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REPRESENTATIONS OF ZERO-CANCELLATIVE POMONOIDS

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Dedicated to Ján Jakubík on the occasion of his 90th birthday in appreciation of his contributions to the fields of the theory of l-groups and their applications

(*Communicated by Anatolij Dvurečenskij*)

ABSTRACT. Several familiar results on representations of MV-algebras shape the idea that the use of solving systems of linear equations can be studied also in the setting of zero-cancellative commutative pomonoids. This paper investigates this idea and shows that for the class of linearly representable zero-cancellative commutative pomonoids the respective results apply as well.

Introduction

Di Nola's Representation Theorem describes MV-algebras as sub-algebras of algebras of functions with values into a non-standard ultrapower of the MV-algebra $[0,1]$. Following Di Nola's work (see [7, 8]), our (see [3]) recent work in the MV-algebra setting and the work of Cignoli and Mundici (see [6]) for totally ordered abelian groups reveals a simple method how to establish this (uniform) representation. This shapes the idea that our method could be used also for other structures than MV-algebras. It is the aim of this paper to examine this in the setting of zero-cancellative commutative pomonoids. In particular, we address the following questions:

(1) For which classes of zero-cancellative commutative pomonoids some kind of embedding finite parts of them into $\mathbb{N}_0$ will work?

(2) Having the Embedding Lemma at hand, is it possible to establish some (uniform) representation via $\mathbb{N}_0$ ($\mathbb{Q}_0^+$, $\mathbb{R}_0^+$)?

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The main idea to answer (1) will be to take the class of linearly representable zero-cancellative commutative pomonoids. Then, we are also able to answer positively (2) for this class. In particular, we obtain that each linearly representable zero-cancellative commutative pomonoid is a cancellative one.

1. Preliminaries

1.1. Zero-cancellative pomonoids

For basic facts about universal algebra we refer to [4], about model theory to [5] and about linear programming to [15]. For definitions concerning operators on Hilbert spaces and related comments see [1] and [12]. Below we recall some details of specific relevance to this paper.

Definition 1.1.

(1) A *commutative pomonoid* is a structure $\mathbf{S} = (S; +, 0; \leq)$, whose reduct $(S; +, 0)$ is a commutative monoid where $\leq$ is a partial order of S for which $+$ is isotone in both of its arguments.

(2) A *zero-cancellative pomonoid* (shortly ZCP) is a commutative pomonoid $\mathbf{S} = (S; +, 0; \leq)$ such that
 (i) $(\forall x \in S)(0 \leq x)$,
 (ii) $(\forall x, y \in S)\big[(x \leq y) \iff (\exists z \in S)(x + z = y)\big]$,
 (iii) $(\forall x, y \in S)\,(x + y = y \implies x = 0)$.

(3) A ZCP $\mathbf{S} = (S; +, 0; \leq)$ is said to be *linearly representable* if there are linearly ordered ZCPs $\mathbf{S}_i$, $i \in I$ such that $\mathbf{S}$ is a subdirect product of $\mathbf{S}_i$, $i \in I$.

(4) A ZCP $\mathbf{S} = (S; +, 0; \leq)$ is said to be *cancellative* if
 (iv) $(\forall x, y, z \in S)\,(x + y = z + y \implies x = z)$.

We denote by $\mathcal{CP}$ the class of all commutative pomonoids, by $\mathcal{ZCP}$ the class of all ZCPs, by $\mathcal{CZCP}$ the class of all cancellative ZCPs, by $\mathcal{LOZCP}$ the class of all linearly ordered ZCPs and by $\mathcal{LRZCP}$ the class of all linearly representable ZCPs. Clearly, $\mathcal{LOZCP} \subseteq \mathcal{LRZCP} \subseteq \mathcal{ZCP} \subseteq \mathcal{CP}$ and $\mathcal{CZCP} \subseteq \mathcal{ZCP} \subseteq \mathcal{CP}$.

Example 1.

(1) Any cancellative ZCP is the positive cone G^+ of any partially ordered commutative group G (see [10: Theorem II.4, Prop. X.1]).

(2) In particular, (1) yields that $\mathbb{N}_0$ ($\mathbb{Q}_0^+$, $\mathbb{R}_0^+$) equipped with the usual addition and order are cancellative ZCPs.

(3) Let $\mathcal{H}$ be a complex Hilbert space and let $D \subseteq \mathcal{H}$ be a linear subspace dense in $\mathcal{H}$ (i.e. $\overline{D} = \mathcal{H}$). Let

$$\mathrm{Lin}_D(\mathcal{H}) = \{A \colon D \to \mathcal{H} \mid A \text{ is a linear operator defined on } D\}.$$

Then $(\mathrm{Lin}_D(\mathcal{H}); +, 0; \leq)$ is a partially ordered commutative group (see [12]) where 0 is the null operator, $+$ is the usual sum of operators defined on D and $\leq$ is defined for all $A, B \in \mathrm{Lin}_D(\mathcal{H})$ by $A \leq B$ iff $B - A$ is positive.

Let

$$\mathcal{G}_D(\mathcal{H}) = \{A\colon D \to \mathcal{H} \mid A \text{ is a positive linear operator defined on } D\}.$$

Then (see [12, 14]) $(\mathcal{G}_D(\mathcal{H}); \oplus, 0; \leq)$ (also called a *generalized effect algebra of positive operators*) is a CZCP where 0 is the null operator, $\oplus$ is the usual sum of operators defined on D. Moreover, $\mathcal{G}_D(\mathcal{H})$ is linearly representable by [13]. Note that respective results hold also for the bounded variant.

One of the convincing arguments for treating (cancellative) ZCPs is that $\mathcal{G}_D(\mathcal{H})$, which has an application in quantum logic, forms a cancellative ZCP.

1.2. Generalized finite embedding theorem

By an *ultrafilter* on a set I we mean an ultrafilter of the Boolean algebra $\mathcal{P}(I)$ of the subsets of I.

Let $\{\mathbf{A}_i \mid i \in I\}$ be a system of algebras of the same type F for $i \in I$. We denote for any $x, y \in \prod\limits_{i\in I} A_i$ the set

$$[\![x = y]\!] = \big\{j \in I : \ x(j) = y(j)\big\}.$$

If F is a filter of $\mathcal{P}(I)$ then the relation θ_F defined by

$$\theta_F = \Big\{\langle x, y\rangle \in \big(\prod_{i\in I} A_i\big)^2 : \ [\![x = y]\!] \in F\Big\}$$

is a congruence on $\prod\limits_{i\in I} \mathbf{A}_i$. For an ultrafilter U of $\mathcal{P}(I)$, an algebra $\big(\prod\limits_{i\in I} \mathbf{A}_i\big)/U := \big(\prod\limits_{i\in I} \mathbf{A}_i\big)/\theta_U$ is said to be an *ultraproduct* of algebras $\{\mathbf{A}_i \mid i \in I\}$. Any ultraproduct of an algebra $\mathbf{A}$ is called an ultrapower of $\mathbf{A}$. The class of all ultraproducts (products, isomorphic images) of algebras from some class of algebras $\mathcal{K}$ is denoted by $\mathrm{P_U}(\mathcal{K})$ ($\mathrm{P}(\mathcal{K})$, $\mathrm{I}(\mathcal{K})$).

Definition 1.2. Let $\mathbf{A} = (A, \mathrm{F})$ be a partial algebra and $X \subseteq A$. Denote the partial algebra $\mathbf{A}|_X = (X, \mathrm{F})$, where for any $f \in F_n$ and all $x_1, \dots, x_n \in X$, $f^{\mathbf{A}|_X}(x_1, \dots, x_n)$ is defined if and only if $f^{\mathbf{A}}(x_1, \dots, x_n) \in X$ holds. Moreover, then we put

$$f^{\mathbf{A}|_X}(x_1, \dots, x_n) := f^{\mathbf{A}}(x_1, \dots, x_n).$$

Definition 1.3. An algebra $\mathbf{A} = (A, \mathrm{F})$ satisfies *the general finite embedding property for the class* $\mathcal{K}$ or shortly *GFEP* (*countable finite embedding property*, shortly *CFEP*) of algebras of the same type if for any finite subset $X \subseteq A$ there are an (countable) algebra $\mathbf{B} \in \mathcal{K}_{Fin}$ and an embedding $\rho\colon \mathbf{A}|_X \to \mathbf{B}$, i.e. an injective mapping $\rho\colon X \to B$ satisfying the property $\rho(f^{\mathbf{A}|_X}(x_1, \dots, x_n)) = f^{\mathbf{B}}(\rho(x_1), \dots, \rho(x_n))$ if $x_1, \dots, x_n \in X$, $f \in \mathrm{F}_n$ and $f^{\mathbf{A}|_X}(x_1, \dots, x_n)$ is defined.

Theorem 1.1. ([2]) *Let* $\mathbf{A} = (A, \mathrm{F})$ *be an algebra and let* $\mathcal{K}$ *be a class of algebras of same type. If* $\mathbf{A}$ *satisfies the general finite embedding property for* $\mathcal{K}$ *then* $\mathbf{A} \in \mathrm{ISP_U}(\mathcal{K})$.

Theorem 1.2. ([2]) *Let* $\mathbf{A} = (A, \mathrm{F})$ *be an algebra such that* F *is finite and let* $\mathcal{K}$ *be a class of an algebras of the same type. If* $\mathbf{A} \in \mathrm{ISP_U}(\mathcal{K})$ *then* $\mathbf{A}$ *satisfies the general finite embedding property for* $\mathcal{K}$.

Remark 1. Let $\{\mathbf{S}_i \mid i \in I\}$ be a collection of commutative pomonoids (ZCPs, CZCPs, LOZCPs), and let U be an ultrafilter on I. The ultraproduct $\big(\prod_{i\in I} \mathbf{S}_i\big)/U$ is again a commutative pomonoid (ZCP, CZCP, LOZCP).

1.3. Farkas' lemma

Let us recall the original formulation of Farkas' lemma [9,15] on rationals:

Theorem 1.3 (Farkas' lemma). *Given a matrix* A *in* $\mathbb{Q}^{m\times n}$ *and* $\boldsymbol{c}$ *a column vector in* $\mathbb{Q}^m$*, then there exists a column vector* $\boldsymbol{x} \in \mathbb{Q}^n$*,* $\boldsymbol{x} \geq \mathbf{0}_n$ *and* $A \cdot \boldsymbol{x} = \boldsymbol{c}$ *if and only if, for all row vectors* $\boldsymbol{y} \in \mathbb{Q}^m$*,* $\boldsymbol{y} \cdot A \geq \mathbf{0}_m$ *implies* $\boldsymbol{y} \cdot \boldsymbol{c} \geq 0$.

In what follows, we will use the following equivalent formulation:

Theorem 1.4 (Theorem of alternatives). *Let* A *be a matrix in* $\mathbb{Q}^{m\times n}$ *and* $\boldsymbol{b}$ *a column vector in* $\mathbb{Q}^n$*. The system* $A \cdot \boldsymbol{x} \leq \boldsymbol{b}$ *has no solution if and only if there exists a row vector* $\boldsymbol{\lambda} \in \mathbb{Q}^m$ *such that* $\boldsymbol{\lambda} \geq \mathbf{0}_m$*,* $\boldsymbol{\lambda} \cdot A = \mathbf{0}_n$ *and* $\boldsymbol{\lambda} \cdot \boldsymbol{b} < 0$.

Remark 2. Since the row vector $\boldsymbol{\lambda} \in \mathbb{Q}^m$ from Theorem 1.4 has non-negative rational components $\lambda_i = \frac{p_i}{q_i}$, $p_i \in \mathbb{N}_0$, $q_i \in \mathbb{N}$ we may assume (by taking the least common multiple q of denominators q_i and multiplying by it the respective conditions for $\boldsymbol{\lambda}$) that $\boldsymbol{\lambda} \in \mathbb{N}_0^m$.

2. The Embedding Lemma

In this section, we use the Farkas' lemma on rationals to prove that any finite partial subalgebra of a linearly ordered ZCP can be embedded into $\mathbb{N}_0$ and hence into the nonnegative rationals $\mathbb{Q}_0^+$ (reals $\mathbb{R}_0^+$).

Lemma 2.1. *Let* $\mathbf{S} = (S; +, 0)$ *be a ZCP,* $X \subseteq S \setminus \{0\}$ *be a finite subset. Then there is a map* $s\colon X \cup \{0\} \longrightarrow \mathbb{N}_0$ *such that*

(1) $s(0) = 0$,

(2) *if* $x, y, x + y \in X \cup \{0\}$ *then* $s(x + y) = s(x) + s(y)$.

(3) *if* $x \in X$ *then* $s(x) > 0$.

Proof. We may assume that $X = \{x_1, \dots, x_n\}$. Let $\boldsymbol{x}$ be the column vector $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ in S^n. We say that a row vector $\mathbf{a} \in \mathbb{N}_0^n$ is *admissible for* X iff any sub-term of the expression $\sum_{i=1}^{n} a_j x_j = (a_1, \dots, a_n) \cdot (x_1, \dots, x_n)^T = \mathbf{a} \cdot (x_1, \dots, x_n)^T$ is in $X \cup \{0\}$. Clearly, the set of all admissible vectors for X is finite. Let
$\mathrm{Adm}(X) := \big\{(\mathbf{a}^1, \mathbf{a}^2) \in \mathbb{N}_0^n \times \mathbb{N}_0^n \mid \mathbf{a}^1, \mathbf{a}^2 \text{ are admissible vectors, } \mathbf{a}^1 \cdot \boldsymbol{x} = \mathbf{a}^2 \cdot \boldsymbol{x}\big\}$.

Thus $\mathrm{Adm}(X)$ is finite and we may write $\mathrm{Adm}(X) = \{(\mathbf{a}_i^1, \mathbf{a}_i^2) \mid i = 1, \dots, m\} \subseteq \mathbb{N}_0^n \times \mathbb{N}_0^n$. Let A^1 be a matrix consisting of rows $\mathbf{a}_i^1$, $i = 1, \dots, m$ and let A^2 be a matrix consisting of rows $\mathbf{a}_i^2$, $i = 1, \dots, m$. Then we have that

$$A^1 \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = A^2 \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Let E_n be the identity matrix of order n and let $\mathbf{0}_{n\times n}$ be the zero matrix of order n. Let us denote by $(*)$ the following system of linear inequalities with variables $z_1, \dots, z_n$ over rationals:

$$\begin{pmatrix} -E_n \\ A^1 - A^2 \\ A^2 - A^1 \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \le \begin{pmatrix} -\mathbf{1}_n \\ \mathbf{0}_m \\ \mathbf{0}_m \end{pmatrix}. \tag{$*$}$$

Then by Farkas' lemma (see Theorem 1.4) for rationals the systems of inequalities $(*)$ does not have a solution in $\mathbb{Q}^n$ if and only if there is a row vector $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_{n+2m}) \in \mathbb{Z}^{n+2m}$, $\boldsymbol{\lambda} \ge \mathbf{0}_{n+2m}$ such that

$$\boldsymbol{\lambda} \cdot \begin{pmatrix} -E_n \\ A^1 - A^2 \\ A^2 - A^1 \end{pmatrix} = 0, \qquad \boldsymbol{\lambda} \cdot \begin{pmatrix} -\mathbf{1}_n \\ \mathbf{0}_m \\ \mathbf{0}_m \end{pmatrix} < 0$$

or equivalently

$$\boldsymbol{\lambda} \cdot \begin{pmatrix} \mathbf{0}_{n\times n} \\ A^1 \\ A^2 \end{pmatrix} = \boldsymbol{\lambda} \cdot \begin{pmatrix} E_n \\ A^2 \\ A^1 \end{pmatrix}, \qquad \boldsymbol{\lambda} \cdot \begin{pmatrix} -\mathbf{1}_n \\ \mathbf{0}_m \\ \mathbf{0}_m \end{pmatrix} < 0. \tag{$**$}$$

Assume that the vector $\boldsymbol{\lambda} \in \mathbb{Z}^{n+2m}$ satisfying $(**)$ exists. Hence there is an index $1 \le j_0 \le n$ such $\lambda_{j_0} \ge 1$.

We then have

$$\boldsymbol{\lambda} \cdot \begin{pmatrix} \mathbf{0}_{n\times n} \\ A^1 \\ A^2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \boldsymbol{\lambda} \cdot \begin{pmatrix} E_n \\ A^2 \\ A^1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

or equivalently

$$\boldsymbol{\lambda} \cdot \begin{pmatrix} \mathbf{0}_n \\ A^1 \cdot \boldsymbol{x} \\ A^2 \cdot \boldsymbol{x} \end{pmatrix} = \boldsymbol{\lambda} \cdot \begin{pmatrix} \boldsymbol{x} \\ A^2 \cdot \boldsymbol{x} \\ A^1 \cdot \boldsymbol{x} \end{pmatrix}. \tag{$***$}$$

Because $A^1 \cdot \boldsymbol{x} = A^2 \cdot \boldsymbol{x}$ we obtain

$$\boldsymbol{\lambda} \cdot \begin{pmatrix} \mathbf{0}_n \\ A^1 \cdot \boldsymbol{x} \\ A^1 \cdot \boldsymbol{x} \end{pmatrix} = \boldsymbol{\lambda} \cdot \begin{pmatrix} \boldsymbol{x} \\ A^1 \cdot \boldsymbol{x} \\ A^1 \cdot \boldsymbol{x} \end{pmatrix} \tag{${}^{**}_{**}$}$$

Let us put

$$y = (\lambda_{n+1}, \cdots, \lambda_{n+2m}) \cdot \begin{pmatrix} A^1 \cdot \boldsymbol{x} \\ A^1 \cdot \boldsymbol{x} \end{pmatrix}.$$

Then we obtain from $\left(\begin{smallmatrix} * * \\ * * \end{smallmatrix}\right)$ that $y = y + \sum_{i=1}^{n} \lambda_j x_j$. Since $\mathbf{S}$ is zero-cancellative, we get that $0 < x_{j_0} \leq \sum_{i=1}^{n} \lambda_j x_j = 0$, a contradiction.

It follows that the system $(*)$ has a rational valued solution $(r_1, \ldots, r_n)$ and from $(*)$ it clearly follows that the solution is positive (more precisely $\mathbf{r} = (r_1, \ldots, r_n) \geq \mathbf{1}_n$). By the same arguments as in Remark 2, we get that system $(*)$ has a solution $(q_1, \ldots, q_n) \in \mathbb{N}^n$. We define the mapping $s \colon X \cup \{0\} \longrightarrow \mathbb{N}_0$ by the following prescription:

$$s(x) = \begin{cases} q_j & \text{if } x = x_j, \\ 0 & \text{if } x = 0. \end{cases}$$

The mapping s evidently satisfies the conditions (1)–(3) of this Lemma. □

Lemma 2.2 (**Embedding Lemma**). *Let $\mathbf{S} = (S; +, \leq, 0)$ be a linearly ordered ZCP and let $X \subseteq S$ be a finite set. Then there exists an embedding $f \colon \mathbf{X} \hookrightarrow \mathbb{N}_0$, where $\mathbf{X}$ is a partial ZCP obtained by the restriction of $\mathbf{S}$ to the set X.*

P r o o f. Let us define a set Y as follows:

$$Y := \big\{ z \in S \mid \big(\exists x, y \in X \cup \{0\}\big)\big[(y \leq x) \ \& \ (x + z = y)\big]\big\} \setminus \{0\} \subseteq S.$$

Evidently, $X \subseteq Y$. Moreover, let $s \colon Y \cup \{0\} \longrightarrow \mathbb{N}_0$ be the respective mapping for the set Y from Lemma 2.1.

Let $f = s|_X$ be the restriction of the mapping s on the set X. Clearly, f is an injective morphism of partial ZCPs. Namely, let $f(x) = f(y)$ for some $x, y \in X$. Assume first that $x \leq y$. Then there is $z \in Y \cup \{0\}$ such that $x + z = y$. It follows that $s(x) + s(z) = s(y)$, i.e., $s(z) = 0$. Therefore $z = 0$ which yields $x = y$. The case $y \leq x$ goes the same way.

A similar argument shows that f preserves order and partial operation $+$. □

3. Variants of Di Nola's Theorem for LRZCPs

In this section, we are going to show some variants Di Nola's representation Theorem for linearly representable ZCPs and its several variants not only via the cancellative ZCP of natural numbers but also via nonnegative rationals (reals).

To prove it, we use the Embedding Lemma obtained in the previous section. First, we establish the CFEP both for linearly ordered and linearly representable ZCPs.

Theorem 3.1.

(1) *The class $\mathcal{LOZCP}$ of linearly ordered ZCPs has the CFEP.*

(2) *The class $\mathcal{LRZCP}$ of linearly representable ZCPs has the CFEP.*

P r o o f.

(1) It follows immediately from Lemma 2.2.

(2) Let $\mathbf{S} = (S; +, \leq, 0)$ be a linearly representable ZCP which is represented by linearly ordered ZCPs $\mathbf{S}_i$, $i \in I$ via an embedding $g \colon S \to \prod_{i \in I} S_i$ and let $X \subseteq S$ be a finite subset. For any $x, y \in X$; $x \neq y$ there is an index $i \in I$ such that $\pi_i(g(x)) \neq \pi_i(g(y))$. Hence there is a finite system of indices $i_1, \dots, i_l$ such that it separates elements from X, i.e., $\mathbf{X} \hookrightarrow \prod_{j=1}^{l} \mathbf{S}_{i_j}$ is an injective mapping.

For any $j \in \{1, \dots, l\}$ there is by Lemma 2.2 an embedding $f_j : \pi_{i_j}(g(X)) \hookrightarrow \mathbb{N}_0$ of partial ZCPs. Consequently there is an embedding $f \colon \mathbf{X} \hookrightarrow \mathbb{N}_0^l$ of partial ZCPs defined by $f(x)(j) = f_j(\pi_{i_j}(g(x)))$. □

Theorem 3.2.

(1) *Any linearly ordered ZCP can be embedded into an ultrapower of* $\mathbb{N}_0$.

(2) *Any linearly representable ZCP can be embedded into a product of ultrapowers of* $\mathbb{N}_0$.

(3) *Any linearly representable ZCP can be embedded into an ultraproduct of finite powers of* $\mathbb{N}_0$.

(4) *Any linearly representable ZCP can be embedded into an ultrapower of the countable power of* $\mathbb{N}_0$.

P r o o f.

(1) It is a direct corollary of Theorem 3.1(1) and Theorem 1.1.

(2) Any linearly representable ZCP is embeddable into a product of linearly ordered ones. The rest follows by (1).

(3) It is a direct corollary of Theorem 3.1(2) and Theorem (1).

(4) It follows from (3). □

Corollary 3.2.1. *Any linearly representable ZCP is cancellative.*

P r o o f. It immediately follows from the fact that $\mathbb{N}_0$ is cancellative and products and ultraproducts of cancellative ZCPs are again cancellative. □

Theorem 3.3.

(1) *Any linearly ordered ZCP can be embedded into an ultrapower of* $\mathbb{Q}_0^+$.

(2) *Any linearly representable ZCP can be embedded into a product of ultrapowers of* $\mathbb{Q}_0^+$.

(3) *Any linearly representable ZCP can be embedded into an ultraproduct of finite powers of* $\mathbb{Q}_0^+$.

(4) *Any linearly representable ZCP can be embedded into an ultrapower of the countable power of* $\mathbb{Q}_0^+$.

P r o o f. (1)–(4) It is a corollary of Theorem 3.2. □

Theorem 3.4.

(1) *Any linearly ordered ZCP can be embedded into an ultrapower of* $\mathbb{R}_0^+$.
(2) *Any linearly representable ZCP can be embedded into a product of ultrapowers of* $\mathbb{R}_0^+$.
(3) *Any linearly representable ZCP can be embedded into an ultraproduct of finite powers of* $\mathbb{R}_0^+$.
(4) *Any linearly representable ZCP can be embedded into an ultrapower of the countable power of* $\mathbb{R}_0^+$.

P r o o f. (1)–(4) It is a corollary of Theorem 3.2. □

Remark 3. It follows that, for any complex Hilbert space $\mathcal{H}$ and any dense linear subspace $D \subseteq \mathcal{H}$, the generalized effect algebra of positive operators $\mathcal{G}_D(\mathcal{H})$ can be embedded as a ZCP into

(1) a product of ultrapowers of $\mathbb{N}_0$ ($\mathbb{Q}_0^+$, $\mathbb{R}_0^+$),
(2) an ultrapower of the countable power of $\mathbb{N}_0$ ($\mathbb{Q}_0^+$, $\mathbb{R}_0^+$).

The same result applies to the generalized effect algebra of positive bounded operators.

4. Uniform Representation of linearly representable ZCPs by regular ultrapowers

This paragraph starts with a general finite α-embedding theorem which is necessary for proving the uniform variants of Di Nola's theorem for linearly representable ZCPs. At first we recall some definitions and then Theorem 4.1 which is taken from [3].

Definition 4.1. ([5]) Let α be a cardinal. A proper filter D over I is said to be *α-regular* if there exists a set $E \subseteq D$ such that $|E| = \alpha$ and each $i \in I$ belongs to only finitely many $e \in E$.

Definition 4.2. Let α be a cardinal, $\mathbf{A} = (A, \mathrm{F})$ an algebra such that $|A| \leq \alpha$. Let $i_A\colon A \to \alpha$ be the respective injective mapping. $\mathbf{A}$ satisfies the *general finite α-embedding* (*countable finite α-embedding property*) *property for the class $\mathcal{K}$* of algebras of the same type if for any finite subset $X \subseteq \alpha$ there are an (countable) algebra $\mathbf{B} \in \mathcal{K}_{Fin}$ and an embedding $\rho\colon \mathbf{A}|_{i_A^{-1}(X)} \to \mathbf{B}$, i.e. an injective mapping $\rho\colon i_A^{-1}(X) \to B$ satisfying the property $\rho(f^{\mathbf{A}|_{i_A^{-1}(X)}}(x_1, \ldots, x_n)) = f^{\mathbf{B}}(\rho(x_1), \ldots, \rho(x_n))$ if $i_A(x_1), \ldots, i_A(x_n) \in X$, $f \in \mathrm{F}_n$ and $f^{\mathbf{A}|_{i_A^{-1}(X)}}(x_1, \ldots, x_n)$ is defined.

Theorem 4.1 (**General finite α-embedding theorem**). *Let α be a cardinal and let $\mathbf{A} = (A, \mathrm{F})$ be an algebra such that $|A| \leq \alpha$. Let $i_A \colon A \to \alpha$ be the respective injective mapping. Let $\mathcal{K}$ be a class of algebras of same type. If $\mathbf{A}$ satisfies the general finite α-embedding property for $\mathcal{K}$ then there is an α-regular ultrafilter over the set $I = \{X \mid X \subseteq \alpha$ and X is finite$\}$ which does not depend on A and algebras $\mathbf{A}_X \in \mathcal{K}$, $X \subseteq \alpha$, X is finite such that A can be embedded into $\big(\prod_{X\in I} \mathbf{A}_X\big)/U$.*

In what follows we present a uniform version of Di Nola's Theorem for linearly representable ZCPs. This enables us to embed all linearly representable ZCPs of a cardinality at most α in an algebra of functions from the set $\binom{\alpha}{2}$ of 2-element subsets of α into a single non-standard ultrapower of the cancellative commutative monoid $\mathbb{N}_0$. Our second goal is to embed all linearly representable ZCPs of a cardinality at most α into a single non-standard ultrapower of the cancellative ZCP $\mathbb{N}_0^{\mathbb{N}}$.

Theorem 4.2. *Let α be a cardinal and let $\mathbf{S} = (S; +, \leq, 0)$ be a linearly ordered ZCP such that $|S| \leq \alpha$, U be the α-regular ultrafilter on the set $I = \{X \mid X \subseteq \alpha$ and X is finite$\}$ from* Theorem 4.1 *which does not depend on* $\mathbf{S}$. *Then*

(1) $\mathbf{S}$ *can be embedded into an ultraproduct of $\mathbb{N}_0$ via the α-regular ultrafilter U.*

(2) $\mathbf{S}$ *can be embedded into the ultrapower $\big(\prod_{X\in I} \mathbb{Q}_0^+\big)/U$.*

(3) $\mathbf{S}$ *can be embedded into the ultrapower $\big(\prod_{X\in I} \mathbb{R}_0^+\big)/U$.*

Proof. (1)–(3) It is a direct corollary of Theorem 3.1(1) and Theorem 4.1. □

Theorem 4.3. *Let α be a cardinal and let $\mathbf{S} = (S; +, \leq, 0)$ be a linearly representable ZCP such that $|S| \leq \alpha$, U be the α-regular ultrafilter on the set $I = \{X \mid X \subseteq \alpha$ and X is finite$\}$ from* Theorem 4.1 *which does not depend on* $\mathbf{S}$. *Then*

(1) $\mathbf{S}$ *can be embedded into a linearly representable ZCP of functions from $\binom{\alpha}{2}$ to the linearly ordered ultrapower $\big(\prod_{X\in I} \mathbb{N}_0\big)/U$.*

(2) $\mathbf{S}$ *can be embedded into a linearly representable ZCP of functions from $\binom{\alpha}{2}$ to the linearly ordered ultrapower $\big(\prod_{X\in I} \mathbb{Q}_0^+\big)/U$.*

(3) $\mathbf{S}$ *can be embedded into a linearly representable ZCP of functions from $\binom{\alpha}{2}$ to the linearly ordered ultrapower $\big(\prod_{X\in I} \mathbb{R}_0^+\big)/U$.*

Proof.

(1) Let $i_S \colon S \to \alpha$ be an injective mapping. Let $\binom{S}{2}$ be the set of all 2-element subsets of S which is evidently non-empty and let $F_0 \in \binom{S}{2}$. Similarly, let $\binom{\alpha}{2}$

be the set of all 2-element subsets of α. Since $\mathbf{S}$ is linearly representable, there are linearly ordered ZCPs $\mathbf{S}_i$, $i \in I$ and an embedding $f\colon \mathbf{S} \to \prod_{i\in I} S_i$. For all $F \in \binom{S}{2}$ there is an index $i_F \in I$ such that $\pi_{i_F} \circ f$ separates elements of F. Let us put $\widehat{S}_{i_F} = (\pi_{i_F} \circ f)(S)$. Then $|\widehat{S}_{i_F}| \leq \alpha$ and $\widehat{\mathbf{S}}_{i_F}$ is a linearly ordered ZCP. It follows that we have an embeding $h\colon \mathbf{S} \to \prod_{F\in\binom{S}{2}} \mathbf{S}_{i_F}$. Moreover, we have an injective mapping $e_S\colon \binom{S}{2} \to \binom{\alpha}{2}$ given by $F \mapsto \{i_S(x) \mid x \in F\} \in \binom{\alpha}{2}$. For any $F \in \binom{S}{2}$ we have from Theorem 4.2 an embedding

$$g_F\colon \mathbf{S}_{i_F} \hookrightarrow \Big(\prod_{X\in I} \mathbb{N}_0\Big)/U.$$

This yields an embedding

$$g\colon \prod_{F\in\binom{S}{2}} \mathbf{S}_{i_F} \hookrightarrow \bigg(\Big(\prod_{X\in I} \mathbb{N}_0\Big)/U\bigg)^{\binom{\alpha}{2}}$$

given as follows:

$$g((x_F)_{F\in\binom{S}{2}})(B) = \begin{cases} g_F(x_F) & \text{if } e_M(F) = B \\ g_{F_0}(x_{F_0}) & \text{otherwise.} \end{cases}$$

The composition of $g \circ h$ gives us the required embedding.

(2), (3) It follows by the same considerations as in (1). □

Going the other way around, we have

Theorem 4.4. *Let α be a cardinal and let $\mathbf{S} = (S; +, \leq, 0)$ be a linearly representable ZCP such that $|S| \leq \alpha$, U be the α-regular ultrafilter on the set $I = \{X \mid X \subseteq \alpha$ and X is finite$\}$ from* Theorem 4.1 *which does not depend on $\mathbf{S}$. Then*

(1) $\mathbf{S}$ *can be embedded into the ultrapower* $\Big(\prod_{X\in I} (\mathbb{N}_0)^{\mathbb{N}}\Big)/U$.

(2) $\mathbf{S}$ *can be embedded into the ultrapower* $\Big(\prod_{X\in I} (\mathbb{Q}_0^+)^{\mathbb{N}}\Big)/U$.

(3) $\mathbf{S}$ *can be embedded into the ultrapower* $\Big(\prod_{X\in I} (\mathbb{R}_0^+)^{\mathbb{N}}\Big)/U$.

Proof.

(1) Let $i_S\colon S \to \alpha$ be an injective mapping and let $X \subseteq S$ be a finite subset. Using the same notation and reasonings as in the proof of Theorem 3.1(2) we have an embedding

$$f\colon \mathbf{X} \hookrightarrow \mathbb{N}_0^l.$$

Moreover, we have also an embedding

$$g\colon \mathbb{N}_0^l \hookrightarrow (\mathbb{N}_0)^{\mathbb{N}}$$

given by:

$$g((x_k)_{k=1}^l)(n) = \begin{cases} x_k & \text{if } k = n \\ x_1 & \text{otherwise.} \end{cases}$$

The composition $\rho_X = g \circ f$ yields an embedding

$$\rho_X : \mathbf{X} \hookrightarrow (\mathbb{N}_0)^{\mathbb{N}}.$$

The remaining part now follows from Theorem 4.1.

(2), (3) It follows by the same considerations as in (1). □

Remark 4. Note first that, for a given infinite cardinal α, Keisler (see [11]) has shown (in a more general form) that if U is regular and κ is an infinite cardinal such that I is taken as in Theorem 4.3, then $\mathrm{card}\big(\prod_{X\in I} \kappa\big)/U) = \mathrm{card}(\kappa^\alpha)$.

This yields that

(1) $\mathrm{card}\Big(\big((\prod_{X\in I} \mathbb{N}_0)/U\big)^{\binom{\alpha}{2}}\Big) = \aleph_0^\alpha = 2^\alpha$ and

(2) $\mathrm{card}\Big(\big(\prod_{X\in I} (\mathbb{N}_0)^{\mathbb{N}}\big)/U\Big) = 2^\alpha$.

It follows that, for a given infinite cardinal α, there is a single linearly representable (and hence cancellative) ZCP of the cardinality 2^α where every linearly representable ZCP of cardinality at most α embeds.

Second, by the same arguments as in [8: Section 4], for every infinite cardinal α there is an iterated ultrapower (see [5: Section 6.5]) $\prod_\alpha$ of $(\mathbb{N}_0)^{\mathbb{N}}$, definable in α, where every linearly representable ZCP of cardinality at most α embeds.

5. Conclusion

Our questions (1) and (2) from Introduction were answered in positive for the class of linearly representable ZCPs. We proved also that any linearly representable ZCP is cancellative. Hence there arise the following open problems:

Problem 5.1. Is every cancellative ZCP linearly representable?

Problem 5.2. Is every ZCP cancellative?

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REFERENCES

[1] BLANK, J.—EXNER, P.—HAVLÍČEK, M.: *Hilbert Space Operators in Quantum Physics* (2nd ed.), Springer, Berlin, 2008.

[2] BOTUR, M.: *A non-associative generalization of Hájek's BL-algebras*, Fuzzy Sets and Systems **178** (2011), 24–37.

[3] BOTUR, M.—PASEKA, J.: *On the extensions of Di Nola's Theorem*, Algebra Universalis (To appear).
[4] BURRIS, S.—SANKAPPANAVAR, H. P.: *A course in Universal Algebra*, Springer-Verlag, New York, 1981.
[5] CHANG, C. C.—KEISLER, H. J.: *Model theory*, Elsevier, Amsterdam, 1973.
[6] CIGNOLI, R.—MUNDICI, D.: *On partial isomorphisms on totally ordered abelian groups and Hájek's completeness theorem for basic logic*, Mult.-Valued Logic **6** (2001), 89–94 (Special issue dedicated to the memory of Grigore Moisil).
[7] DI NOLA, A.: *Representation and reticulation by quotients of MV-algebras*, Ric. Mat. **XL** (1991), 291–297.
[8] DI NOLA, A.—LENZI, G.—SPADA, L.: *Representation of MV-algebras by regular ultrapowers of* $[0,1]$, Arch. Math. Logic **49** (2010), 491–500.
[9] FARKAS, G.: *Über die Theorie der einfachen Ungleichungen*, J. Reine Angew. Math. **124** (1902), 1–27.
[10] FUCHS, L.: *Partially Ordered Algebraic Systems*, Pergamon Press, Oxford, 1963.
[11] KEISLER, H. J.: *A Survey of Ultraproducts.* Proc. Internat. Congr. Logic, Methodology and Philosophy of Science, North-Holland, Amsterdam, 1965, pp. 112–126.
[12] PASEKA, J.: *PT-Symmetry in (generalized) effect algebras*, Internat. J. Theoret. Phys. **50** (2011), 1198–1205.
[13] PASEKA, J.: *On realization of generalized effect algebras*, Rep. Math. Phys. **70** (2012), 375–384.
[14] POLAKOVIČ, M.—RIEČANOVÁ, Z.: *Generalized effect algebras of positive operators densely defined on Hilbert spaces*, Internat. J. Theoret. Phys. **50** (2011), 1167–1174.
[15] SCHRIJVER, A.: *Theory of Linear and Integer Programming.* Wiley-Intersci. Ser. Discrete Math. Optim., John Wiley & Sons, Hoboken, NJ, 1998.

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