p - q + (1 - (p + q)) = 1 - 2q$ is the symmetry corresponding to the projection $1 - q$. $\square$

If t is a partial symmetry, then we refer to the symmetry $s := t + (1 - t^2)$ in Theorem 5.2 as the *canonical extension of t to a symmetry s.*

If $s \in A$ is a symmetry, then the quadratic mapping $J_s\colon A \to A$ is called the *symmetry transformation* corresponding to s.

5.3. Theorem. *Let s be a symmetry, $a, b \in A$, and $e, f \in P$. Then:*

(i) *The symmetry transformation $J_s(a) := sas$ is an order, linear, and Jordan automorphism of A and $(J_s)^{-1} = J_s$.*

(ii) *J_s restricted to P is an OML-automorphism of P.*

(iii) *If $ab = ba$, then $J_s(ab) = J_s(a)J_s(b) = J_s(b)J_s(a) \in A$.*

(iv) *If $J_s(a)J_s(b) = J_s(b)J_s(a)$, then $ab = ba \in A$.*

(v) *$(J_s(a))^{\mathrm{o}} = J_s(a^{\mathrm{o}})$.*

Proof.

(i) As J_s is a quadratic mapping, it is both linear and order preserving. Also, for $a \in A$, $J_s(a^2) = sa^2s = sassas = (J_s(a))^2$, and it follows that J_s is a Jordan homomorphism of A. Since $J_s(J_s(a)) = ssass = a$, it follows that J_s is its own inverse on A; hence it is a linear, order, and Jordan automorphism.

(ii) If $e \in P$, then $(J_s(e))^2 = J_s(e^2) = J_s(e)$, so J_s maps P into (and clearly onto) P. Thus the restriction of J_s to P is an order automorphism of P, and as $J_s(1 - e) = 1 - J_s(e)$, it is an OML automorphism.

(iii) If $ab = ba$, then $ab = ba = a \circ b = b \circ a$, so part (iii) follows from the fact that J_s is a Jordan automorphism.

(iv) Assume the hypothesis of (iv). Then by (iii) with a replaced by $J_s(a)$ and b replaced by $J_s(b)$, we have $ab = J_s(J_s(a))J_s(J_s(b)) = J_s(J_s(a)J_s(b)) = J_s(J_s(b)J_s(a)) = J_s(J_s(b))J_s(J_s(a)) = ba$.

(v) By (ii), $J_s(a^{\mathrm{o}}) \in P$, and since $aa^{\mathrm{o}} = a^{\mathrm{o}}a = a$, (iii) implies that $J_s(a)J_s(a^{\mathrm{o}}) = J_s(aa^{\mathrm{o}}) = J_s(a)$. Suppose that $f \in P$ with $J_s(a)f = J_s(a)$. It will be sufficient to prove that $J_s(a^{\mathrm{o}}) \le f$. We have $fJ_s(a) = J_s(a) = J_s(a)f$. Put $e := J_s(f)$. Then $e \in P$ and $J_s(e) = f$, so $J_s(e)J_s(a) = J_s(a)J_s(e)$; hence, by (iv), $ea = ae$, and therefore by (iii), $J_s(a) = J_s(a)f = J_s(a)J_s(e) = J_s(ae)$, whence, $a = ae$ and therefore $a^{\mathrm{o}} \le e$, whereupon $J_s(a^{\mathrm{o}}) \le J_s(e) = f$. □

5.4. Definition. A symmetry $s \in A$ is said to *exchange the projections $e, f \in P$* iff $ses = f$, i.e., iff $J_s(e) = f$. A partial symmetry t is said to *exchange the projections $e, f \in P$* iff both $tet = f$ and $tft = e$ hold.

Let s be a symmetry and let $e, f \in P$. Clearly, s exchanges e and f iff $J_s(e) = f$ iff $J_s(f) = e$ iff s exchanges f and e.

5.5. Theorem. *Let $t \in A$ be a partial symmetry that exchanges the projections $e, f \in P$ and let $s := t + (1 - t^2)$ be the canonical extension of t to a symmetry. Then s exchanges e and f.*

P r o o f. Assume the hypotheses and let $u := t^2$. Then $u \in P$ and we have $e = tft = t^2et^2 = ueu$, so $e = ue = eu$, and therefore $(1-u)e = e(1-u) = 0$. Consequently, $ses = (t+(1-u))e(t+(1-u)) = tet = f$. □

The following lemma provides a weak version of finite additivity for the relation of exchangeability by a symmetry.

5.6. Lemma. *Let $e, e_1, e_2, f, f_1, f_2 \in P$ with $e_1 \perp f_2$, $e_2 \perp f_1$, $e_1 \perp e_2$, $f_1 \perp f_2$, $e = e_1 + e_2$ and $f = f_1 + f_2$, and suppose that e_i and f_i are exchanged by a symmetry $s_i \in A$ for $i = 1, 2$. Then there is a symmetry $s \in A$ exchanging e and f.*

P r o o f. Let $p_i = e_i \vee f_i$ for $i = 1, 2$. As $e_1 \le e_2^{\perp}, f_2^{\perp}$, we have $e_1 \le p_2^{\perp}$. Also, $f_1 \le e_2^{\perp}, f_2^{\perp}$, so $f_1 \le p_2^{\perp}$, and it follows that $p_1 = e_1 \vee f_1 \le p_2^{\perp}$, whence $p_1p_2 = 0$. Set $u = s_1p_1$ and $v = s_2p_2$. From $s_1p_1s_1 = s_1(e_1 \vee f_1)s_1 = s_1e_1s_1 \vee s_1f_1s_1 = e_1 \vee f_1 = p_1$, it follows that s_1 commutes with p_1. Likewise s_2 commutes with p_2, whence both u and v belong to A and are partial symmetries with $u^2 = p_1$, $v^2 = p_2$, $ue_1u = p_1f_1p_1 = f_1$, $ve_2v = p_2f_2p_2 = f_2$, and $uv = 0$. Straightforward calculation using the data above shows that $s := u + v + (1 - p_1 - p_2)$ is a symmetry and that $ses = f$. □

5.7. Lemma. *If $e, f \in P$ and $a := e - f$, then $e, f \in C(|a|)$.*

P r o o f. We have $a^2 = (e-f)^2 = e - ef - fe + f$, from which it follows that $ea^2 = a^2e = e - efe$ and $fa^2 = a^2f = f - fef$. Thus, $e, f \in C(a^2)$, and as $|a| \in CC(a^2)$ it follows that $e, f \in C(|a|)$. □

5.8. Theorem. *If $e, f \in P$, there exists a symmetry $s \in A$ such that $sefes = fef$, i.e., $J_s(efe) = J_s(J_e(f)) = J_f(e) = fef$.*

P r o o f. Let $e, f \in P$. By Lemma 5.7 with f replaced by $1-f$ and $a := e+f-1$, we have $e, f \in C(|a|)$. Put $t := \operatorname{sgn}(a)$, so that $t^2 = a^{\circ} \in P$. Thus, t is a partial symmetry with $|a| = at = ta$, and we have

$$|a|f = taf = (te - t(1-f))f = tef \quad \text{and} \quad f|a| = fat = f(et - (1-f)t) = fet.$$

Therefore, since $ea = e + ef - e = ef$ and $|a|$ commutes with e and f,

$$t(efe)t = (tef)et = |a|fet = |a|f|a| = f|a||a| = fet|a| = fea = fef.$$

Now let $s := t + (1-t^2)$ be the canonical extension of t to a symmetry (Theorem 5.2). Since $tef = |a|f$, and $af = ef + f - f = ef$ it follows that

$$t^2efe = t(tef)e = t|a|fe = afe = efe,$$

so $(1-t^2)efe = 0$, and therefore $sefes = tefet = fef$. □

5.9. Theorem. *Let $e, f \in P$. Then:*

(i) *$\phi_e f$ and $\phi_f e$ are exchanged by a symmetry in A.*

(ii) *(Symmetry Parallelogram Law) $e - (e \wedge f))$ and $(e \vee f) - f$ are exchanged by a symmetry in A.*

(iii) *If e and f are complements in P, then e and $f^\perp$ are exchanged by a symmetry in A.*

(iv) *If $e \not\perp f$, there are nonzero subelements $0 \neq e_1 \leq e$ and $0 \neq f_1 \leq f$ that are exchanged by a symmetry.*

Proof.

(i) According to [6: Definition 4.8, Theorem 5.6], $(efe)^{\mathrm{o}} = \phi_e(f)$ and $(fef)^{\mathrm{o}} = \phi_f(e)$. Also, by Theorem 5.8, there exists a symmetry $s \in A$ such that $J_s(efe) = fef$, whence by Theorem 5.3 we have $J_s(\phi_e f) = J_s((efe)^{\mathrm{o}}) = (J_s(efe))^{\mathrm{o}} = (fef)^{\mathrm{o}} = \phi_f e$.

(ii) We have $e \wedge (e \wedge f)^\perp = e \wedge (e^\perp \vee f^\perp) = \phi_e(f^\perp)$ and $(e \vee f) \wedge f^\perp = \phi_{f^\perp}(e)$, so (ii) follows from (i).

(iii) If $e \wedge f = 0$ and $e \vee f = 1$, then the symmetry s in (ii) satisfies $ses = J_s(e) = f^\perp$.

(iv) If $e \not\perp f$, then $0 \neq e_1 := \phi_e f \leq e$ and $0 \neq f_1 := \phi_f(e) \leq f$, and by (i), e_1 and f_1 are exchanged by a symmetry. □

5.10. Lemma. *If the projections e and f are complements in P and $s \in A$ is a symmetry exchanging e and f, then e and f are perspective. In fact, the projection $p := \frac{1}{2}(1 + s) \in P$ corresponding to the symmetry s is a common complement of both e and f.*

Proof. Let $q := e \wedge p$. Then $q \leq e$ and $q \leq p$, so $eq = pq = q$. Thus, $sq = (2p - 1)q = q$, so $fq = sesq = seq = sq = q$, whence $q \leq f$. But $q \leq e$, so $e \wedge p = q \leq e \wedge f = 0$. Now let $r := e^\perp \wedge p^\perp$. Then $er = pr = 0$, so $sr = (2p - 1)r = -r$, whence $fr = sesr = -ser = 0$, and we have $r \leq f^\perp$. But $r \leq e^\perp$, so $r \leq e^\perp \wedge f^\perp = 0$. Therefore, p is a complement of e, and by a similar argument, p is also a complement of f. □

In the following theorem we improve the result in Lemma 5.10 by dropping the hypothesis that e and f are complements, and by concluding that e and f are not only perspective, but strongly perspective.

5.11. Theorem. *Let $e, f \in P$ be exchanged by a symmetry $s \in A$. Then e and f are strongly perspective in P. In fact, if $p := e \vee f$, $r := p - (e \wedge f)$, $t := rsr$, and $q := \frac{1}{2}(r + t)$, then t is a symmetry in rAr, q is a projection in $P[0, r]$, and $k := q \vee (r^\perp \wedge p)$ is a common complement of e and f in $P[0, p]$.*

Proof. Assume the hypotheses. By Theorem 5.3(ii), $sps = s(e \vee f)s = J_s(e \vee f) = J_s(e) \vee J_s(f) = ses \vee sfs = f \vee e = p$, whence $sp = ps$. Likewise, $s(e \wedge f)s = e \wedge f$ and it follows that $srs = r$, whence $t = rsr = sr = rs$. Therefore, $t \in rAr$ with $t^2 = r$, i.e., t is a symmetry in the synaptic algebra rAr. Clearly, both e and f commute with r, whence $t(e \wedge r)t = rs(er)sr = rsesr = rfr = fr = rf = f \wedge r$, and therefore t exchanges the projections $e \wedge r$ and $f \wedge r$ in $P[0, r]$.

We have $(e \wedge r) \wedge (f \wedge r) = (e \wedge f) \wedge p \wedge (e \wedge f)^{\perp} = 0$, and as $r \leq p$, $(e \wedge r) \vee (f \wedge r) = (e \vee f) \wedge r = p \wedge r = r$, so $e \wedge r$ and $f \wedge r$ are complements in $P[0, r]$. Thus, working in the synaptic algebra rAr with unit r, and applying Lemma 5.10, we find that $e \wedge r$ and $f \wedge r$ are perspective in $P[0, r]$ with $q = \frac{1}{2}(r+t)$ as a common complement. Therefore,

$$(e \wedge r) \vee k = (e \wedge r) \vee q \vee (r^{\perp} \wedge p)) = r \vee (p - r) = p.$$

Also, as $q \leq r \leq p$ and $e \wedge r \leq r \leq p$, it follows that both q and $e \wedge r$ commute with $r^{\perp} \wedge p$, whence

$$(e \wedge r) \wedge k = (e \wedge r) \wedge (q \vee (r^{\perp} \wedge p)) = ((e \wedge r) \wedge q) \vee (e \wedge r \wedge r^{\perp} \wedge p) = 0.$$

Likewise, $(f \wedge r) \vee k = p$ and $(f \wedge r) \wedge k = 0$. □

5.12. Theorem. *Let $e, f \in P$. Then:*

(i) *If e and f are perspective, then there are symmetries $s_1, s_2 \in A$ with $s_2s_1es_1s_2 = J_{s_2}(J_{s_1}(e)) = f$.*

(ii) *Suppose e and f are orthogonal and there are symmetries $s_1, s_2 \in A$ with $s_2s_1es_1s_2 = f$. Then there is a symmetry s exchanging e and f.*

(iii) *If e and f are both perspective and orthogonal, then there is a symmetry s exchanging e and f.*

Proof.

(i) Let p be a common complement of e and f. By Theorem 5.9(iii), there exist symmetries $s_1, s_2 \in A$ with $s_1es_1 = p^{\perp} = s_2fs_2$, and it follows that $s_2s_1es_1s_2 = f$.

(ii) Assume the hypotheses of (ii). Then $e = s_1s_2fs_2s_1$. Let

$$x := s_2s_1e \qquad \text{and} \qquad y := es_1s_2.$$

Here x and y belong to the enveloping algebra R, but not necessarily to A; however, $x + y \in A$ by Lemma 2.4. As $ef = fe = 0$, it follows that $xf = fy = 0$. We have $xy = f$, $yx = e$, $xe = x$, and $fx = s_2s_1es_1s_2s_2s_1e = x$, so $x^2 = xefx = 0$. Also, $ey = y$, and $yf = es_1s_2s_2s_1es_1s_2 = y$, so $y^2 = yfey = 0$. Furthermore, $ye = es_1s_2e = s_1s_2fs_2s_1s_1s_2e = s_1s_2fe = 0$ and $ex = es_2s_1e = es_2s_1s_1s_2fs_2s_1 = efs_2s_1 = 0$.

Now put $s := (x + y) + 1 - e - f$. Using the data above, a straightforward computation shows that s is a symmetry in A and s exchanges e and f.

(iii) Part (iii) follows from (i) and (ii). □

Theorem 5.15 below, is a version of additivity for projections exchanged by a symmetry, but it requires completeness of P and the rather strong hypothesis that the suprema of the two orthogonal families involved are themselves orthogonal. The next two lemmas will aid in its proof.

5.13. Lemma. *Let $e, f \in P$ with $e \perp f$, suppose that e and f are exchanged by a symmetry $s \in A$, put $x := se$, $y := es$, and $p := \frac{1}{2}(x + y + e + f)$. Then:*

(i) $xy = f$ *and* $yx = e$.

(ii) $x = xe = fx$, $y = ey = yf$, *and* $x^2 = y^2 = ex = xf = ye = fy = 0$.

(iii) $p \in P$.

(iv) $2epe = e$ *and* $2pep = p$.

(v) $2fpf = f$ *and* $2pfp = p$.

Proof.

(i) We have $ses = f$, $sfs = e$, so $xy = f$. Also $yx = es^2e = e^2 = e$.

(ii) Clearly, $xe = x$, $ey = y$, and since $ef = fe = 0$, $xf = 0$, and $fy = 0$. Moreover, $x^2 = sese = fe = 0$, $y^2 = eses = fs = fes = 0$, $ex = ese = sfsse = sfe = 0$, $yf = esf = esses = es = y$, and similarly, $y^2 = 0$, $ye = 0$, and $fx = x$.

(iii) We have $x + y \in A$, whence $p \in A$. A straightforward computation using the data in (ii) shows that $p^2 = p$.

(iv) Since $ex = ye = 0$ and $ef = 0$, it follows that $2epe = exe + eye + e + efe = e$. Similarly, $2pe = xe + ye + e = x + e$, so $2pep = xp + ep = \frac{1}{2}(x^2 + xy + xe + xf + ex + ey + e) = \frac{1}{2}(f + x + y + e) = p$.

(v) As in the proof of (iii), the proof of (iv) is a straightforward computation using the data in (ii). □

5.14. Lemma. *Suppose that P is a complete OML, let $(q_i)_{i \in I}$ be an orthogonal family in P, put $q := \bigvee_{i \in I} q_i$, let $i \in I$, $r \in P$, and suppose that $q_j r = 0$ for all $j \in I$ with $j \neq i$. Then $qr = q_i r$ and $rq = rq_i$.*

Proof. Define $q_i' := \bigvee_{j \in I, j \neq i} q_j$. As $q_j \perp q_i$ for all $j \in I$ with $j \neq i$, it follows that $q_i \perp q_i'$ with $q = q_i \vee q_i' = q_i + q_i'$. Also, since $q_j \perp r$ for $j \in I$ with $j \neq i$, it follows that $q_i' \perp r$, whence $q_i' r = r q_i' = 0$, and therefore $qr = (q_i + q_i')r = q_i r$ and $rq = r(q_i + q_i') = rq_i$. □

5.15. Theorem. *Suppose that the OML P is complete, let $(e_i)_{i \in I}$ and $(f_i)_{i \in I}$ be orthogonal families in P with $e = \bigvee_{i \in I} e_i$ and $f = \bigvee_{i \in I} f_i$. Then, if e and f are orthogonal and if, for each $i \in I$, e_i and f_i are exchanged by a symmetry $s_i \in A$, then there is a symmetry $s \in A$ exchanging e and f.*

Proof. Our proof is suggested by the proof of [19: Lemma 3.1]. We begin by noting that $e \perp f$ implies $e_i \perp f_j$ for all $i, j \in I$. Also, for $i \in I$, we have $s_ie_is_i = f_i$ and $s_if_is_i = e_i$. Let $x_i := s_ie_i$, $y_i := e_is_i$, and $p_i := \frac{1}{2}(x_i+y_i+e_i+f_i)$. By parts (i) and (iii) of Lemma 5.13, $y_ix_i = e_i$, $x_iy_i = f_i$, and $p_i \in P$. Put $p := \bigvee_{i\in I} p_i$ and $s := 2p - 1$. We are going to show that s is the required symmetry.

We claim that $(p_i)_{i\in I}$ is an orthogonal family. Indeed, suppose $i, j \in I$ with $i \neq j$. Then $4p_ip_j = (x_i + y_i + e_i + f_i)(x_j + y_j + e_j + f_j)$ and it will be sufficient to show that the sixteen terms that result from an expansion of the latter product are all zero. This follows from the facts that, for $i \neq j$, $e_ie_j = e_if_j = f_ie_j = f_if_j = 0$ together with the data in Lemma 5.13(ii). For instance, $x_ix_j = x_ie_if_jx_j = 0$.

As in the argument above, $p_je_i = 0$ for $i, j \in I$ with $j \neq i$, and it follows from Lemma 5.14 with $(q_i)_{i\in I} = (e_i)_{i\in I}$ and $r = p_i$ that $ep_i = e_ip_i$ and $p_ie = p_ie_i$ for all $i \in I$. Likewise, by Lemma 5.14, this time with $(q_i)_{i\in I} = (p_i)_{i\in I}$ and $r = e_i$ we have $pe_i = p_ie_i$ and $e_ip = e_ip_i$ for all $i \in I$.

By Lemma 5.13(iii), $2e_ip_ie_i = e_i$, whence $2epe_i = 2ep_ie_i = 2e_ip_ie_i = e_i$, and we have $(2ep - 1)e_i = 0$, whereupon $(2ep - 1)^{\text{o}}e_i = 0$ for all $i \in I$. Therefore, $e_i \leq ((2ep - 1)^{\text{o}})^{\perp}$ for all $i \in I$, and it follows that $e \leq ((2ep - 1)^{\text{o}})^{\perp}$, whence $(2ep - 1)^{\text{o}}e = 0$, and consequently, $(2ep - 1)e = 0$, i.e., $2epe = e$. By similar arguments, $2pep = p$, $2fpf = f$, and $2pfp = p$.

Let us write $h = ses = (2p - 1)e(2p - 1) = 4pep - 2ep - 2pe + e$, noting that, since s is a symmetry, h is a projection. Using the facts that $ef = fe = 0$, $2pep = p$, and $2fpf = f$ we find that

$$fhf = f(4pep - 2ep - 2pe + e)f = 4fpepf = 2fpf = f.$$

Similarly using the facts that $ef = fe = 0$, $2pfp = p$, and $2epe = e$,

$$\begin{aligned} hfh &= (2p-1)e(2p-1)f(2p-1)e(2p-1) \\ &= (2p-1)(2ep-e)f(2pe-e)(2p-1) \\ &= (2p-1)(4epfpe)(2p-1) = (2p-1)(2epe)(2p-1) \\ &= (2p-1)e(2p-1) = h. \end{aligned}$$

Therefore $f(1 - h)f = f - fhf = f - f = 0$, whence $f(1 - h) = 0$, so $f \leq h$. Likewise, since $hfh = h$, it follows that $h \leq f$, and we have $h = f$. □

6. Central orthocompleteness

If P is regarded as an effect algebra (Remark 3.4) then the following definition of central orthocompleteness for the OML P is equivalent to the effect-algebra definition of central orthocompleteness [11: Definition 6.1].

6.1. Definition. A family $(p_i)_{i\in I}$ in the OML P is *centrally orthogonal* iff there is a pairwise orthogonal family $(c_i)_{i\in I}$ in the center $P\cap C(A)$ of P such that $p_i \le c_i$ for all $i \in I$. The projection lattice P is *centrally orthocomplete* iff every centrally orthogonal family $(p_i)_{i\in I}$ in P has a supremum $p = \bigvee_{i\in I} p_i$ in P.

Obviously, if P is complete as a lattice, then it is centrally orthocomplete.

6.2. Standing Assumption. For the remainder of this article, we assume that the OML P of projections in A is centrally orthocomplete.

6.3. Theorem. ([11: Theorem 6.8 (i)]) *The center $P\cap C(A)$ of P is a complete boolean algebra.*

6.4. Lemma. *Let $d \in P\cap C(A)$ and let c be a projection in the synaptic algebra $dAd = dA = Ad$. Then:*

(i) *c is a central projection in dA iff c is a central projection in A.*

(ii) *The OML $P[0,d]$ of projections in dA is centrally orthocomplete.*

Proof.

(i) Suppose that c is a central projection in dA and let $a \in A$. Then, as $c \le d$, we have $c = cd = dc$ and $ca = cda = dac = ac$, so $c \in P\cap C(A)$. The converse is obvious, and (i) is proved. □

6.5. Lemma. ([11: Theorem 6.8 (ii)]) *For each $p \in P$, there is a smallest central projection $c \in P\cap C(A)$ such that $p \le c$.*

6.6. Definition. If $a \in A$, then the smallest central projection $c \in P\cap C(A)$ such that $a^{\circ} \le c$ is called the *central cover* of a and denoted by $\gamma(a)$. The mapping $\gamma\colon A \to P\cap CP$ is called the *central cover mapping*.

Since a° is the smallest projection $p \in P$ such that $ap = a$, it follows that γa is the smallest central projection $c \in P\cap C(A)$ such that $ac = a$. Moreover, by [11: Theorems 5.2, 6.10], the central cover mapping γ has the following properties.

6.7. Theorem. *Let $p, q \in P$. Then:*

(i) *$\gamma 1 = 1$, $\gamma p = 0 \iff p = 0$, and $\gamma(P) := \{\gamma p : p \in P\} = P\cap C(A)$.*

(ii) *$\gamma(\gamma p) = \gamma p$ and $p \le q \implies \gamma p \le \gamma q$.*

(iii) *$\gamma(p \wedge \gamma q) = \gamma p \wedge \gamma q$.*

(iv) *$\gamma p \perp q \iff \gamma p \perp \gamma q \iff p \perp \gamma q \implies p \perp q$.*

(v) *If $(p_i)_{i\in I}$ is a family of elements in P and the supremum $\bigvee_{i\in I} p_i$ exists in P, then $\bigvee_{i\in I} \gamma p_i$ exists in P and $\gamma(\bigvee_{i\in I} p_i) = \bigvee_{i\in I} \gamma p_i$.*

6.8. Theorem.

(i) *The center $P \cap C(A)$ of P is sup/inf-closed in P.*

(ii) *Let $(c_i)_{i \in I}$ be a family of elements in the center $P \cap C(A)$ of P. Since $P \cap C(A)$ is a complete boolean algebra, the supremum p and the infimum q of $(c_i)_{i \in I}$ exist in $P \cap C(A)$; moreover, p and q are, respectively, the supremum and the infimum of $(c_i)_{i \in I}$ in P.*

Proof. Part (i) follows from the fact that the center of any OML is sup/inf-closed in the OML. Using the central cover mapping, one proceeds as in the proof of [11: Theorem 5.2(xiii)] to prove part (ii). □

7. Equivalence of projections

The assumption that P is centrally orthocomplete is still in force. In the next definition, we denote the composition of symmetry transformations by juxtaposition.

7.1. Definition. Let $\mathcal{J}$ be the set of all mappings $J\colon A \to A$ of the form $J = J_{s_n} J_{s_{n-1}} \cdots J_{s_1}$ where $s_1, s_2, \ldots, s_n$ are symmetries in A.

Thus, $\mathcal{J}$ is the group under composition generated by the symmetry transformations on A. As a consequence of Theorem 5.3, the transformations $J \in \mathcal{J}$ have the following properties.

7.2. Theorem. *Let $J \in \mathcal{J}$, $a, b \in A$, and $e, f \in P$. Then:*

(i) *The J is an order, linear, and Jordan automorphism of A.*

(ii) *J restricted to P is an OML-automorphism of P.*

(iii) *If $ab = ba$, then $J(ab) = J(a)J(b) = J(b)J(a) \in A$.*

(iv) *If $J(a)J(b) = J(b)J(a)$, then $ab = ba \in A$.*

(v) *$(J(a))^{\mathrm{o}} = J(a^{\mathrm{o}})$.*

7.3. Definition. Let $p, q \in P$.

(i) The projections p and q are *$\mathcal{J}$-equivalent*, in symbols $p \sim q$, iff there exists $J \in \mathcal{J}$ such that $J(p) = q$. If $p \sim q$ and $\mathcal{J}$-equivalence is understood, we may simply say that p and q are *equivalent*.

(ii) The projections p and q are *related* iff there are nonzero projections $p_1 \le p$ and $q_1 \le q$ such that $p_1 \sim q_1$. If p and q are not related, we say that they are *unrelated*.

(iii) p is *invariant* iff it is unrelated to its orthocomplement $p^{\perp}$.

(iv) If there exists a projection $q_1 \le q$ such that $p \sim q_1$, we say that p is *sub-equivalent* to q, in symbols, $p \preceq q$.

We note that $\sim$ is the transitive closure of the relation of being exchangeable by a symmetry and that, as a consequence of Theorems 5.11 and 5.12, two projections p and q are equivalent iff they are projective in the OML P.

Now we investigate the extent to which the equivalence relation $\sim$ is a *Sherstnev-Kalinin* (SK-) *congruence* on the OML P [13: §7]. By definition, an SK-congruence satisfies axioms SK1–SK4 in [13: Definition 7.2, Remarks 7.3]. For these axioms we have:

• (SK1) Obviously, *if* $e \in P$ *then* $e \sim 0 \implies e = 0$.

• (SK2) Axiom SK2, complete additivity (and even finite additivity) of $\sim$, is problematic. Theorem 5.15 which assumes completeness of the OML P, is a weak substitute for axiom SK2 and Lemma 5.6 is a weak substitute for finite additivity.

• (SK3d) Axiom SK3d (divisibility) holds, in fact we have the following *complete divisibility* property [21: p. 4]: *If* $(e_i)_{i\in I}$ *is an orthogonal family in* P, $p \in P$, *and* $p \sim \bigvee_{i\in I} e_i$, *then there exists an orthogonal family* $(p_i)_{i\in I}$ *such that* $p = \bigvee_{i\in I} p_i$ *and* $p_i \sim e_i$ *for all* $i \in I$. Indeed, if $J \in \mathcal{J}$ with $p = J(\bigvee_{i\in I} e_i) = \bigvee_{i\in I} J(e_i)$, then $p_i := J(e_i) \sim e_i$ for all $i \in I$.

• (SK3e) Combining Theorems 3.3 and 5.12(i), we find that $\sim$ satisfies axiom SK3e: *If* $p, q, e, f \in P$, $p \perp q$, $e \perp f$, *and* $p \vee q = e \vee f$, *then there exist* $p_1, p_2, q_1, q_2 \in P$ *such that* $p_1 \perp p_2$, $q_1 \perp q_2$, $p_1 \perp q_1$, $p_2 \perp q_2$, $p_1 \vee p_2 = p$, $q_1 \vee q_2 = q$, $p_1 \vee q_1 \sim e$, *and* $p_2 \vee q_2 \sim f$.

• (SK4) *Non-orthogonal projections are related, in fact, they have nonzero subprojections that are exchanged by a symmetry* (Theorem 5.9(iv)).

As we shall see, in spite of the fact that $\sim$ may not qualify as an SK-congruence, it does enjoy a number of important properties.

7.4. Theorem. *Let* $0 \neq e, f \in P$ *with* $e \sim f$. *Then* e *and* f *have nonzero subprojections that are exchanged by a symmetry.*

Proof. As $e \sim f$, there are symmetries $s_1, s_2, \dots, s_n \in A$ such that

$$f = s_n s_{n-1} \cdots s_2 s_1 e s_1 s_2 \cdots s_{n-1} s_n.$$

The proof is by induction on n. For $n = 1$ the desired conclusion is obvious. Suppose that the conclusion holds for all sequences of symmetries of length $n-1$ and let $r := s_n f s_n = s_{n-1} \cdots s_2 s_1 e s_1 s_2 \cdots s_{n-1}$. By the induction hypothesis, there are nonzero subprojections $p \le e$ and $q \le r$ and a symmetry $s \in S$ such that $sps = q$. Let $k := s_n q s_n \le s_n r s_n = f$. If $p \not\perp k$, then by SK4, there are nonzero subprojections $p_1 \le p \le e$ and $k_1 \le k \le f$ that are exchanged by a symmetry, and our proof is complete. Thus, we can and do assume that $p \perp k$. Therefore, since $k = s_n s p s s_n$, it follows from Theorem 5.12(ii) that p and k are exchanged by a symmetry. □

In [13: §7], for an SK-congruence, the infimum of all the invariant elements that dominate an element is called the *hull* of that element. Thus, by the following theorem, the central cover mapping is an analogue for the equivalence relation $\sim$ of a hull mapping for an SK-congruence.

7.5. Theorem. *Let $h \in P$. Then the following conditions are mutually equivalent*

(i) *h is invariant.*

(ii) *If $p \in P$, s is a symmetry in A, and $sps \leq h$, then $p \leq h$.*

(iii) *If $p \in P$ and $p \preceq h$, then $p \leq h$.*

(iv) *If $q \in P$, then $q \wedge h = 0 \implies q \perp h$.*

(v) *h is central.*

Proof.

(i) $\implies$ (ii). Let h be invariant $p \in P$, and let $s \in A$ be a symmetry such that $h_1 := sps \leq h$. Aiming for a contradiction, we assume that p is related to $h^\perp$, i.e., there exist subprojections $0 \neq p_1 \leq p$ and $0 \neq q_1 \leq h^\perp$ with $p_1 \sim q_1$. Now $h_1 = sps = s(p_1 \vee (p \wedge p_1^\perp))s = sp_1s \vee s(p \wedge p_1^\perp)s$, whence $0 \neq h_2 := sp_1s \leq h_1 \leq h$. But then $h \geq h_2 \sim p_1 \sim q_1 \leq h^\perp$, so h is related to $h^\perp$, contradicting the invariance of h. Therefore, p is unrelated to $h^\perp$ and it follows from SK4 that $p \perp h^\perp$, i.e., $p \leq h$.

(ii) $\implies$ (iii). If $p \preceq h$, there are symmetries $s_1, s_2, \dots s_n \in A$ such that $s_n s_{n-1} \cdots s_1 p s_1 s_2 \cdots s_n \leq h$, whence (iii) follows from (ii) by induction on n.

(iii) $\implies$ (iv). Assume that (iii) holds and that $q \not\perp h$. By SK4, there exist subprojections $0 \neq q_1 \leq q$ and $0 \neq h_1 \leq h$ with $q_1 \sim h_1$. Then $q_1 \preceq h$, so $q_1 \leq h$ by (iii), and we have $0 \neq q_1 \leq q \wedge h$, whence $q \wedge h \neq 0$.

(iv) $\implies$ (v). Assume that (iv) holds and let $q \in P$. Then $\phi_{q^\perp}(h^\perp) \wedge h = (q \vee h^\perp) \wedge q^\perp \wedge h = 0$, so $\phi_{q^\perp}(h^\perp) \leq h^\perp$ by (iii). Therefore, $q^\perp$ is compatible with $h^\perp$, and it follows that q is compatible with h. Since q is an arbitrary element of P, it follows that h is in the center of P.

(iv) $\implies$ (i). Assume that $h \in P \cap C(A)$ and, aiming for a contradiction, suppose that h is related to $h^\perp$. Then there exist subprojections $0 \neq h_1 \leq h$ and $0 \neq q_1 \leq h^\perp$ such that $h_1 \sim q_1$. Thus there are symmetries $s_1, s_2, \dots, s_n \in A$ such that $q_1 = s_n s_{n-1} \cdots s_1 h_1 s_1 \cdots s_{n-1} s_n$. As $h_1 \leq h$, we have $s_1 h_1 s_1 \leq s_1 h s_1 = h s_1^2 = h$, and by induction on n, $q_1 \leq h$. Consequently, $q_1 \leq h \wedge h^\perp = 0$, contradicting $q_1 \neq 0$. □

7.6. Corollary. *If $p \in P$ and $c \in P \cap C(A)$, then p and c are unrelated iff $p \perp c$.*

Proof. If p and c are unrelated, then $p \perp c$ by SK4. Conversely, suppose $p \perp c$, $p_1 \leq p$, $c_1 \leq c$ and $p_1 \sim c_1$. Then $p_1 \preceq c$, so $p_1 \leq c$ by Theorem 7.5. But $p_1 \leq p \leq c^\perp$, so $p_1 \leq c \wedge c^\perp = 0$; hence p and c are unrelated. □

7.7. Theorem. *Suppose that P is a complete OML and $p \in P$. Then $\gamma p = \bigvee\{q \in P : q \preceq p\}$.*

Proof. Since P is complete, $h := \bigvee\{q \in P : q \preceq p\}$ exists in P. If $q \preceq p$, then since $p \leq \gamma p$, we have $q \preceq \gamma p \in P \cap C(A)$, whence $q \leq \gamma p$ by Theorem 7.5. Therefore $h \leq \gamma p$. We claim that h is a central projection. Indeed, suppose $r \in P$, $s \in A$ is a symmetry, and $srs \leq h$. By Theorem 7.5 it will be sufficient to show that $r \leq h$. But $srs \leq h$ implies that $r \leq shs = \bigvee\{sqs : q \preceq p\}$, and since $q \preceq p$ implies $sqs \preceq p$, it follows that $\bigvee\{sqs : q \preceq p\} \leq h$. Therefore, $r \leq h$, whence $h \in P \cap C(A)$. Obviously, $p \leq h$, so $\gamma p \leq \gamma h = h$, and we have $h = \gamma p$. □

7.8. Corollary. *Let P be a complete OML. Then:*

(i) *If $e, f \in P$, then $\gamma e \perp \gamma f$ iff $e \perp \gamma f$ iff e and f are unrelated.*

(ii) *P is irreducible, i.e., $P \cap C(A) = \{0, 1\}$, iff every pair of nonzero projections are related.*

Proof.

(i) If $\gamma e \perp \gamma f$, than as $e \leq \gamma e$, it follows that $e \perp \gamma f$. If $e \perp \gamma f$, then since $\gamma f \in P \cap C(A)$, Corollary 7.6 implies that e and γf are unrelated, and since $f \leq \gamma f$, it follows that e is unrelated to f. To complete the proof of (i) it will be sufficient to prove that $\gamma e \not\perp \gamma f$ implies that e is related to f. So assume that $\gamma e \not\leq (\gamma f)^\perp$. Then $e \not\leq (\gamma f)^\perp$, else $\gamma e \leq \gamma(\gamma f)^\perp = (\gamma f)^\perp$. Thus by Theorem 7.7 and De Morgan duality, $e \not\leq (\gamma f)^\perp = \bigwedge\{q^\perp : q \preceq f\}$, whence there exists a projection $q \preceq f$ such that $e \not\perp q$. Therefore, by SK4, e is related to q, and since $q \preceq f$, it follows that e is related to f.

(ii) Part (ii) follows immediately from (i). □

By [13: Definition 7.14], an SK-congruence is a *dimension equivalence relation* (DER) iff unrelated elements have orthogonal hulls. Therefore, by Corollary 7.8(i), if P is complete, then the equivalence relation $\sim$ is an analogue of a DER.

8. The case of a complete projection lattice

Some of the results above, notably Theorems 5.15 and 7.7 require the completeness of the OML P. In this section, we present some additional results involving exchangeability of projections by symmetries that also require completeness of P.

8.1. Standing Assumption. In this section, we assume that P is a complete lattice. Therefore, P is centrally orthocomplete, and the central cover mapping $\gamma\colon A \to P \cap C(A)$ exists.

8.2. Theorem. *Suppose that $p \in P$, and S is the set of all symmetries in A. Then $\gamma p = \bigvee\{sqs : \; s \in S, \; q \in P, \; \text{and} \; q \leq p\}$.*

Proof. Let $h := \bigvee\{sqs : \; q \in P \text{ and } q \leq p\}$. If $s \in S$, $q \in P$, and $q \leq p$, then $q \leq \gamma p$, whence $sqs \leq s(\gamma p)s = \gamma p$, and therefore $h \leq \gamma p$. Aiming for a contradiction, we assume that $h \neq \gamma p$, i.e., that $r := \gamma p - h = \gamma p \wedge h^{\perp} \neq 0$. Then $r \not\leq (\gamma p)^{\perp}$ so by Corollary 7.8, r and p are related. Therefore, there exist $0 \neq r_1 \leq r$ and $0 \neq p_1 \leq p$ with $r_1 \sim p_1$; hence, by Theorem 7.4 there exist $0 \neq r_2 \leq r_1 \leq r \leq h^{\perp}$ and $0 \neq p_2 \leq p_1 \leq p$ such that r_2 and p_2 are exchanged by a symmetry $s \in A$. Thus, $r_2 = sp_2s \leq h$, so $r_2 \leq h \wedge h^{\perp} = 0$, contradicting $r_2 \neq 0$. □

8.3. Lemma. *If $e, f \in P$ are orthogonal projections, then there exist projections $e_1, e_2, f_1, f_2 \in P$ such that $e_1 \perp e_2$, $f_1 \perp f_2$, $e = e_1 \vee e_2 = e_1 + e_2$, $f = f_1 \vee f_2 = f_1 + f_2$, e_1 and f_1 are exchanged by a symmetry, and e_2 is unrelated to f_2, whence $\gamma e_2 \perp \gamma f_2$.*

Proof. Let $(e_i, f_i)_{i \in I}$ be a maximal family of pairs of projections such that $(e_i)_{i \in I}$ is an orthogonal family of subprojections of e, $(f_i)_{i \in I}$ is an orthogonal family of subprojections of f, and for each $i \in I$, there is a symmetry $s_i \in A$ that exchanges e_i and f_i. We can assume that the natural numbers $1, 2, 3$ and 4 do not belong to the indexing set I.

Put $e_1 := \bigvee_{i \in I} e_i$ and $f_1 := \bigvee_{i \in I} f_i$. Then $e_1 \leq e$ and $f_1 \leq f$, so $e_1 \perp f_1$. By Theorem 5.15, e_1 and f_1 are exchanged by a symmetry. Let $e_2 := e - e_1$ and $f_2 := f - f_1$. Then $e_2 \leq e \wedge e_i^{\perp}$ and $f_2 \leq f \wedge f_i^{\perp}$ for all $i \in I$. Suppose that e_2 is related to f_2; then they have nonzero subprojections $0 \neq e_3 \leq e_2$ and $0 \neq f_3 \leq f_2$ with $e_3 \sim f_3$. Thus, by Theorem 7.4, there are nonzero subprojections $0 \neq e_4 \leq e_3 \leq e_2$ and $0 \neq f_4 \leq f_3 \leq f_2$ that are exchanged by a symmetry $s_4 \in A$. Evidently, $e_4 \leq e \wedge e_i^{\perp}$ and $f_4 \leq f \wedge f_i^{\perp}$ for all $i \in I$. But then we can enlarge the family $(e_i, f_i)_{i \in I}$ by appending the pair (e_4, f_4), contradicting maximality. Therefore e_2 is unrelated to f_2, whence $\gamma e_2 \perp \gamma f_2$ by Corollary 7.8. □

In the next theorem we improve Lemma 8.3, by removing the hypothesis that e and f are orthogonal.

8.4. Theorem. *If e and f are any two projections, then we can write orthogonal sums $e = e_1 + e_2$ and $f = f_1 + f_2$, where e_1 and f_1 are exchanged by a symmetry and $\gamma e_2 \perp \gamma f_2$.*

Proof. Write $e_{11} := \phi_e(f) = e \wedge (e_{12})^{\perp}$, where $e_{12} := e \wedge f^{\perp}$, and write $f_{11} := \phi_f(e) = f \wedge (f_{12})^{\perp}$, where $f_{12} := e^{\perp} \wedge f$. Then e_{12} commutes with both e and $(e_{12})^{\perp}$, whence $e = e_{11} \vee e_{12} = e_{11} + e_{12}$, and likewise $f = f_{11} + f_{12}$. Also by Theorem 5.9(i) e_{11} and f_{11} are exchanged by a symmetry s_1. Moreover, since

$e_{12} \perp f_{12}$, Lemma 8.3 provides orthogonal decompositions $e_{12} = e_{13} + e_2$ and $f_{12} = f_{13} + f_2$ where e_{13} and f_{13} are exchanged by a symmetry s_2 and $\gamma e_2 \perp \gamma f_2$. Thus $e = e_{11} + e_{12} = e_{11} + e_{13} + e_2$ and $f = f_{11} + f_{12} = f_{11} + f_{13} + f_2$.

Since $e_{13} \perp f_{11}$ and $e_{11} \perp f_{13}$, Lemma 5.6 provides a symmetry s exchanging $e_1 := e_{11} + e_{13}$ and $f_1 := f_{11} + f_{13}$. □

8.5. Lemma. *If e and f are orthogonal projections, then there is a central projection $h \in P$ such that eh and a subprojection of fh are exchanged by a symmetry s, and $f(1-h)$ and a subprojection of $e(1-h)$ are exchanged by the symmetry s.*

Proof. Let $e = e_1 + e_2$, $f = f_1 + f_2$ be the decompositions of Theorem 8.4 and set $h = \gamma f_2$. Then h is a central projection, $f_2 h = f_2$, $h \perp \gamma e_2$, and there is a symmetry $s \in A$ with $se_1 s = f_1$. Thus, $eh = e_1 h + e_2 h = e_1 h + e_2 \gamma e_2 h = e_1 h$, and $s(eh)s = s(e_1 h)s = f_1 h \leq fh$. Also $f(1-h) = f_1(1-h) + f_2(1-h) = f_1(1-h)$ and $sf(1-h)s = sf_1(1-h)s = e_1(1-h) \leq e(1-h)$. □

In the next theorem we improve Lemma 8.5 by removing the hypothesis that e and f are orthogonal.

8.6. Theorem (Generalized Comparability). *Given any two projections e, f there is a central projection h and a symmetry s with $s(eh)s \leq fh$ and $sf(1-h)s \leq e(1-h)$.*

Proof. By Theorem 5.9(i) the subprojections $e_1 := \phi_e(f) \leq e$ and $f_1 := \phi_f(e) \leq f$ are exchanged by a symmetry s_1. Set $e_2 = e \wedge f^\perp$ and $f_2 = e^\perp \wedge f$. Then $e_1 = e \wedge e_2^\perp$, $f_1 = f \wedge f_2^\perp$, $e_1 \perp e_2$, $f_1 \perp f_2$, $e = e_1 + e_2$, and $f = f_1 + f_2$. Since $e_2 \perp f_2$, Lemma 8.5 applies giving a symmetry s_2 and a central projection h with $f_3 := s_2(e_2 h)s_2 \leq f_2 h$ and $e_3 := s_2(f_2(1-h))s_2 \leq e_2(1-h)$. We note that $f_3(1-h) = 0$ and $e_3 h = 0$.

We claim that the projections e_1, $e_2 h$ and e_3 are pairwise orthogonal. Indeed, as $e_1 \perp e_2$, we have $e_1 \perp e_2 h$. Also, $e_1 \perp e_2(1-h)$, so $e_1 \perp e_3$. Moreover, $e_3 \leq 1-h$, so $e_2 h \perp e_3$. Thus, $e_1 + (e_2 h + e_3)$ is a projection. Similarly, $f_1 + (f_3 + f_2(1-h))$ is a projection. Since $e_1 \leq e \leq e \vee f^\perp = f_2^\perp$ and $f_3 \leq f_2$ it follows that $e_1 \perp (f_3 + (1-h)f_2)$. Similarly, $f_1 \perp (e_2 h + e_3)$, Thus, as s_1 exchanges e_1 and f_1 and s_2 exchanges $e_2 h + e_3$ and $f_3 + f_2(1-h)$, it follows from Lemma 5.6 that there is a symmetry $s \in S$ such that

$$s(e_1 + (e_2 h + e_3))s = f_1 + (f_3 + f_2(1-h)), \tag{1}$$

$$\text{whence} \quad s(f_1 + (f_3 + f_2(1-h)))s = e_1 + (e_2 h + e_3). \tag{2}$$

Multiplying both sides of (1) by h, and both sides of (2) by $1-h$, we find that $s(eh)s = (f_1 + f_3)h \leq fh$ and $s(f(1-h))s = (e_1 + e_3)(1-h) \leq e(1-h)$. □

For the case under consideration in which P is a complete OML, the generalized comparability theorem above can be used to prove that P has the relative center property. Our proof of the following theorem is suggested by the proof of [3: Proposition 1] in which $\sim$ is replaced by strong perspectivity.

8.7. Theorem. *The OML P has the relative center property.*

Proof. Let $p \in P$ and suppose that d is a central element of the p-interval $P[0,p]$. Then pAp is a synaptic algebra with unit p, $P[0,p]$ is the projection lattice of pAp, and d commutes with every element of pAp.

Applying Theorem 8.6 to the projections d and $p \wedge d^{\perp} = p - d$, we find that there is a symmetry $s \in A$ and central element $h \in P \cap C(A)$ such that $sdhs \leq (p-d)h$ and $s(p-d)(1-h)s \leq d(1-h)$. Put $q := dh \vee sdhs \in P$. Then $sqs = sdhs \vee dh = q$, so $sq = qs$. Also, $dh \leq d \leq p$, $sdhs \leq (p-d)h \leq p-d \leq p$, and we have $dh \leq q \leq p$. Let $s_0 := sq = qs$ and $t := s_0 + (p-q)$. Then $s_0^2 = q$, $t^2 = p$, and $sdhs = s_0dhs_0 = tdht$. Also, $ptp = t$, so $t \in pAp$, and it follows that $dt = td$; hence $dh = tdht = sdhs \leq (p-d)h \leq d^{\perp}$, and it follows that $dh = 0$. An analogous argument shows that $(p-d)(1-h) = 0$, and it follows that $d = d + (p-d)(1-h) = p(1-h)$. □

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