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BASES OF MEASURABILITY IN BOOLEAN ALGEBRAS

Miroslav Repický

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ABSTRACT. In the paper we find characterizations of some notions studied by Kułaga [5] and we generalize his results. In particular, we characterize regularity and completeness of factor subalgebras via stability of the decidability operator and we discuss some possibilities in defining the notions of first category and Baire property in Boolean algebras.

1. Introduction

The notion of a "base of measurability" which we deal with in the present paper is a more general analogue to the "category base" from [6]. It allows to describe by algebraic means in a general way some common facts concerning measurability in the sense of Lebesgue measure and the Baire category. Here the term "measurability" resembles rather the Baire measurability than a measurability given by some real-valued function although both these notions of measurability contain the same aspect. The meaning of the term "mesurability" is related to the meaning presented in the work [3] where a "measurable space with negligibles" is a triple $(X, \Sigma, \mathcal{I})$ where X is a set, Σ is a σ-algebra of subsets of X, and $\mathcal{I}$ is a σ-ideal of $\mathcal{P}(X)$ generated by $\mathcal{I} \cap \Sigma$. In literature there are many examples of bases of measurability and some of them have weaker properties than a category base. For example, the systems of Laver perfect sets or Miller perfect sets are not known to be category bases without assuming additional axioms of ZFC although assuming Martin's axiom they are such bases. However, the notions of measurability defined by these systems are attractive regardless the additional axioms.

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In Section 2, following [5], we introduce the base of measurability in a Boolean algebra A as a subset of $P \subseteq A^+$ together with the ideal of negligible sets $s(P)$ and the subalgebra $\operatorname{dec}(P) \subseteq A$ of decidable elements according to P. It is quite a natural requirement for P that its elements are decidable, which is expressed by the term P is "separable" in the sense of [5], which means that each element of P "separates" parts of a two-element partition of a dense subset of P. In all reasonable cases a base can be replaced by a "separable" one with the same $s(P)$ and $\operatorname{dec}(P)$. To distinguish the exceptional cases of P we introduce other three properties — "eligible", "stable", and "weakly stable" — according to the possibility to replace P by a "separable" one so that $s(P)$ or $\operatorname{dec}(P)$ is either changed or not. We show that the four considered properties of bases are distinct.

Section 3 is devoted to characterizations of inclusions $s(P) \subseteq s(Q)$ and $\operatorname{dec}(P) \subseteq \operatorname{dec}(Q)$. By some obvious reasons general characterizations of these inclusions are possible only for complete Boolean algebras A.

It is a natural and basic task to find a characterization of pairs of ideals I and subalgebras B for a given Boolean algebra A so that $I = s(P)$ and $B = \operatorname{dec}(P)$ for some P. In Section 4 we prove that $s(P)$ is the least "P-regular" ideal, i.e., $P/s(P)$ is a regular subset of $A/s(P)$, and $\operatorname{dec}(P)/s(P)$ is a regular subalgebra of $A/s(P)$ closed under infinite operations of $A/s(P)$ (written $\operatorname{dec}(P)/s(P) \subseteq_{rc} A/s(P)$) if and only if P is "stable". The "if" direction was obtained by Kułaga as well as the characterization of pairs of $I = s(P)$ and $B = \operatorname{dec}(P)$ in the case that we allow "stable" bases only (Corollary 4.17). The question for "non-stable" bases remains open. The importance of the relation $\subseteq_{rc}$ can be seen in the Marczewski Theorem 7.2, where the "covering" subalgebra necessarily is a $\subseteq_{rc}$-subalgebra. We prove that for every ideal $I \subseteq A$ disjoint from P, the ideal $s(P \mathbin{\triangle} I)$ is the least "P-regular" ideal containing I. We characterize the pairs of ideals I and subalgebras $B \subseteq A$ such that $\operatorname{dec}(B \setminus I) = B \mathbin{\triangle} I$ and $s(B \setminus I) = I$ and prove that $s(P)$ is the intersection of all ideals I maximal with the property that $P \cap I = \emptyset$.

In Section 5 we consider several disjointness properties of a base $P \subseteq A^+$ fulfilled by every category base. One of them is called the "strong reduction property" and it is somewhat weaker then the "disjoint refinement property" introduced by Kułaga. The factor algebra B/I is complete if and only if $B \setminus I$ has the "strong reduction property" in B. We also prove that a base P has the "strong reduction property" in A if and only if P is "eligible" and $\operatorname{dec}(P)/s(P)$ is complete. This improves the result of Kułaga saying that $\operatorname{dec}(P)/s(P)$ is complete provided that P has the "disjoint refinement property". Moreover, we prove that if P has the "strong reduction property" in A, then for all "P-regular" ideals I in A, the factor algebras $\operatorname{dec}(P)/I$ are mutually isomorphic complete Boolean algebras and $\operatorname{dec}(P)/I \subseteq_{rc} A/I$ (Corollary 5.12).

In Section 6 we deal with a possibility to define the ideal of meager elements and the subalgebra of Baire elements of a Boolean algebra A for a base P. Let us recall that for a category base the ideal I of meager sets is defined as the σ-ideal generated by negligible sets and the definition of the class of Baire sets corresponds to our definition of $\mathrm{Baire}(P, I)$. Then the ideal of meager sets satisfies the property expressed in Banach theorem ([1]). We call the ideals in Boolean algebras with this property "P-Banach" ideals (Definition 6.1). A "P-Banach" ideal need not be "P-regular" because it may happen that $P \cap I \neq \emptyset$. But, if we put P_I = the maximal open subset of P disjoint from I, then for P "eligible", I is "P-Banach" $\iff$ I is "P_I-Banach" $\iff$ I is "P_I-regular". This allows to apply the results of the previous sections for "P-Banach" ideals. We consider two ways of definition of the ideal of meager elements in A. The first ideal, denoted by $I(P)$, is closely related to the definition used by Kułaga for σ-complete Boolean algebras and also related to the definition of meager sets for category bases. The second ideal, denoted by $I^{\mathrm{BM}}(P)$, is defined by a modification of the Banach-Mazur game for the Boolean algebra A. The ideal $I^{\mathrm{BM}}(P)$ seems to be more appropriate than $I(P)$ because $I^{\mathrm{BM}}(P)$ is always "P-Banach" while to prove that $I(P)$ is "P-Banach" we need to assume the "orthogonalization property" for P. For the ideal $I = I^{\mathrm{BM}}(P)$ we have $\mathrm{Baire}(P, I) = \mathrm{dec}(P_I \triangle I)$ whenever P is "eligible" and, if P has the "strong reduction property", then $\mathrm{Baire}(P, I) = \mathrm{dec}(P_I) \triangle I$ (Theorem 6.14). The same results for $I(P)$ require stronger assumptions on P and have its origin in [5]. We prove that $I(P) = I^{\mathrm{BM}}(P)$, if P has the "orthogonalization property" and A is ω-distributive. Therefore $I(P) = I^{\mathrm{BM}}(P)$ for every category base.

In Section 7 we prove that if $P \subseteq \mathcal{P}(X)$ has the "strong reduction property", then $\mathrm{Baire}(P, I)$ is closed under the Suslin operation $\mathcal{A}$ for every "P-Banach" σ-ideal I in $\mathcal{P}(X)$. In particular, $\mathrm{Baire}(P, I^{\mathrm{BM}}(P))$ is closed under the operation $\mathcal{A}$. This is a slight generalization of the theorem known for category bases.

Section 8 contains a list of several examples of bases of measurability.

2. Base of measurability

Let P be a partially ordered set, $D \subseteq P$, and let $x \in P$. We say that D is

a *dense* subset of P if $(\forall u \in P)(\exists v \in D)(v \leq u)$;

a *pre-dense* subset of P if $(\forall u \in P)(\exists v \in D)(\exists r \in P)(r \leq u \ \& \ r \leq v)$;

pre-dense below x in P if $(\forall u \leq x)(\exists v \in D)(\exists r \in P)(r \leq u \ \& \ r \leq v)$;

an *open* subset of P, if $(\forall u \in P)(\forall v \in D)(u \leq v \implies u \in D)$.

If E, F are subsets of P, then $E \leq F$ means that $u \leq v$ for all $u \in E$ and $v \in F$. The notions of dense and pre-dense subsets of a Boolean algebra A are always referred to $P = A \setminus \{0\}$.

The operations on a Boolean algebra are denoted by usual symbols $\vee$, $\wedge$, and $-$ (we shall not use these symbols as logical connectives); the symbols $\cup$, $\cap$, and $\setminus$ denote the corresponding set-theoretical operations; $u\triangle v = (u-v)\vee(v-u)$. If E and F are subsets of a Boolean algebra, then $E^+ = E \setminus \{0\}$ and $E \perp F$ means that $u \wedge v = 0$ for all $u \in E$ and $v \in F$. Instead of $E \le \{u\}$ and $E \perp \{u\}$ we shall write $E \le u$ and $E \perp u$. Let A be a Boolean algebra and let $I \subseteq A$ be an ideal. For a set $X \subseteq A$ we denote $X/I = \{[x]_I : x \in X\}$ where $[x]_I = \{y \in A : x \triangle y \in I\}$. Then A/I is a Boolean algebra and if B is a subalgebra of A, then B/I is a subalgebra of A/I. The mapping $h\colon A \to A/I$ defined by $h(x) = [x]_I$ is the natural homomorphism and $h(X) = X/I$; then $h^{-1}(X/I) = \bigcup(X/I)$.

Definition 2.1. ([5]) Let A be a Boolean algebra and let $\emptyset \ne P \subseteq A^+$.

(1) $$\begin{aligned}\operatorname{dec}(P) &= \operatorname{dec}_A(P)\\ &= \big\{x \in A : (\forall u\in P)(\exists v\in P)[\,v \le u \ \&\ (v \le x \text{ or } v \wedge x = 0)]\big\}.\end{aligned}$$

(2) $s(P) = s_A(P) = \big\{x \in A : (\forall u \in P)(\exists v \in P)[\,v \le u \ \&\ v \wedge x = 0]\big\}$.

$\operatorname{dec}(P)$ is the set of *decidable elements* or also *measurable elements* and $s(P)$ is the set of *negligible elements* in the *base of measurability* P.

The motivation for $\operatorname{dec}(P)$ and $s(P)$ comes from well-known Marczewski's properties (s) and (s^0) of sets in [9].

If A is a subalgebra of a Boolean algebra A' and $P \subseteq A$, then

$$\operatorname{dec}_A(P) = \operatorname{dec}_{A'}(P) \cap A \qquad \text{and} \qquad s_A(P) = s_{A'}(P) \cap A.$$

Definition 2.2. ([5]) Let A be a Boolean algebra and let $P, Q \subseteq A^+$.

(1) $P \preceq Q$ if $(\forall u \in Q)(\exists v \in P)(v \le u)$.
(2) $P \sim Q$ if $P \preceq Q$ and $Q \preceq P$.
(3) $P^{\uparrow} = \big\{x \in A : (\exists u \in P)(u \le x)\big\}$.

Lemma 2.3. *Let A be a Boolean algebra and let $P, Q \subseteq A^+$.*

(1) $\operatorname{dec}(P)$ *is a subalgebra of A, $s(P) \subseteq \operatorname{dec}(P)$ is an ideal in A, $P \cap s(P) = \emptyset$, and $I \cap \operatorname{dec}(P) \subseteq s(P)$ for every open subset I of A disjoint from P.*
(2) $P \preceq \operatorname{dec}(P) \setminus s(P)$.
(3) $P \sim Q$ *if and only if* $P^{\uparrow} = Q^{\uparrow}$.
(4) $\operatorname{dec}(P) = \operatorname{dec}(P^{\uparrow})$ *and* $s(P) = s(P^{\uparrow})$.
(5) *If $P \sim Q$, then* $\operatorname{dec}(P) = \operatorname{dec}(Q)$ *and* $s(P) = s(Q)$.
(6) $\operatorname{dec}\big(\bigcup \mathcal{A}\big) \supseteq \bigcap_{Q\in\mathcal{A}} \operatorname{dec}(Q)$ *and* $s\big(\bigcup\mathcal{A}\big) \supseteq \bigcap_{Q\in\mathcal{A}} s(Q)$ *for any $\mathcal{A} \subseteq \mathcal{P}(A^+)$. If every $Q \in \mathcal{A}$ is an open subset of $\bigcup\mathcal{A}$, then the equalities hold.*
(7) $\operatorname{dec}(P) \subseteq \operatorname{dec}(\operatorname{dec}(P) \setminus s(P))$ *and* $s(P) \subseteq s(\operatorname{dec}(P) \setminus s(P))$.
(8) *If* $\operatorname{dec}(\operatorname{dec}(P) \setminus s(P)) = \operatorname{dec}(P)$, *then* $s(\operatorname{dec}(P) \setminus s(P)) = s(P)$.

P r o o f. Conditions (1)–(4) are easy consequences of definitions; (5) is a consequence of (3) and (4). For the inclusions in (6) notice that, if $D(Q)$ denotes a dense subset of $Q \in \mathcal{A}$ (witnessing that a fixed $x \in A$ belongs to $\operatorname{dec}(Q)$ resp. to $s(Q)$), then $\bigcup_{Q \in \mathcal{A}} D(Q)$ is a dense subset of $\bigcup \mathcal{A}$ (and hence $x \in \operatorname{dec}\big(\bigcup \mathcal{A}\big)$ resp. $x \in s\big(\bigcup \mathcal{A}\big)$). For the equality notice that if all $Q \in \mathcal{A}$ are open subsets of $\bigcup \mathcal{A}$ and D is a dense subset of $\bigcup \mathcal{A}$, then $D \cap Q$ is a dense subset of Q for all $Q \in \mathcal{A}$. Properties (7) and (8) are consequences of the next more general observation: If B is a Boolean algebra and $I \subseteq B$ is an ideal, then $B \subseteq \operatorname{dec}(B \setminus I)$ and $I = s(B \setminus I) \cap B$. In particular, if $B = \operatorname{dec}(B \setminus I)$, then $s(B \setminus I) = s(B \setminus I) \cap B = I$. □

Definition 2.4. Let A be a Boolean algebra and let $P \subseteq A^+$.

(1) P is *separable* if $P \subseteq \operatorname{dec}(P)$.

(2) P is *eligible* if $P \sim \operatorname{dec}(P) \setminus s(P)$ (if $\operatorname{dec}(P) \setminus s(P) \preceq P$).

(3) P is *stable* if $\operatorname{dec}(\operatorname{dec}(P) \setminus s(P)) = \operatorname{dec}(P)$.

(4) P is *weakly stable* if $s(\operatorname{dec}(P) \setminus s(P)) = s(P)$.

(5) P is *nowhere eligible* if for every $u \in P$ there is no $v \in \operatorname{dec}(P) \setminus s(P)$ such that $v \leq u$.

(6) P is *potentially eligible* in A, if P is eligible in the Boolean completion of A.

Every separable set is eligible, every eligible set is stable, and every stable set is weakly stable (by Lemma 2.3 (2), (5), and (8), respectively). Every eligible set is potentially eligible. Moreover, if A is a subalgebra of a Boolean algebra A' and $P \subseteq A$, then the following holds.

(i) P is separable in A if and only if P is separable in A'.

(ii) If P is eligible in A, then P is eligible in A'.

$\operatorname{dec}_A(P) \setminus s_A(P) \subseteq \operatorname{dec}_{A'}(P) \setminus s_{A'}(P)$ and hence if P is eligible in A, then $\operatorname{dec}_{A'}(P) \setminus s_{A'}(P) \preceq \operatorname{dec}_A(P) \setminus s_A(P) \preceq P$.

P is potentially eligible if and only if for every $u \in P$ there is a nonempty open subset $E \leq u$ of P such that the set $E \cup \{v \in P : \ v \perp E\}$ is an open dense subset of P.

Theorem 2.5.

(1) *P is eligible if and only if $P \sim Q$ for some separable set Q.*

(2) *P is stable if and only if* $\operatorname{dec}(P) = \operatorname{dec}(Q)$ *and* $s(P) = s(Q)$ *for some separable set Q.*

Proof.

(1) If $P \sim Q$ with Q separable, then by Lemma 2.3, $P \sim Q \sim \operatorname{dec}(Q) \setminus s(Q) = \operatorname{dec}(P) \setminus s(P)$.

(2) If $\operatorname{dec}(P) = \operatorname{dec}(Q)$ and $s(P) = s(Q)$ with separable Q, then $\operatorname{dec}(P) = \operatorname{dec}(Q) = \operatorname{dec}(\operatorname{dec}(Q) \setminus s(Q)) = \operatorname{dec}(\operatorname{dec}(P) \setminus s(P))$.

The inverse implications hold by definitions since $\operatorname{dec}(P) \setminus s(P)$ is separable. □

If P is weakly stable, then $\operatorname{dec}(P) \subseteq \operatorname{dec}(Q)$ and $s(P) = s(Q)$ for some separable set Q (take $Q = \operatorname{dec}(P) \setminus s(P)$). The converse of this implication does not hold because of the characterization in Theorem 4.13 below.

Remark 2.6.

(1) If B is a subalgebra of a Boolean algebra A and $I \subseteq B$ is an ideal, then $B \setminus I$ is a separable subset of A. In particular, B^+ is separable.

(2) Let B be a subalgebra of a Boolean algebra A, let $I_B = \{x \in B :\ (\forall y \leq x)(y \in B)\}$, and let $B \neq A$. Then I_B is an ideal in A, $A \setminus B \sim A \setminus I_B$, and therefore $\operatorname{dec}(A \setminus B) = \operatorname{dec}(A \setminus I_B) = A$ and $s(A \setminus B) = s(A \setminus I_B) = I_B$.

(3) In the Boolean algebra $A = \mathcal{P}(\mathbb{R})$ there is an eligible set P which is not separable. Let P be the system of all sets of reals with positive inner measure. Then $\operatorname{dec}_A(P)$ is the σ-algebra of measurable sets, $s_A(P)$ is the σ-ideal of measure zero sets, and $P \setminus \operatorname{dec}(P) \sim P \sim \operatorname{dec}(P) \setminus s(P)$.

(4) Let $P, Q \subseteq A^+$ and $P, Q \neq \emptyset$. If P and Q are open disjoint subsets of $P \cup Q$ (i.e., $u \nleq v$ and $v \nleq u$ for $u \in P$ and $v \in Q$) and $u \wedge v \neq 0$ for all $u \in P$ and $v \in Q$, then $\operatorname{dec}(P \cup Q) = \operatorname{dec}(P) \cap \operatorname{dec}(Q) = s(P \cup Q) \cup \{x \in A :\ -x \in s(P \cup Q)\}$ and $s(P \cup Q) = s(P) \cap s(Q)$ (see also Lemma 2.3 (6)). Hence, $P \cup Q$ is stable and nowhere eligible because $\operatorname{dec}(P \cup Q) \setminus s(P \cup Q)$ is a filter disjoint from $P \cup Q$. $P \cup Q$ is neither potentially eligible.

(5) Let $u \in A \setminus \{0, 1\}$. The sets $\{u\}$ and $\{u, -u\}$ are separable, $\operatorname{dec}(\{u\}) = \{x \in A :\ x \leq -u \text{ or } u \leq x\}$, $s(\{u\}) = \{x \in A :\ x \leq -u\}$, $\operatorname{dec}(\{u, -u\}) = \operatorname{dec}(\{u\}) \cap \operatorname{dec}(\{-u\}) = \{0, 1, u, -u\}$, $s(\{u, -u\}) = s(\{u\}) \cap s(\{-u\}) = \{0\}$.

(6) Let $P = B \setminus I$ where B is a subalgebra of a Boolean algebra A and I is an ideal in B. Assume that $x \in A$ is a *Bernstein element* for P, i.e., for all $u \in P$, $u \wedge x \neq 0$ and $u - x \neq 0$. Denote $Q_x = P_x \cup P_{-x}$ where $P_x = \{u \wedge x :\ u \in P\}$. Then $\operatorname{dec}(P) \subsetneq \operatorname{dec}(P_x)$ and $s(P) \subsetneq s(P_x)$ because $x \in \operatorname{dec}(P_x) \cap \operatorname{dec}(P_{-x})$, $x \in s(P_{-x})$, and $-x \in s(P_x)$. By Lemma 2.3 (6), $\operatorname{dec}(Q_x) = \operatorname{dec}(P_x) \cap \operatorname{dec}(P_{-x})$ and $s(Q_x) = s(P_x) \cap s(P_{-x})$ because P_x and P_{-x} are open subsets of Q_x. Therefore $\operatorname{dec}(P) \subsetneq \operatorname{dec}(Q_x)$ while $s(P) = s(Q_x)$. (To prove $s(Q_x) \subseteq s(P)$ assume that $a \in s(Q_x)$ and $u \in P$ are given. Since $a \in s(P_x)$ and $a \in s(P_{-x})$ there are $v_0, v_1 \in P$ such that $v_0 \wedge x \leq u \wedge x$, $(v_0 \wedge x) \wedge a = 0$, and $v_1 - x \leq v_0 - x$, $(v_1 - x) \wedge a = 0$. For $w = u \wedge v_0 \wedge v_1$ we have $w \leq u$ and $w \wedge a = 0$. Finally $w \in P$ because x is Bernstein for P and hence $v_0 - u \in I$ and $v_1 - v_0 \in I$.)

(7) Assume that $P, Q \subseteq A^+$, $P, Q \neq \emptyset$, and $u \wedge v \neq 0$ for all $u \in P$ and $v \in Q$. Denote $R = \{u \wedge v : u \in P \ \& \ v \in Q\}$. Then $\mathrm{dec}(R) \supseteq \mathrm{dec}(P) \cup \mathrm{dec}(Q)$ and $s(R) \supseteq s(P) \cup s(Q)$.

(8) The Boolean algebra $A = \mathcal{P}(\mathbb{R})$ has a subset P which is not weakly stable and a weakly stable subset Q which is not stable. Fix a sequence $\langle I_n : n \in \omega \rangle$ of pairwise disjoint nonempty open intervals, let $u_n = \bigcup_{k \geq n} I_k \cap \mathbb{Q}$ for $n \in \omega$, and let $M \subseteq u_0$ selects exactly one point from each $I_n \cap \mathbb{Q}$. Let $\mathcal{N}$ be the ideal of nowhere dense subsets of $\mathbb{R}$. Denote $P_0 = \big\{I : I$ is an open interval and $|\{k : I \cap I_k \neq \emptyset\}| \leq 1\big\}$, $P_1 = \{u_n : n \in \omega\}$, $P_2 = \{u_n \setminus X : n \in \omega \ \& \ X \in \mathcal{N}\}$, and define $P = P_0 \cup P_1$ and $Q = P_0 \cup P_2$.

(a) $P_0 \subseteq \mathrm{dec}(P) \setminus s(P)$, $P_1 \cap \mathrm{dec}(P) = \emptyset$, and $P_0 \sim \mathrm{dec}(P) \setminus s(P)$. Clearly, $M \notin s(P)$ because no u_n is disjoint from M but $M \in s(P_0) = s(\mathrm{dec}(P) \setminus s(P))$. Therefore P is not weakly stable.

(b) $P_0 \subseteq \mathrm{dec}(Q) \setminus s(Q)$, $P_2 \cap \mathrm{dec}(Q) = \emptyset$, and $P_0 \sim \mathrm{dec}(Q) \setminus s(Q)$. The set Q is weakly stable because $\mathcal{N} \subseteq s(Q) \subseteq s(\mathrm{dec}(Q) \setminus s(Q)) = s(P_0) = \mathcal{N}$ but Q is not stable because $\bigcup_{n \in \omega} I_{2n} \in \mathrm{dec}(P_0) \setminus \mathrm{dec}(Q)$.

(9) The inclusion $\mathrm{dec}(P) \subseteq \mathrm{dec}(Q)$ does not imply the inclusion $s(P) \subseteq s(Q)$. For example, let A be the algebra of Borel sets of reals, $\mathcal{M}$ be the σ-ideal of meager sets, $\mathcal{N}$ be the σ-ideal of measure zero sets, $P = A \setminus \mathcal{M}$, $Q = A \setminus \mathcal{N}$. There are no inclusions between the ideals $\mathcal{M} \cap A$ and $\mathcal{N} \cap A$.

Lemma 2.7. *Let A be a Boolean algebra, let I be an ideal in A, and let $P \subseteq A^+$. The following conditions are equivalent:*

(1) $\mathrm{dec}(P) = A$ *and* $s(P) = I$.

(2) $s(P) \subseteq I$, $P \cap I = \emptyset$, *and* $\mathrm{dec}(P) \setminus I$ *is a dense subset of* $A \setminus I$.

(3) P *is a dense subset of* $A \setminus I$.

Proof. The implication (1) $\implies$ (2) is trivial.

(2) $\implies$ (3) If (2) holds, then $P \preceq \mathrm{dec}(P) \setminus s(P) \preceq \mathrm{dec}(P) \setminus I \preceq A \setminus I$ and hence P is a dense subset of $A \setminus I$.

(3) $\implies$ (1) If P is a dense subset of $A \setminus I$, then $P \sim A \setminus I$ and hence $\mathrm{dec}(P) = \mathrm{dec}(A \setminus I) = A$ and $s(P) = s(A \setminus I) = I$. □

Question 2.8. Let B be a subalgebra of a Boolean algebra A and let $I, J \subseteq B$ be ideals in A. Under what conditions the equalities $\mathrm{dec}(B \setminus I) = \mathrm{dec}(B \setminus J) = B$ imply $I = J$? (If $B = A$, then $s(B \setminus I) = I$.)

Theorem 2.9. ([5]) *If P and Q are eligible, then $P \sim Q$ if and only if* $\mathrm{dec}(P) = \mathrm{dec}(Q)$ *and* $s(P) = s(Q)$.

Proof. Assume that $\mathrm{dec}(P) = \mathrm{dec}(Q)$ and $s(P) = s(Q)$. Then $P \sim \mathrm{dec}(P) \setminus s(P) = \mathrm{dec}(Q) \setminus s(Q) \sim Q$. The inverse implication holds by Lemma 2.3 (5). □

For subsets B, I, P of a Boolean algebra A we define

$$\mathrm{cmpl}(B) = \max\big\{\kappa:\ \kappa \le |A|^{+}\ \ \&\ \ (\forall R \in [B]^{<\kappa})\big[\textstyle\bigvee^{A} R \text{ exists } \to \bigvee^{A} R \in B\big]\big\},$$
$$\mathrm{cov}(I, P) = \max\big\{\kappa:\ \kappa \le |A|^{+}\ \ \&\ \ (\forall R \in [I]^{<\kappa})\big[\textstyle\bigvee^{A} R \text{ exists } \to \bigvee^{A} R \notin P\big]\big\}.$$

Clearly, $\mathrm{cmpl}(B) = \mathrm{cov}(B, A \setminus B)$. We shall use these definitions only in the case when B is a subalgebra or an ideal and I is an ideal. If $P \preceq Q$ and I is an ideal in A, then $\mathrm{cov}(I, P) \le \mathrm{cov}(I, Q)$.

Theorem 2.10. *Let A be a Boolean algebra and let $P \subseteq A^{+}$.*

(1) $\min\{\mathrm{cmpl}(\mathrm{dec}(P)), \mathrm{cov}(s(P), P)\} \le \mathrm{cmpl}(s(P)) \le \mathrm{cov}(s(P), P)$.
(2) *If P is stable, then* $\mathrm{cmpl}(s(P)) = \min\{\mathrm{cmpl}(\mathrm{dec}(P)), \mathrm{cov}(s(P), P)\}$.
(3) *If $A \setminus \mathrm{dec}(P) \preceq P$, then* $\mathrm{cmpl}(\mathrm{dec}(P)) \le \mathrm{cmpl}(s(P))$.
(4) *If P is stable and $A \setminus \mathrm{dec}(P) \preceq P$, then* $\mathrm{cmpl}(\mathrm{dec}(P)) = \mathrm{cmpl}(s(P)) = \mathrm{cov}(s(P), P)$.

Proof. (1) Use the facts $P \preceq \mathrm{dec}(P) \setminus s(P)$ and $s(P) \cap P = \emptyset$.

(2) By (1) it remains to prove $\mathrm{cmpl}(s(P)) \le \mathrm{cmpl}(\mathrm{dec}(P))$. By Theorem 2.5 we can assume that P is a separable subset of A. We follow the proof of [5: Lemma 4.1]: Suppose that $R \subseteq \mathrm{dec}(P)$, $|R| < \mathrm{cmpl}(s(P))$, and $x = \bigvee R$. We prove that $x \in \mathrm{dec}(P)$. Let $u \in P$. If $u \wedge r \in s(P)$ for all $r \in R$, then $u \wedge x \in s(P)$ by completeness of $s(P)$ and so there is $v \le u$ in P such that $v \wedge x = 0$. If $u \wedge r \notin s(P)$ for some $r \in R$, then $u \wedge r \in \mathrm{dec}(P) \setminus s(P)$ as P is separable. There is $v \in P$ with $v \le u \wedge r$ because $P \preceq \mathrm{dec}(P) \setminus s(P)$ and so $v \le u$ and $v \le x$.

(3) $A \setminus \mathrm{dec}(P) \preceq \mathrm{dec}(P) \setminus s(P)$ because $P \preceq \mathrm{dec}(P) \setminus s(P)$. If $R \subseteq s(P)$, $x = \bigvee R$, and $x \in \mathrm{dec}(P) \setminus s(P)$, we find $y \in A \setminus \mathrm{dec}(P)$ such that $y \le x$. Let $R' = \{r \wedge y:\ r \in R\}$. Then $R' \subseteq s(P)$ and $\bigvee R' = y \notin \mathrm{dec}(P)$.

(4) follows by (2) and (3). □

Definition 2.11. Let A be a Boolean algebra and let $\emptyset \ne P \subseteq A^{+}$. We say that P is a *π-complete* subset of A, if for every finite set $P_0 \subseteq P$ either $\bigwedge P_0 = 0$, or there is a lower bound for P_0 in P.

If P is π-complete, then P is separable. The inverse is not true because if B is a subalgebra of A and $I \ne \{0\}$ is a non-maximal ideal in B, then $B \setminus I$ is separable but not π-complete.

Lemma 2.12. *P is π-complete if and only if P is separable and*

$$(\forall P_0 \in [P]^{<\omega})\big[\textstyle\bigwedge P_0 = 0 \ \ \textit{or} \ \ \bigwedge P_0 \notin s(P)\big].$$

Remark 2.13. For $P \subseteq A^+$ let $\overline{P}$ be the closure of P under finite meets and let $I_P \subseteq s(P)$ be the ideal in A generated by $(\overline{P} \cup \{0\}) \cap s(P)$.

(1) If P is separable and I is an ideal in A disjoint from P with $I_P \subseteq I$, then P/I is a π-complete subset of $(A/I)^+$.

(2) If P is separable, then P is π-complete if and only if $I_P = \{0\}$.

(3) If $I_P = \{0\}$, then P is π-complete if and only if P is separable.

Remark 2.14. Let P be a subset of a Boolean algebra A and let

$$H_0 = \big\{x \in A:\ (\forall u \in P)(u \nleq x)\big\},$$
$$H_1 = \big\{x \in A:\ (\forall u \in P)(\forall v \in P)(v \leq u \vee x \implies v \wedge u \notin H_0)\big\}$$
$$H_2 = \big\{x \in A:\ (\forall u \in P)(\forall v \in P)(v \leq -u \vee x \implies v - u \notin H_0)\big\}.$$

H_0, H_1, H_2 are open subsets of A and H_0 is the largest open subset of A disjoint from P. A referee suggested to note that P is separable if and only if for all $u, v \in P$, $u \wedge v \notin H_0$ or $u - v \notin H_0$. The following holds:

(1) $H_1 \subseteq H_0 \iff H_1 \cap P = \emptyset \iff P$ does not generate a filter $\iff H_1 \neq A$.

(2) $H_2 \subseteq H_0$.

(3) P is separable $\iff H_0 \subseteq H_1 \iff H_0 \subseteq H_2$.

3. Comparing of bases of measurability

In the present section we give some characterizations of the inclusions $s(P) \subseteq s(Q)$ and $\mathrm{dec}(P) \subseteq \mathrm{dec}(Q)$. These results are not necessary for reading other parts of the paper so the reader can skip them if he wishes.

Definition 3.1. Let A be a Boolean algebra and let $P, Q \subseteq A^+ = A \setminus \{0\}$.

(1) $P \trianglelefteq Q$ if for every dense subset D of P the set $D' = \big\{u \in Q:\ (\exists v \in D)(u \leq v)\big\}$ is a dense subset of Q.

(2) $P \sqsubseteq_1 Q$ if for every dense subset D of P the set $D' = \{u \in Q:\ D$ is pre-dense below u in $A\}$ is a dense subset of Q.

(3) A pair $\{D_1, D_2\}$ of subsets of P is said to be a *cut of P in A* if $D_1 \cup D_2$ is a dense subset of P and $D_1 \perp D_2$. We write $P \sqsubseteq_2 Q$ if for every cut $\{D_1, D_2\}$ of P, the pair $\{D_1', D_2'\}$ is a cut of Q, where $D_i' = \{u \in Q:$ D_i is pre-dense below u in $A\}$ for $i = 1, 2$.

Lemma 3.2. *The relations $\trianglelefteq$, $\sqsubseteq_1$, $\sqsubseteq_2$ are all reflexive and transitive.*

P r o o f. To see that $\trianglelefteq$, $\sqsubseteq_1$, $\sqsubseteq_2$ are reflexive notice that for $Q = P$ in Definition 3.1 (1), (2), and (3) we have, respectively, $D \subseteq D'$ and $D_1 \subseteq D_1'$, $D_2 \subseteq D_2'$. We are going to prove that $\trianglelefteq$, $\sqsubseteq_1$, $\sqsubseteq_2$ are transitive and let $P, Q, R \subseteq A^+$ be arbitrary nonempty.

Let us assume that $P \trianglelefteq Q \trianglelefteq R$ and we prove $P \trianglelefteq R$. Let D be a dense subset of P. Then $D' = \big\{u \in Q :\ (\exists v \in D)(u \leq v)\big\}$ is a dense subset of Q. Now $\big\{u \in R :\ (\exists v \in D)(u \leq v)\big\} \supseteq \big\{u \in R :\ (\exists v \in D)(\exists v' \in Q)(u \leq v' \leq v)\big\}$ $= \big\{u \in R :\ (\exists v' \in D')(u \leq v')\big\}$ is a dense subset of R.

Let us assume that $P \sqsubseteq_1 Q \sqsubseteq_1 R$ and we prove $P \sqsubseteq_1 R$. Let D be a dense subset of P. Then $D' = \{u \in Q :\ D \text{ is pre-dense below } u\}$ is a dense subset of Q, $D'' = \{u \in R :\ D' \text{ is pre-dense below } u\}$ is a dense subset of R and $\{u \in R :\ D \text{ is pre-dense below } u\} \supseteq D''$ because D is pre-dense below each member of D'.

The arguments for transitivity of $\sqsubseteq_2$ are similar to the arguments for transitivity of $\sqsubseteq_1$. □

Lemma 3.3. *Let $P, Q \subseteq A^+$.*

(1) $P \trianglelefteq Q$ *implies* $P \sqsubseteq_2 Q$

(2) $P \sqsubseteq_2 Q$ *implies* $P \sqsubseteq_1 Q$.

(3) $P \sqsubseteq_1 Q$ *implies* $s(P) \subseteq s(Q)$.

(4) $P \sqsubseteq_2 Q$ *implies* $s(P) \subseteq s(Q)$ *and* $\operatorname{dec}(P) \subseteq \operatorname{dec}(Q)$.

P r o o f. (1) is easy and (2) holds because, if D is a dense subset of P, then $\{D, \emptyset\}$ is a cut of P.

(3) Let $P \sqsubseteq_1 Q$ and let $x \in s(P)$. Then $D = \{u \in P :\ u \wedge x = 0\}$ is dense in P. If $u \in A$ is such that D is pre-dense below u, then $u \wedge x = 0$. Therefore the set $\{u \in Q :\ u \wedge x = 0\} \supseteq \{u \in Q :\ D \text{ is pre-dense below } u\}$ is dense in Q and so $x \in s(Q)$.

(4) Let $P \sqsubseteq_2 Q$. Then $s(P) \subseteq s(Q)$ by (2) and (3). We prove $\operatorname{dec}(P) \subseteq \operatorname{dec}(Q)$. Let $x \in \operatorname{dec}(P)$ and set $D_1 = \{u \in P :\ u \leq x\}$, $D_2 = \{u \in P : u \wedge x = 0\}$. Then $\{D_1, D_2\}$ is a cut of P and the set $\{u \in Q :\ u \leq x$ or $u \wedge x = 0\}$ is a dense subset of Q because it contains $D_1' \cup D_2'$ where $\{D_1', D_2'\}$ is the cut of Q from definition of the relation $\sqsubseteq_2$. This proves that $x \in \operatorname{dec}(Q)$. □

Theorem 3.4. *Let A be a complete Boolean algebra and let $P, Q \subseteq A^+$.*

(1) $P \sqsubseteq_1 Q$ *if and only if* $s(P) \subseteq s(Q)$.

(2) $P \sqsubseteq_2 Q$ *if and only if* $s(P) \subseteq s(Q)$ *and* $\operatorname{dec}(P) \subseteq \operatorname{dec}(Q)$.

P r o o f. The "only if" parts follow by assertions (3) and (4) of Lemma 3.3. It remains to prove the "if" parts.

(1) Let us assume that $s(P) \subseteq s(Q)$ and let D be a dense subset of P. Let $x = -\bigvee D$. Then $x \in s(P)$ and hence also $x \in s(Q)$. We prove that the set $D' = \{u \in Q : D \text{ is pre-dense below } u \text{ in } A\}$ is a dense subset of Q. Let $u \in Q$. There is $v \in Q$ such that $v \le u$ and $v \wedge x = 0$. Then $v \le \bigvee D$ which means that D is pre-dense below v and $v \in D'$.

(2) Let us assume that $s(P) \subseteq s(Q)$, $\mathrm{dec}(P) \subseteq \mathrm{dec}(Q)$, let $\{D_1, D_2\}$ be a cut of P and let D'_1, D'_2 be the sets corresponding to D_1, D_2 in definition of $\sqsubseteq_2$. We prove that $\{D'_1, D'_2\}$ is a cut of Q. Clearly, it is enough to prove that $D'_1 \cup D'_2$ is a dense subset of Q. Let $x_i = \bigvee D_i$ and $E_i = \{u \in Q : u \le x_i\}$ for $i = 1, 2$. Then $x_i \in \mathrm{dec}(P)$ and $-(x_1 \vee x_2) \in s(P)$ and so $x_i \in \mathrm{dec}(Q)$ and $-(x_1 \vee x_2) \in s(Q)$. It follows that $E_1 \cup E_2$ is a dense subset of Q. If $u \in E_i$, then $u \le \bigvee D_i$ and hence D_i is pre-dense below u in A. Therefore $D'_1 \cup D'_2 \supseteq E_1 \cup E_2$ is a dense subset of Q. □

In the case when A is complete the conditions in definitions of $\sqsubseteq_1$ and $\sqsubseteq_2$ can be simplified a bit and it is not difficult to see the following:

Theorem 3.5. *Let A be a complete Boolean algebra and let $P, Q \subseteq A^+$.*

1. *$s(P) \subseteq s(Q)$ if and only if for every dense subset D of P there exists a dense subset D' of Q such that $\bigvee D' \le \bigvee D$.*
2. *$s(P) \subseteq s(Q)$ and $\mathrm{dec}(P) \subseteq \mathrm{dec}(Q)$ if and only if for every cut $\{D_1, D_2\}$ of P there exists a cut $\{D'_1, D'_2\}$ of Q such that $\bigvee D'_1 \le \bigvee D_1$ and $\bigvee D'_2 \le \bigvee D_2$.*
3. *$\mathrm{dec}(P) \subseteq \mathrm{dec}(Q)$ if and only if for every cut $\{D_1, D_2\}$ of P and every $x \in s(P)$ there exists a cut $\{D'_1, D'_2\}$ of Q such that $\bigvee D'_1 \le \bigvee D_1 \vee x$ and $\bigvee D'_2 \wedge (\bigvee D_1 \vee x) = 0$.*

For a π-complete set see Definition 2.11:

Theorem 3.6. *Let $P \subseteq Q$ and let Q be a π-complete subset of a Boolean algebra A. The following conditions are equivalent:*

1. *$P \trianglelefteq Q$.*
2. *$P \sqsubseteq_2 Q$.*
3. *$P \sqsubseteq_1 Q$.*
4. *$s(P) \subseteq s(Q)$.*
5. *$s(P) \subseteq s(Q)$ and $\mathrm{dec}(P) \subseteq \mathrm{dec}(Q)$.*

Proof. The implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) and (2) $\Longrightarrow$ (5) $\Longrightarrow$ (4) hold by Lemma 3.3. We prove (4) $\Longrightarrow$ (1). Let $s(P) \subseteq s(Q)$ and let $D \subseteq P$ be a dense subset of P. We prove that $D' = \big\{u \in Q : (\exists v \in D)(u \le v)\big\}$ is a dense subset of Q. Let $u \in Q$. As $u \notin s(Q)$, $u \notin s(P)$, and hence there is $v \in D$ such that $u \wedge v \ne 0$. As $P \subseteq Q$, $v \in Q$ and by π-completeness of Q we can find $w \in Q$ such that $w \le u \wedge v$. Then $w \in D'$ and $w \le u$. □

4. Towards a characterization of $\operatorname{dec}(P)$ and $s(P)$

Let us recall that a Boolean algebra B is a regular subalgebra of a Boolean algebra A if B is a subalgebra of A and every partition of the unit in B is a partition of the unit in A (see e.g. [8]).

Definition 4.1. Let A be a Boolean algebra, let $P \subseteq A^+$, and let B be a subalgebra of A. We say that P is a *regular* subset of A if $P \subseteq A^+$ and every dense subset of P is a pre-dense subset of A. We say that B is an A-*complete* subalgebra of A, if B is closed under the least upper bounds computed in A. We write $P \subseteq_r A$, if P is a regular subset of A, we write $B \subseteq_r A$, if B is a regular subalgebra of A, and we write $B \subseteq_{rc} A$, if B is a regular A-complete subalgebra of A.

If B is a subalgebra of A, then $B^+ \subseteq_r A$ (regular subset) if and only if $B \subseteq_r A$ (regular subalgebra).

For subsets C, D of a Boolean algebra A we denote $C \triangle D = \{c \triangle d : \ c \in C$ and $d \in D\}$, and similarly we define $C - D$, $C \wedge D$, and $C \vee D$.

Lemma 4.2. *Let P be a π-complete subset of a Boolean algebra A and let $\overline{P}$ be the closure of P under the least upper bounds computed in A.*

(1) $\overline{P} \subseteq \operatorname{dec}(P)$ *and if* $P \subseteq Q \subseteq \overline{P} \setminus \{0\}$, *then* $P \sim Q$ *and* Q *is* π*-complete.*

(2) *If* $s(P) = \{0\}$, *then* $\operatorname{dec}(P)$ *is an* A*-complete subalgebra of* A.

(3) *If* A *is complete, then* $\operatorname{dec}(P) = \overline{P} \triangle s(P) = \overline{P} \vee s(P)$.

Proof.

(1) If $x \in \overline{P}$ and $x = \bigvee P_0$ for some $P_0 \subseteq P$, then for every $u \in P$ either there is $v \in P_0$ such that $u \wedge v \neq 0$ or $u \wedge x = 0$. In the former case there is $w \leq u$ in P such that $w \leq u$ and $w \leq x$. It follows that $x \in \operatorname{dec}(P)$. Therefore $\overline{P} \subseteq \operatorname{dec}(P)$.

(2) If $s(P) = \{0\}$, then $\operatorname{dec}(P) = \operatorname{dec}(\operatorname{dec}(P)^+)$ because P is stable. By (1), $\overline{\operatorname{dec}(P)^+} \subseteq \operatorname{dec}(P)$ because $\operatorname{dec}(P)^+$ is a π-complete subset of A, and hence $\operatorname{dec}(P)$ is A-complete.

(3) Let A be complete. If $x \in \operatorname{dec}(P)$ and $a = \bigvee\{p \in P : \ p \leq x\}$, then $a \in \overline{P}$ and $x - a \in s(P)$ because $x - a \in \operatorname{dec}(P)$. Hence $x = a \triangle (x - a) = a \vee (x - a)$. Therefore $\operatorname{dec}(P) = \overline{P} \triangle s(P) = \overline{P} \vee s(P)$. □

***Example* 4.3.**

(1) ([5: Example 2.1]) Let X be a topological space, let $\operatorname{Open}(X)$ be the system of open sets in X and let $\mathcal{N}(X)$ be the system of nowhere dense subsets of X. Then $A = \mathcal{P}(X)$ is a complete Boolean algebra, $P = \operatorname{Open}(X) \setminus \{\emptyset\}$ is a π-complete subset of A, $\overline{P} = \operatorname{Open}(X)$, $s(P) = \mathcal{N}(X)$, and $\operatorname{dec}(P) = \operatorname{Open}(X) \triangle \mathcal{N}(X) = \operatorname{Open}(X) \vee \mathcal{N}(X)$ is the field of subsets of X with nowhere dense boundary.

(2) ([5: Example 5.1]) Let I be an ideal on a space X such that I does not contain any nonempty open set. Let P be the family of al sets of the form $U \setminus F$ where U is a nonempty open set and $F \in I$. Then P is a π-complete subset of $A = \mathcal{P}(X)$. $A \in \operatorname{dec}(P)$ if and only if there is $N \in \mathcal{N}(X)$, a disjoint system of open sets $\mathcal{A}$, and $M_U \in I$ for all $U \in \mathcal{A}$ such that $A = N \cup \bigcup_{U \in \mathcal{A}} U \setminus M_U$; the sets in $s(P)$ are of the form $N \cup \bigcup_{U \in \mathcal{A}} U \cap M_U$. Especially, if I is the ideal of meager sets in a space X with no meager nonempty open set, then $\operatorname{dec}(P)$ is the family of all sets with the Baire property and $s(P)$ is the σ-ideal of meager sets.

Lemma 4.4. *Let A be a Boolean algebra and let $P \subseteq A^+$.*

(1) *$s(P) = \{0\}$ if and only if P is a regular subset of A. In particular, $s(\operatorname{dec}(P)^+) = \{0\}$ if and only if $\operatorname{dec}(P) \subseteq_r A$.*

(2) *$s(P) = s(\operatorname{dec}(P)^+)$ if and only if P is weakly stable and $s(P) = \{0\}$.*

(3) *$\operatorname{dec}(P) = \operatorname{dec}(\operatorname{dec}(P)^+)$ if and only if $\operatorname{dec}(P) \subseteq_{rc} A$. More generally, if B is a subalgebra of A, then $B = \operatorname{dec}(B^+)$ if and only if $B \subseteq_{rc} A$.*

Proof.

(1) P is not a regular subset of A if and only if there is a dense subset D of P and $x \neq 0$ such that $u \wedge x = 0$ for every $u \in D$.

(2) If $s(P) = s(\operatorname{dec}(P)^+)$, then $s(P) = s(P) \cap \operatorname{dec}(P) = s(\operatorname{dec}(P)^+) \cap \operatorname{dec}(P) = \{0\}$, and consequently, P is also weakly stable; if P is weakly stable and $s(P) = \{0\}$, then $s(P) = s(\operatorname{dec}(P) \setminus s(P)) = s(\operatorname{dec}(P)^+)$.

(3) If $\operatorname{dec}(B^+) = B$, then $s(B^+) = \{0\}$ and hence B is a regular subalgebra of A. By Lemma 4.2 (2), B is also A-complete, because B^+ is π-complete. Conversely, if $B \subseteq_{rc} A$, then B^+ is a dense subset of $\operatorname{dec}(B^+)$ because $s(B^+) = \{0\}$ (by (1)), $B \subseteq \operatorname{dec}(B^+)$, and $B^+ \preceq \operatorname{dec}(B^+) \setminus s(B^+) = \operatorname{dec}(B^+)^+$. Hence, $\operatorname{dec}(B^+) = B$, by A-completeness of B. □

We will return to the conditions that Lemma 4.4 deals with in Remark 4.20.

Lemma 4.5. *Let A be a Boolean algebra, let $P \subseteq A^+$, let I be an ideal in A disjoint from P, and let $h\colon A \to A/I$ be the natural homomorphism. Then $P/I \subseteq (A/I)^+$ and the following assertions hold:*

(1) *$\operatorname{dec}_A(P) \subseteq \operatorname{dec}_A(P \triangle I) = h^{-1}(\operatorname{dec}_{A/I}(P/I))$.*

(2) *$s_A(P) \subseteq s_A(P \triangle I) = h^{-1}(s_{A/I}(P/I))$.*

Proof. (1) For $x \in A$ the following four conditions are equivalent:

(i) $x \in \operatorname{dec}_A(P \triangle I)$.

(ii) $(\forall (u,s) \in P \times I)(\exists (v,t) \in P \times I)[v \triangle t \leq u \triangle s \ \&\ (v \triangle t \leq x \text{ or } v \triangle t \leq -x)]$.

(iii) $(\forall u \in P)(\exists v \in P)[v - u \in I \ \&\ (v - x \in I \text{ or } v \wedge x \in I)]$.

(iv) $[x]_I \in \operatorname{dec}_{A/I}(P/I)$.

This proves the equality $\operatorname{dec}_A(P \triangle I) = h^{-1}(\operatorname{dec}_{A/I}(P/I))$.

By definition, $x \in \operatorname{dec}_A(P)$ if and only if

(v) $(\forall u \in P)(\exists v \in P)\big[v \leq u \ \ \& \ \ (v \leq x \ \text{ or } \ v \wedge x = 0)\big]$.

If (v) holds, for some $x \in A$, then the existential quantifier in (ii) will be satisfied for $t = s \wedge v$ because $v \triangle t = v - s \leq v$. This proves the inclusion $\operatorname{dec}_A(P) \subseteq \operatorname{dec}_A(P \triangle I)$.

(2) This proof is similar to the proof of (1). □

Remark 4.6. The assertions of Lemma 4.5 can be read also as:

(1′) $\operatorname{dec}_A(P)/I \subseteq \operatorname{dec}_A(P \triangle I)/I = \operatorname{dec}_{A/I}(P/I)$.

(2′) $s_A(P)/I \subseteq s_A(P \triangle I)/I = s_{A/I}(P/I)$.

For $Q = h(P)$ they can be read also as:

(1″) $h(\operatorname{dec}_A(P)) \subseteq \operatorname{dec}_{h(A)}(h(P))$ and $\operatorname{dec}_A(h^{-1}(Q)) = h^{-1}(\operatorname{dec}_{h(A)}(Q))$.

(2″) $h(s_A(P)) \subseteq s_{h(A)}(h(P))$ and $s_A(h^{-1}(Q)) = h^{-1}(s_{h(A)}(Q))$.

Definition 4.7. Let A be a Boolean algebra and let $P \subseteq A^+$. An ideal $I \subseteq A$ is said to be a *P-regular ideal in A*, if $I \cap P = \emptyset$ and $P/I \subseteq_r A/I$.

Lemma 4.8. *Let A be a Boolean algebra and let $P \subseteq A^+$.*

(1) *An ideal I in A is P-regular in A if and only if $I \cap P = \emptyset$ and for all $x \in A$, $x \in I$ if and only if $(\forall u \in P)(\exists v \in P)[v - u \in I \ \ \& \ \ v \wedge x \in I]$.*

(2) *If $I \subseteq J$ are ideals in A disjoint from P, then J is P-regular if and only if J is $(P \triangle I)$-regular.*

Proof. By Lemma 4.4 (1), P/I is regular if and only if $s_{A/I}(P/I) = \{0_{A/I}\}$ (this is the assertion (1)); (2) holds by definition since $(P \triangle I)/J = P/J$. □

Lemma 4.9. *Let A be a Boolean algebra, let $P \subseteq A^+$, and let I be an ideal in A disjoint from P. The following conditions are equivalent:*

(1) *I is a P-regular ideal in A.*

(2) *$s(P \triangle I) = I$.*

(3) *$s(P \triangle I) \subseteq I$.*

Proof. By Lemma 4.5 (2) and Lemma 4.4 (1) it is obvious that P/I is a regular subset of A/I if and only if $s(P \triangle I) = I$. As $I \subseteq s(P \triangle I)$, all conditions in the lemma are equivalent. □

The following lemma is concerned with the inclusions in Lemma 4.5 and in Remark 4.6.

Lemma 4.10. *Let A be a Boolean algebra, let $P \subseteq A^+$, and let I be an ideal in A disjoint from P.*

(1) *The following conditions are equivalent:*
  (a) $\mathrm{dec}_A(P) = \mathrm{dec}_A(P \triangle I)$.
  (b) $s_A(P) = s_A(P \triangle I)$.
  (c) $I \subseteq s_A(P)$.
  (d) $P \sim P \triangle I$.

(2) *The following conditions are equivalent:*
  (a) $\mathrm{dec}_A(P)/I = \mathrm{dec}_{A/I}(P/I)$.
  (b) $\mathrm{dec}_A(P)/I = \mathrm{dec}_A(P \triangle I)/I$.
  (c) $\mathrm{dec}_A(P \triangle I) = \mathrm{dec}_A(P) \triangle I$.

(3) *The following conditions are equivalent:*
  (a) $s_A(P)/I = s_{A/I}(P/I)$.
  (b) $s_A(P)/I = s_A(P \triangle I)/I$.
  (c) $s_A(P \triangle I) = s_A(P) \triangle I$.
  (d) $s_A(P) \triangle I$ *is a P-regular ideal in A.*

Moreover, every condition in (1) *implies the conditions in* (2) *and every condition in* (2) *implies the conditions in* (3).

P r o o f. The proof of (1a) $\Longrightarrow$ (1c) and (1b) $\Longrightarrow$ (1c). As $I \subseteq s_A(P \triangle I) \subseteq \mathrm{dec}_A(P \triangle I)$, and by Lemma 2.3 (1), $I \cap \mathrm{dec}_A(P) \subseteq s_A(P)$, each of the equalities $\mathrm{dec}_A(P) = \mathrm{dec}_A(P \triangle I)$ and $s_A(P) = s_A(P \triangle I)$ implies $I \subseteq s_A(P)$.

The proof of (1c) $\Longrightarrow$ (1d). Let $I \subseteq s(P)$. $P \triangle I \preceq P$ because $P \subseteq P \triangle I$. We prove $P \preceq P \triangle I$. Let $u \in P$ and $s \in I$. As $s \in s(P)$, there is $v \le u$ in P such that $v \wedge s = 0$ and so $v \le u \triangle s$.

The implications (1d) $\Longrightarrow$ (1a) and (1d) $\Longrightarrow$ (1b) hold by Lemma 2.3 (3).

The equivalences (2a) $\Longleftrightarrow$ (2b) and (3a) $\Longleftrightarrow$ (3b) hold by Lemma 4.5. (2c) $\Longleftrightarrow$ (2b) and (3c) $\Longleftrightarrow$ (3b) hold because $I \subseteq s_A(P \triangle I) \subseteq \mathrm{dec}_A(P \triangle I)$, and (3d) $\Longleftrightarrow$ (3c) holds by Lemma 4.9 because $s_A(P \triangle s_A(P) \triangle I) = s_A(P \triangle I)$.

Clearly, (1a) implies (2b). We prove (2c) $\Longrightarrow$ (3c). $s(P \triangle I) \cap \mathrm{dec}(P) = s(P)$, by Lemma 2.3 (1). If $\mathrm{dec}(P \triangle I) = \mathrm{dec}(P) \triangle I$, then $s(P \triangle I) = (\mathrm{dec}(P) \triangle I) \cap s(P \triangle I) = (\mathrm{dec}(P) \cap s(P \triangle I)) \triangle (I \cap s(P \triangle I)) = s(P) \triangle I$. □

Lemma 4.11. *Let A be a Boolean algebra, let $P \subseteq A^+$, and let $I \subseteq s(P)$ be an ideal in A. Then P is a separable* (*eligible, stable, weakly stable, resp.*) *subset of A if and only if P/I is a separable* (*eligible, stable, weakly stable, resp.*) *subset of A/I.*

P r o o f. The definitions of these properties can be verified using Lemma 4.5. For properties "stable" and "weakly stable" Lemma 4.5 should be applied twice together with Lemma 4.10. For example:

$$\begin{aligned}\mathrm{dec}_A(P) &= h^{-1}(\mathrm{dec}_{h(A)}(h(P))),\\ \mathrm{dec}_A(\mathrm{dec}_A(P)\setminus s_A(P)) &= h^{-1}(\mathrm{dec}_{h(A)}(h(\mathrm{dec}_A(P)\setminus s_A(P))))\\ &= h^{-1}(\mathrm{dec}_{h(A)}(\mathrm{dec}_{h(A)}(h(P))\setminus s_{h(A)}(h(P)))).\end{aligned}$$

(The first equality holds because $I \subseteq s(P)$; for the second equality the inclusions $I \subseteq s(P) \subseteq s(\mathrm{dec}(P) \setminus s(P))$ should be exploited.) Therefore P is stable (left-hand sides equal) if and only if $h(P) = P/I$ is stable (right-hand sides equal). Weak stability can be verified in the same way. □

Theorem 4.12. *Let A be a Boolean algebra, let $P \subseteq A^+$, and let I be an ideal in A disjoint from P.*

(1) *$s(P)$ is the least P-regular ideal and*

$$\mathrm{dec}(P) = \mathrm{dec}(P \vartriangle s(P)) = \bigcup \mathrm{dec}_{A/s(P)}(P/s(P)).$$

(2) *$s(P \vartriangle I)$ is the least P-regular ideal containing I and*

$$\mathrm{dec}(P \vartriangle I) = \mathrm{dec}(P \vartriangle s(P \vartriangle I)) = \bigcup \mathrm{dec}_{A/s(P\vartriangle I)}(P/s(P \vartriangle I)).$$

Proof. If J is a P-regular ideal containing I, then by Lemma 4.5 (2) and Lemma 4.9, $s(P) \subseteq s(P \vartriangle J) = J$. On the other hand, by Lemma 4.10 (1), $s(P \vartriangle s(P)) = s(P)$. Therefore $s(P)$ is the least P-regular ideal. Then $s(P \vartriangle I)$ is $(P \vartriangle I)$-regular; it is P-regular because $I \subseteq s(P \vartriangle I)$ and $(P \vartriangle I)/s(P \vartriangle I) = P/s(P \vartriangle I)$. The ideal $s(P \vartriangle I)$ is the least one because if J is a P-regular ideal containing I, then $s(P \vartriangle I) \subseteq s(P \vartriangle J) = J$. By an obvious application of Lemma 4.5 and Lemma 4.10 (1) we obtain the remaining equalities. □

Theorem 4.13. *Let A be a Boolean algebra and let $P \subseteq A^+$.*

(1) *$P/s(P) \subseteq_r A/s(P)$.*

(2) *$\mathrm{dec}(P)/s(P) \subseteq_r A/s(P)$ if and only if P is a weakly stable subset of A.*

(3) *$\mathrm{dec}(P)/s(P) \subseteq_{rc} A/s(P)$ if and only if P is a stable subset of A.*

Proof. (1) holds by Theorem 4.12 (1).

(2) Let $\mathrm{dec}(P)/s(P) \subseteq_r A/s(P)$. Then $(\mathrm{dec}(P) \setminus s(P))/s(P) \subseteq_r A/s(P)$. By the minimality property of the ideal $s(\mathrm{dec}(P) \setminus s(P))$ from Theorem 4.12 (1) we obtain $s(\mathrm{dec}(P)\setminus s(P)) \subseteq s(P)$ and consequently P is weakly stable. Conversely, if P is weakly stable, then by (1) for $P' = \mathrm{dec}(P) \setminus s(P)$ we have $(\mathrm{dec}(P) \setminus s(P))/s(P) \subseteq_r A/s(P)$ because $s(P') = s(P)$.

(3) By (1), $s(P/s(P)) = \{0/s(P)\}$. Therefore, by Lemma 4.4 (3) applied for the factor algebra $A/s(P)$, $P/s(P)$ is stable if and only if $\mathrm{dec}(P/s(P)) \subseteq_{rc} A/s(P)$. By implication (1)(c) $\Longrightarrow$ (2)(a) from Lemma 4.10 for $I = s(P)$ we have $\mathrm{dec}(P)/s(P) = \mathrm{dec}(P/s(P))$, and by Lemma 4.11, P is stable if and only if $P/s(P)$ is stable. □

Lemmas 4.5, 4.9, and 4.10 (1) have the following special variants. Let us note that the inclusion $I \subseteq s(B \setminus I)$ is fulfilled for ideals I in A with base $I \cap B$.

Lemma 4.14. *Let B be a subalgebra of a Boolean algebra A, let I be an ideal in A, and let $h\colon A \to A/I$ be the natural homomorphism. Then*

(1) $\mathrm{dec}_A(B \setminus I) \subseteq \mathrm{dec}_A((B \triangle I) \setminus I) = h^{-1}(\mathrm{dec}_{A/I}((B/I)^+))$.

(2) $s_A(B \setminus I) \subseteq s_A((B \triangle I) \setminus I) = h^{-1}(s_{A/I}((B/I)^+))$.

(3) *B/I is a regular subalgebra of A/I if and only if $s_A((B \triangle I) \setminus I) = I$.*

(4) *In each of the inclusions in* (1) *and* (2) *the equality holds if and only if $I \subseteq s_A(B \setminus I)$.*

Proof. It is enough to consider $P = B \setminus I$ because $(B \setminus I) \triangle I = (B \triangle I) \setminus I$. □

In the same way also other parts of Lemma 4.10 can be rephrased. Theorem 4.12 (1) has the following form:

Corollary 4.15. *Let B be a subalgebra of a Boolean algebra A and let I be an ideal in A. Then $s(B \setminus I)$ is the least ideal J in A with respect to the inclusion such that $J \cap B = I \cap B$ and B/J is a regular subalgebra of A/J.*

Proof. By Theorem 4.12, $s(B \setminus I)$ is the least ideal J such that $J \cap (B \setminus I) = \emptyset$ and $(B \setminus I)/J \subseteq_r A/J$. Then $J \cap B = I \cap B$ and $(B \setminus I)/J = (B/J)^+$. □

Theorem 4.16. *Let B be a subalgebra of a Boolean algebra A and let I be an ideal in A. The following conditions are equivalent:*

(1) $\mathrm{dec}(B \setminus I) = B \triangle I$ *and* $s(B \setminus I) = I$.

(2) $\mathrm{dec}(B \setminus I) = B \triangle I$ *and* $I \subseteq s(B \setminus I)$.

(3) $B/I \subseteq_{rc} A/I$ *and* $I \subseteq s(B \setminus I)$.

Proof. We prove (2) $\Longrightarrow$ (3) $\Longrightarrow$ (1). If (2) holds, then we can apply the equality in Lemma 4.14 (1) and hence $\mathrm{dec}((B/I)^+) = \mathrm{dec}(B \setminus I)/I = (B \triangle I)/I = B/I$. By Lemma 4.4 (3), $B/I \subseteq_{rc} A/I$.

If (3) holds, then by Lemma 4.4 (3) and the equality in Lemma 4.14 (1), $B/I = \mathrm{dec}((B/I)^+) = \mathrm{dec}(B \setminus I)/I$. It follows that $\mathrm{dec}(B \setminus I) \subseteq B \triangle I \subseteq B \triangle s(B \setminus I) \subseteq \mathrm{dec}(B \setminus I)$. Therefore $\mathrm{dec}(B \setminus I) = B \triangle I$. The equality $s(B \setminus I) = I$ follows by minimality of $s(B \setminus I)$ in Corollary 4.15. □

Corollary 4.17. ([5: Theorem 2.7]) *Let A be a Boolean algebra, let B be a subalgebra of A, and let I be an ideal in A such that $I \subseteq B$. The following conditions are equivalent:*

(1) *There exists a stable set $P \subseteq A^+$ such that* $\mathrm{dec}(P) = B$ *and* $s(P) = I$.

(2) *There exists a separable set $P \subseteq A^+$ such that* $\mathrm{dec}(P) = B$ *and* $s(P) = I$.

(3) $\mathrm{dec}(B \setminus I) = B$.

(4) $B/I \subseteq_{rc} A/I$.

Proof. (1) $\iff$ (2) holds by Theorem 2.5; (1) $\iff$ (3) holds by stability of P and $B \setminus I$; (3) $\iff$ (4) holds by equivalence (1) $\iff$ (3) in Theorem 4.16, because now $B = B \triangle I$ and $s(B \setminus I) = I$. □

Lemma 4.18. *Let A be a Boolean algebra, let $P \subseteq A^+$, and let I be an ideal in A disjoint from P.*

(1) *If P is separable, then also $P \triangle I$ is separable.*

(2) *If P is eligible, then also $P \triangle I$ is eligible.*

Proof.

(1) If $P \subseteq \mathrm{dec}(P)$, then $P \triangle I \subseteq \mathrm{dec}(P) \triangle I \subseteq \mathrm{dec}(P \triangle I)$.

(2) Let P be eligible and Q separable with $P \sim Q$ (see Theorem 2.5). Then $P \triangle I \sim Q \triangle I$, and by (1), $Q \triangle I$ is separable. Therefore $P \triangle I$ is eligible. □

The inverse implications do not hold by Lemma 4.21.

Corollary 4.19. *Let A be a Boolean algebra, let P be an eligible subset of A^+, and let $I \subseteq A$ be an ideal disjoint from P. Then conditions* (2a)–(2c) *in* Lemma 4.10 *are equivalent to condition*

(d) $(\mathrm{dec}(P) \triangle I)/(s(P) \triangle I) \subseteq_{rc} A/(s(P) \triangle I)$.

Proof. We prove (c) $\iff$ (d). By Lemma 4.18, $P \triangle I$ is eligible and hence stable. If $\mathrm{dec}(P \triangle I) = \mathrm{dec}(P) \triangle I$, then by (3c), $s(P \triangle I) = s(P) \triangle I$, and hence by Theorem 4.13, $(\mathrm{dec}(P) \triangle I)/(s(P) \triangle I) = \mathrm{dec}(P \triangle I)/s(P \triangle I) \subseteq_{rc} A/(s(P) \triangle I)$. Conversely, assume (d) holds. By Lemma 2.3 (1), $\mathrm{dec}(P) \setminus (s(P) \triangle I) = \mathrm{dec}(P) \setminus s(P)$ because $s(P) \triangle I \subseteq s(P \triangle I)$ is disjoint from $P \subseteq P \triangle I$. By (d) and the equivalence (3) $\equiv$ (4) of Corollary 4.17 we have $\mathrm{dec}(P) \triangle I = \mathrm{dec}((\mathrm{dec}(P) \triangle I) \setminus (s(P) \triangle I)) = \mathrm{dec}((\mathrm{dec}(P) \setminus s(P)) \triangle I) = \mathrm{dec}(P \triangle I)$ because $P \sim \mathrm{dec}(P) \setminus s(P)$ and $P \triangle I \sim (\mathrm{dec}(P) \setminus s(P)) \triangle I$. □

Remark 4.20. Let A be a Boolean algebra and let us consider the following conditions for $P \subseteq A^+$:

(α) P is a regular subset of A.

(β) $\mathrm{dec}(P) \subseteq_{rc} A$.

(γ) $\mathrm{dec}(P)$ is a regular subalgebra of A.

There are six reasonable combinations of these properties because (β) $\Longrightarrow$ (γ) holds. Let us also note that (α) $\Longrightarrow$ (β) holds for stable P and (α) $\Longrightarrow$ (γ) holds for weakly stable P (see Lemma 4.4). Here are the examples:

($\alpha\beta\gamma$) Let $B \subseteq_{rc} A$. Then $P = B^+$ is a regular subset of A and $\mathrm{dec}(P) = B \subseteq_{rc} A$.

($\alpha\bar{\beta}\gamma$) If P is a non-stable weakly stable subset of A, then $P' = P/s(P)$ is a regular non-stable weakly stable subset of $A/s(P)$. Consequently, (α) and (γ) holds for P', but by Theorem 4.13, (β) does not hold for P'.

($\alpha\bar{\beta}\bar{\gamma}$) If P is a non-weakly stable subset of A, then $P' = P/s(P)$ is a regular non-weakly stable subset of $A/s(P)$. Consequently, (α) holds and (β) and (γ) do not hold for P'.

($\bar{\alpha}\beta\gamma$) If I is a nontrivial ideal in A, then $P = A \setminus I$ is not a regular subset of A, but $\mathrm{dec}(P) = A \subseteq_{rc} A$. We can also consider A to be a subalgebra of a Boolean algebra A' which is a direct sum of A and another Boolean algebra. Then the above P is not a regular subset of A' and $\mathrm{dec}_{A'}(P) = A \subseteq_{rc} A'$.

($\bar{\alpha}\bar{\beta}\gamma$) Let P and A satisfy ($\alpha\bar{\beta}\gamma$) and let A' be the disjoint sum of A and another Boolean algebra. Then P and A' satisfy ($\bar{\alpha}\bar{\beta}\gamma$).

($\bar{\alpha}\bar{\beta}\bar{\gamma}$) Let P and A satisfy ($\alpha\bar{\beta}\bar{\gamma}$) and let A' be the disjoint sum of A and another Boolean algebra. Then P and A' satisfy ($\bar{\alpha}\bar{\beta}\bar{\gamma}$).

Lemma 4.21. *Let A be a Boolean algebra, let $P \subseteq A^+$ and let I be an ideal in A disjoint from P maximal with respect to the inclusion. Then $P \triangle I$ is separable, $\mathrm{dec}(P \triangle I) = A$, $s(P \triangle I) = I$, and hence I is P-regular. If P is separable, then $\mathrm{dec}(P) \triangle I$ is a dense subalgebra of A.*

Proof. $P \triangle I$ is a dense subset of $A \setminus I$ because every $x \in A$ with no element from $P \triangle I$ below belongs to I. Hence $P \triangle I$ is separable and, by Lemma 2.7, $\mathrm{dec}(P \triangle I) = A$ and $s(P \triangle I) = I$. By Lemma 4.9, I is P-regular. If P is separable, then $P \triangle I \subseteq \mathrm{dec}(P) \triangle I$. □

Definition 4.22. Let $s^*_A(P)$ be the intersection of all ideals in A disjoint from P that are maximal with respect to the inclusion.

Lemma 4.23. $s^*(P) = s(P)$.

Proof. By Lemma 4.21 and by Theorem 4.12, $s(P) \subseteq s^*(P)$ because $s(P)$ is the least P-regular ideal in A disjoint from P. Assume that $x \in s^*(P) \setminus s(P)$. Let $u \in P$ be such that $(\forall v \in P)[v \leq u \implies v \wedge x \neq 0]$ and let $I \supseteq s(P) \cup \{u - x\}$ be an ideal disjoint from P maximal with respect to the inclusion. Then $u \in I$ because $u - x \in I$ and $x \in s^*(P) \subseteq I$. This contradiction proves that $s^*(P) = s(P)$. □

Question 4.24. Let A be a complete Boolean algebra. By Lemma 4.4 and Theorem 4.13, B is a complete subalgebra of A if and only if there is a regular stable subset $P \subseteq A^+$ such that $B = \mathrm{dec}(P)$. What subalgebras of A have the form $\mathrm{dec}(P)$ for some regular subset P of A?

5. Completeness of $\mathrm{dec}(P)/s(P)$

Definition 5.1. Let A be a Boolean algebra, let $P \subseteq A^+$, let κ be an infinite cardinal, and let $\overline{A}$ be the Boolean completion of A.

(1) We say that P has the *disjoint refinement property* (see [5]), if for every open dense subset D of P there exists an antichain $E \subseteq D$ in A which is P-maximal, i.e., $(\forall u \in P)(\exists v \in E)(\exists w \in P)(w \leq u \wedge v)$. (Then, $E \subseteq \mathrm{dec}(P)$.)

(2) We say that P has the *reduction property*, if for every open set $D \subseteq P$ there exists a dense subset E of P such that $E \cap D \perp E \setminus D$ in A.

(3) We say that P has the *strong reduction property in B* where $B \subseteq A$, if for every open set $D \subseteq P$ there exist an $x \in B$ and a dense subset E of P such that $E \cap D \leq x$ and $E \setminus D \perp x$ in A. (Clearly, $x \in \mathrm{dec}(P)$.)

(4) We say that P has the *orthogonalization property* if there is $Q \subseteq (\overline{A})^+$ such that $P \sim Q$ and Q has the disjoint refinement property in $\overline{A}$.

(5) We say that P is *κ-resolute*, if for any set $S \subseteq P$ of pairwise disjoint elements of size $< \kappa$, the set $D = \big\{u \in P:\ (\exists v \in S)(u \leq v) \text{ or } (\forall v \in S)(u \wedge v = 0)\big\}$ is a dense subset of P.

Remark 5.2. Let us note that a *category base* (see [6]), in our terminology, is a pair $(X, \mathcal{C})$ such that $\mathcal{C}$ is a $|\mathcal{C}|$-resolute subset of $\mathcal{P}(X) \setminus \{\emptyset\}$ and $X = \bigcup \mathcal{C}$.

The following lemma is easy and we leave the proof to the reader.

Lemma 5.3. *Let A be a Boolean algebra and let $P, Q \subseteq A^+$ be such that $P \sim Q$. If Q has the reduction property (strong reduction property in A, resp.), then P has the reduction property (strong reduction property in A, resp.).*

Lemma 5.4. *Let A be a Boolean algebra, let $P \subseteq A^+$, and let $\overline{A}$ be the Boolean completion of A.*

(1) *If P is a π-complete subset of A^+, then P is κ-resolute for all κ and P has the disjoint refinement property.*

(2) *If P is separable and $s(P)$ is a κ-complete ideal, then P is κ-resolute.*

(3) *If P is κ-resolute, then P is separable.*

(4) *P is ω-resolute if and only if P is separable.*

(5) *If P has the strong reduction property in A, then P has the reduction property.*

(6) *P has the reduction property if and only if P has the strong reduction property in $\overline{A}$.*

(7) *P has the strong reduction property in A if and only if P is eligible and $\mathrm{dec}(P) \setminus s(P)$ has the strong reduction property in A.*

(8) *If P has the reduction property, then P is potentially eligible.*

(9) *If P has the disjoint refinement property, then* $\operatorname{dec}(P)\cap P$ *is a dense subset of P with the disjoint refinement property and also* $\operatorname{dec}(P)\setminus s(P)$ *has the disjoint refinement property.*

(10) *If P is eligible and* $\operatorname{dec}(P)\setminus s(P)$ *has the disjoint refinement property, then P has the orthogonalization property.*

(11) *If P has the orthogonalization property, then P is potentially eligible and P has the reduction property.*

(12) *If P has a separable dense subset, then $P/s(P)$ has the disjoint refinement property. In fact, the following conditions are equivalent:*
   (a) *$P/s(P)$ has the disjoint refinement property.*
   (b) *$P/s(P)$ has a separable dense subset.*
   (c) *$P \triangle s(P)$ has a separable dense subset.*
   (d) *$P/s(P)$ has a π-complete dense subset.*

P r o o f. (1)–(6) are easy consequences of definitions, (8) is a consequence of (7) and (6), and (10) holds by definition.

(7) By Lemma 5.3, the strong reduction property is retained in similar sets and it is enough to prove that $\operatorname{dec}(P)\setminus s(P)\preceq P$, whenever P has the strong reduction property in A. Let $u\in P$. The set $D=\{v\in P:\ v\le u\}$ is an open subset of P. Let $x\in A$ and E be a dense subset of P such that $E\cap D\le x$ and $E\setminus D\perp x$. Then also $E\cap D\le x\wedge u$ and $E\setminus D\perp x\wedge u$. Hence $x\wedge u\in\operatorname{dec}(P)\setminus s(P)$ and $x\wedge u\le u$.

(9) If P has the disjoint refinement property, then for every $u\in P$ there is a P-maximal antichain $E\subseteq\big\{v\in P:\ (v\le u)\ \text{or}\ (\forall v'\le v)(v'\not\le u)\big\}$. Then $E\subseteq\operatorname{dec}(P)$ and E contains a $v\le u$. Therefore $\operatorname{dec}(P)\cap P$ is a dense subset of P and a dense subset of $\operatorname{dec}(P)\setminus s(P)$ and $\operatorname{dec}(P)\cap P$ has the disjoint refinement property.

(11) We prove that P has the reduction property. Then, by (8), P is potentially eligible. By Lemma 5.3, the reduction property is retained in similar sets and, by (6), without loss of generality we can assume that P has the disjoint refinement property. If D is an open subset of P, then $D'=D\cup\big\{u\in P:\ (\forall v\in D)(v\not\le u)\big\}$ is open dense subset of P. Let E' be a disjoint refinement for D'. Then the set $E=\big\{u\in P:\ (\exists v\in E')(u\le v)\big\}$ is dense.

(12) We prove (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) $\Longrightarrow$ (d) $\Longrightarrow$ (a). If $P/s(P)$ has the disjoint refinement property, then by (7), $P/s(P)$ has a separable dense subset $Q/s(P)$ for some $Q\subseteq P$. Then $Q\triangle s(P)$ is a separable dense subset of $P\triangle s(P)$. It follows that $Q/s(P)$ is a π-complete dense subset of $P/s(P)$ and, by (1), this implies that $P/s(P)$ has the disjoint refinement property. □

If P is κ-resolute and every disjoint system $E \subseteq P$ has size $< \kappa$, then P is λ-resolute for all λ and the following lemma can be applied.

Lemma 5.5. ([5,6]) *Let A be a Boolean algebra and let $P \subseteq A^+$.*

(1) *If P is a κ-resolute subset of A^+ and P has a dense subset of size $\leq \kappa$, then P has the disjoint refinement property.*

(2) *If P is a separable subset of A^+ and P has a dense subset of size $\leq \omega$, then P has the disjoint refinement property.*

Proof. We prove (1); (2) is a special case of (1) by Lemma 5.4 (3). Let us assume that $D = \{u_\alpha : \alpha < \kappa\}$ is a dense subset of P. By induction on $\alpha < \kappa$ construct a sequence of elements $v_\alpha \in D$ such that for each $\alpha < \kappa$, $v_\alpha \leq u_\alpha$ and either $v_\alpha = v_\xi$ for some $\xi < \alpha$ and then there is $u \in P$ such that $u \leq u_\alpha \wedge v_\xi$, or $v_\alpha \wedge v_\xi = 0$ for all $\xi < \alpha$. In the induction step use the κ-resoluteness of P. Then $E = \{v_\xi : \xi < \kappa\}$ is a P-maximal antichain. □

Remark 5.6. If I is a κ-complete ideal in a Boolean algebra B and B/I has κ-c.c., then $B \setminus I$ has the disjoint refinement property. By Lemma 5.4 (11), B/I has also the reduction property. If, moreover, B is κ-complete, then $B \setminus I$ has the strong reduction property in B. By the next theorem this holds because B/I is complete.

Theorem 5.7. *Let B be a Boolean algebra and let I be an ideal in B.*

(1) *B/I is complete if and only if $B \setminus I$ has the strong reduction property in B.*

(2) *If B is complete, then B/I is complete if and only if $B \setminus I$ has the reduction property.*

Proof.

(1) Assume that B/I is complete and let D be an open subset of $B \setminus I$. Let $x \in B$ be such that $[x]_I$ is the least upper bound for D/I. Let $E_1 = \{u \in D : u \leq x\}$, $E_2 = \{u \in B \setminus I : u \wedge x = 0\}$, and $E = E_1 \cup E_2$. Then E is a dense subset of $B \setminus I$, $E \cap D = E_1 \leq x$, and $E \setminus D = E_2 \perp x$. Therefore $B \setminus I$ has the strong reduction property.

Conversely, let us assume that $B \setminus I$ has the strong reduction property. We prove that for every $R \subseteq B \setminus I$, the set R/I has the least upper bound in B/I. The set $D = \{u \in B \setminus I : (\exists r \in R)(u \leq r)\}$ is open subset of $B \setminus I$ and hence there are a dense set $E \subseteq B \setminus I$ and $x \in B$ such that $E \cap D \leq x$ and $E \setminus D \perp x$.

We claim that $r - x \in I$ for all $r \in R$. Otherwise there is $u \in E$ such that $u \leq r - x$. Then $u \in E \cap D$ and $u \leq x$ which is a contradiction. Hence $[x]_I$ is an upper bound for R/I and we prove that $[x]_I$ is the least upper bound. Let $y \in B$ be arbitrary such that $r - y \in I$ for all $r \in R$ and we show that $x - y \in I$. On the contrary assume that $x - y \in B \setminus I$ and hence there is $u \in E$ such that $u \leq x - y$. If $u \in E \cap D$, then there is $r \in R$ such that $u \leq r$ and so

$u \le r \wedge (x - y) \le r - y \in I$; if $u \in E \setminus D$, then $u = u \wedge (x - y) \le u \wedge x = 0$. This is a contradiction because $E \cap I = \emptyset$.

(2) As B is complete, by Lemma 5.4 (6), $B \setminus I$ has the reduction property (in B) if and only if it has the strong reduction property in B. □

One of the results of [5] says that if A is a complete Boolean algebra and P is a separable subset of A^+ with the disjoint refinement property, then $\mathrm{dec}(P)/s(P)$ is complete. Now we have a closer approximation of the property "$\mathrm{dec}(P)/s(P)$ is complete":

Theorem 5.8. *Let A be a Boolean algebra and let $P \subseteq A^+$. Then P has the strong reduction property in A if and only if P is eligible and the Boolean algebra* $\mathrm{dec}(P)/s(P)$ *is complete.*

Proof. By Lemma 5.4 (7) and Theorem 5.7, the following conditions are equivalent:

P has the strong reduction property in A (equivalently, in $\mathrm{dec}(P)$).

P is eligible and $\mathrm{dec}(P) \setminus s(P)$ has the strong reduction property in $\mathrm{dec}(P)$.

P is eligible and $\mathrm{dec}(P)/s(P)$ is complete.

□

Remark 5.9. Let B be a subalgebra of A and let I be an ideal in A. Then $I \cap B$ is an ideal in B and $B/(I \cap B) \simeq B/I = (B \triangle I)/I$. By Theorem 5.7, $B \setminus I$ has the strong reduction property in B if and only if $(B \triangle I) \setminus I$ has the strong reduction property in $B \triangle I$.

Theorem 5.10. *Let B be a subalgebra of a Boolean algebra A and let I be an ideal in B. If B/I is complete, then* $\mathrm{dec}(B \setminus I) = B \triangle s(B \setminus I)$ *and* $B/I \simeq B/s(B \setminus I) \subseteq_{rc} A/s(B \setminus I)$.

Proof. $B/s(B \setminus I)$ is isomorphic to the complete algebra B/I because $s(B \setminus I) \cap B = I$ and hence it is complete. By Theorem 5.7, $B \setminus I$ has the strong reduction property in A and, by Theorem 5.8, $\mathrm{dec}(B \setminus I)/s(B \setminus I)$ is complete. By Lemma 2.3 (2) the separable set $B \setminus I$ is a dense subset of $\mathrm{dec}(B \setminus I) \setminus s(B \setminus I)$. Since $B \setminus s(B \setminus I) = B \setminus I$, the algebra $B/s(B \setminus I)$ is a dense subalgebra of $\mathrm{dec}(B \setminus I)/s(B \setminus I)$ and being complete, these two algebras are equal. It follows that $\mathrm{dec}(B \setminus I) = B \triangle s(B \setminus I)$ and, by Theorem 4.16, $B/s(B \setminus I) \subseteq_{rc} A/s(B \setminus I)$. □

Theorem 5.11. *Let A be a Boolean algebra, let $P \subseteq A^+$, and let I be an ideal in A disjoint from P.*

(1) *If P has the strong reduction property in A, then P is stable and $P \triangle I$ has the strong reduction property in* $\mathrm{dec}(P)$.

(2) *If P is stable and $P \triangle I$ has the strong reduction property in* $\mathrm{dec}(P)$, *then*

(a) $\operatorname{dec}(P \triangle I) = \operatorname{dec}(P) \triangle s(P \triangle I)$,
(b) $\operatorname{dec}(P)/s(P) \simeq \operatorname{dec}(P)/s(P \triangle I) = \operatorname{dec}(P \triangle I)/s(P \triangle I)$ *and these Boolean algebras are complete,*
(c) $\operatorname{dec}(P)/s(P) \subseteq_{rc} A/s(P)$ *and* $\operatorname{dec}(P)/s(P \triangle I) \subseteq_{rc} A/s(P \triangle I)$.

P r o o f.

(1) If P has the strong reduction property in A, then P has the strong reduction property also in $\operatorname{dec}(P)$. P is stable by Lemma 5.4. Let D be an open subset of $P \triangle I$. The set $H = \{u \in P : (\exists x \in I)(u - x \in D)\}$ is an open subset of P. Hence there is $z \in \operatorname{dec}(P)$ and a dense subset E of P such that $E \cap H \leq z$ and $E \setminus H \perp z$. For every $u \in E \cap H$ choose $x_u \in I$ such that $u - x_u \in D$. Let

$$E_1 = \{v \in D : (\exists u \in E \cap H)(v \leq u - x_u)\}, \qquad E_2 = (E \setminus H) - I.$$

Then $E_1 \leq z$, $E_1 \subseteq D$, $E_2 \perp z$, $E_2 \cap D = \emptyset$, and $E_1 \cup E_2$ is a dense subset of $P \triangle I$. (For $v \triangle x \in P \triangle I$ there is $u \in E$ such that $u \leq v$ and $u - x \leq v \triangle x$. If $u \in E \cap H$, then $u - x_u - x \in E_1$; if $u \in E \setminus H$, then $u - x \in E_2$.) This proves that $P \triangle I$ has the strong reduction property in $\operatorname{dec}(P)$.

(2) We prove $\operatorname{dec}(P\triangle I) \subseteq \operatorname{dec}(P)\triangle s(P\triangle I)$ (the reverse inclusion is obvious). Let $x \in \operatorname{dec}(P \triangle I)$. Then the union D of the open subsets $D_1 = \{u \in P \triangle I : u \leq x\}$ and $D_2 = \{u \in P \triangle I : u \wedge x = 0\}$ of $P \triangle I$ is open dense. As $P \triangle I$ has the strong reduction property in $\operatorname{dec}(P)$, there are $y \in \operatorname{dec}(P)$ and a dense set $E \subseteq P\triangle I$ such that $E \cap D_1 \leq y$ and $E \setminus D_1 \perp y$. The set $E' = (E\cap D_1) \cup (E \cap D_2)$ is dense and $E' \perp x \triangle y$. Therefore $x \triangle y \in s(P \triangle I)$ and $x \in \operatorname{dec}(P) \triangle s(P \triangle I)$ because $x = y \triangle (x \triangle y)$.

By Lemma 2.3, $s(P \triangle I) \cap \operatorname{dec}(P) = s(P)$ because $s(P) \subseteq s(P \triangle I)$ and $P \cap s(P\triangle I) \subseteq (P\triangle I) \cap s(P\triangle I) = \emptyset$. Therefore $\operatorname{dec}(P)/s(P) \simeq \operatorname{dec}(P)/s(P\triangle I) = \operatorname{dec}(P \triangle I)/s(P \triangle I)$. Since $P \triangle I$ has the strong reduction property in $\operatorname{dec}(P)$, by Theorem 5.8 these algebras are complete and since P and $P \triangle I$ are stable, by Theorem 4.13, $\operatorname{dec}(P)/s(P) \subseteq_{rc} A/s(P)$ and $\operatorname{dec}(P\triangle I)/s(P\triangle I) \subseteq_{rc} A/s(P \triangle I)$. □

In a similar way it is possible to prove that if P has the reduction property or the disjoint refinement property, then $P \triangle I$ has, respectively, the reduction property or the disjoint refinement property (see [5] for the case of disjoint refinement property).

Recall that P-regular ideals are the ideals of the form $s(P \triangle I)$ for an ideal $I \subseteq A^+$ disjoint from P. We can rephrase Theorem 5.11 as follows:

Corollary 5.12. *Let A be a Boolean algebra, let $P \subseteq A^+$ have the strong reduction property in A, and let I be a P-regular ideal in A. Then* $\operatorname{dec}(P \triangle I) = \operatorname{dec}(P) \triangle I$, $\operatorname{dec}(P)/I \subseteq_{rc} A/I$, $\operatorname{dec}(P)/s(P) \subseteq_{rc} A/s(P)$, $\operatorname{dec}(P)/I \simeq \operatorname{dec}(P)/s(P)$, *and* $\operatorname{dec}(P)/I$ *and* $\operatorname{dec}(P)/s(P)$ *are complete.*

***Example* 5.13.** Let $A = \mathcal{P}(\mathbb{R})$ and let I_0 be either the ideal $\mathcal{N}$ of measure zero sets or the ideal $\mathcal{M}$ of meager sets of reals or $\mathcal{N} \cap \mathcal{M}$. The factor algebra Borel$/I_0$ is complete hence the set $P = \text{Borel} \setminus I_0$ has the strong reduction property in $\mathcal{P}(\mathbb{R})$. One can verify that $s(P) = I_0$ and hence $\operatorname{dec}(\text{Borel} \setminus I_0) = \text{Borel} \bigtriangleup I_0$. If an ideal $I \subseteq \mathcal{P}(\mathbb{R})$ is disjoint from P and maximal with respect to the inclusion, then I is P-regular and hence $\text{Borel}/I_0 \cong \text{Borel}/I \subseteq_{rc} \mathcal{P}(\mathbb{R})/I$.

Corollary 5.14. *Let $P \subseteq A^+$ have the strong reduction property in A and let I be a P-regular ideal in A. Then* $\operatorname{cmpl}(\operatorname{dec}(P) \bigtriangleup I) \geq \operatorname{cmpl}(I)$.

Proof. By Theorem 2.10 (2) for $P \bigtriangleup I$ instead of P because $\operatorname{dec}(P \bigtriangleup I) = \operatorname{dec}(P) \bigtriangleup I$ and $s(P \bigtriangleup I) = I$. □

***Example* 5.15.** Let $A = \mathcal{P}(\mathbb{R})$ and $P = \text{Open}(\mathbb{R}) \setminus \{\emptyset\}$. Then P has the strong reduction property, $\operatorname{dec}(P)$ is the algebra of sets with nowhere dense boundary (see Example 4.3 (1)). The ideal $\mathcal{M}$ of meager sets is P-regular because $\mathcal{M} = s(P \triangle \mathcal{M})$. It is a well known fact that if the ideal $\mathcal{M}$ is κ-complete, then the algebra of sets with the Baire property $\operatorname{dec}(P) \bigtriangleup \mathcal{M} = \text{Open}(\mathbb{R}) \bigtriangleup \mathcal{M}$ is κ-complete.

6. The first category ideal

Definition 6.1. Let A be a Boolean algebra and let $P \subseteq A^+$. We say that an ideal I in A is a *P-Banach ideal*, if for every $x \in A$,

$$x \in I \iff (\forall u \in P)(\exists v \in P)[v \leq u \ \ \& \ \ v \wedge x \in I]. \tag{$*$}$$

We define $\text{Baire}(P, I)$ to be the set of all $x \in A$ such that

$$(\forall u \in P)(\exists v \in P)\big[v \leq u \ \ \& \ \ (v \wedge x \in I \ \text{ or } \ v - x \in I)\big]. \tag{$**$}$$

We define $P_I = \big\{u \in P : \ (\forall v \in P)[v \leq u \implies v \notin I]\big\}$.

For example, the σ-ideal of meager sets in a topological space is a P-Banach ideal where P is the family of open sets (see [1]).

Remark 6.2. By ($*$), the property "P-Banach ideal" is absolute for subalgebras: If $P \subseteq A \subseteq A'$ and I is a P-Banach ideal in A', then $I \cap A$ is a P-Banach ideal in A. By Lemma 4.8 the property "P-regular ideal" is absolute in the same sense.

Notice that if I is disjoint from P, then for all $x \in A$,

$$x \in s(P \bigtriangleup I) \iff (\forall u \in P)(\exists v \in P)[v - u \in I \ \ \& \ \ v \wedge x \in I], \tag{$\dagger$}$$

$$x \in \operatorname{dec}(P \bigtriangleup I) \iff (\forall u \in P)(\exists v \in P)\big[v - u \in I \ \ \& \ \ (v \wedge x \in I \ \text{ or } \ v - x \in I)\big]. \tag{$\ddagger$}$$

Lemma 6.3. *Let I be an ideal in A and let $P \subseteq A^+$.*

(1) $I \cap P_I = \emptyset$.
(2) $s(P) = s(P_I) \cap s(P \cap I)$ *and* $\mathrm{dec}(P) = \mathrm{dec}(P_I) \cap \mathrm{dec}(P \cap I)$.
(3) *I is P-Banach if and only if I is P_I-Banach.*
(4) $\mathrm{Baire}(P, I) = \mathrm{Baire}(P_I, I)$.
(5) *$s(P)$ is the least P-Banach ideal and $s(P_I)$ is the least P-Banach ideal containing $P \cap I$.*
(6) *If P has any of the following properties, then P_I has this property, too: separable, π-complete, eligible, disjoint refinement property, reduction property, strong reduction property, orthogonalization property, κ-resolution property.*
(7) *If P has the strong reduction property in A, then there is $x \in \mathrm{dec}(P)$ such that $s(P_I) = s(P) \triangle J_x$ and $\mathrm{dec}(P_I) = \mathrm{dec}(P) \triangle J_x$, where $J_x = \{y \in A : y \leq x\}$. If, moreover, I is P-Banach, then $x \in I$.*

P r o o f. (1) is trivial and (2) follows by Lemma 2.3 (6). In all proofs use the fact that P_I is an open subset of P and $P_I \cup (P \cap I)$ is a dense subset of P. In (5) we can verify by $(*)$ that $s(P)$ is the least P-Banach ideal, $s(P_I)$ is a P-Banach ideal, and $s(P_I) \subseteq I$ for every P-Banach ideal I. In (6) assume, for example, that P is eligible. For every $u \in P_I$ there is $x \in \mathrm{dec}(P) \setminus s(P)$ such that $x \leq u$. By (2), $x \in \mathrm{dec}(P_I)$ and $x \notin s(P_I)$ because there is $v \in P$ such that $v \leq x$ and $v \in P_I$ because $v \leq u$. Therefore P_I is eligible.

(7) Let P have the strong reduction property in A. There are $x \in \mathrm{dec}(P)$ and a dense set $E \subseteq P$ such that $E \cap I \leq x$ and $E \setminus I \perp x$. As $E \setminus I$ is a dense subset of P_I, $x \in s(P_I)$, and by (2), $s(P) \triangle J_x \subseteq s(P_I)$ and $\mathrm{dec}(P) \triangle J_x \subseteq \mathrm{dec}(P_I)$. To prove the inverse inclusions, using the set E verify that, if $y \in s(P_I)$, then $y - x \in s(P)$ and, if $y \in \mathrm{dec}(P_I)$, then $y - x \in \mathrm{dec}(P)$. If I is P-Banach, then $s(P_I) \subseteq I$ and hence $x \in I$. □

Lemma 6.4. *Let A be a Boolean algebra, let $P \sim Q$ be subsets of A^+, and let I be an ideal in A.*

(1) *I is P-regular if and only if I is Q-regular.*
(2) *I is P-Banach if and only if I is Q-Banach.*
(3) $\mathrm{Baire}(P, I) = \mathrm{Baire}(Q, I)$.

P r o o f. Recall that I is P-regular if and only if $I \cap P = \emptyset$ and $s(P \triangle I) \subseteq I$. Therefore all assertions can be verified by characterizations $(*)$, $(**)$, $(\dagger)$, $(\ddagger)$. □

Theorem 6.5. *Let I be an ideal in a Boolean algebra A and let $P \subseteq A^+$ be eligible.*

(1) *I is P-Banach if and only if I is P_I-regular if and only if $I = s(P_I \triangle I)$.*

(2) $\mathrm{Baire}(P, I) = \mathrm{dec}(P_I \triangle I)$.

(3) *$s(P_I \triangle I)$ is the least P-Banach ideal containing I disjoint from P_I.*

P r o o f. There is a separable set Q such that $P \sim Q$. Then $P_I \sim Q_I$, Q_I and $Q_I \triangle I$ are separable, and $P_I \triangle I \sim Q_I \triangle I$. By Lemma 6.3 and Lemma 6.4 we can reduce the proof of to a separable set Q. Therefore, without loss of generality let us assume that P is separable and I is disjoint from P; hence $P_I = P$.

Every $x \in A$ satisfying the right-hand side of $(*)$ or $(**)$ satisfies, respectively, also the right-hand side of (†) or (‡). If $u, v \in P$ and $v - u \in I$, by separability of P there is $v' \leq v$ in P such that $v' \leq u$. Hence the right-hand sides of the equivalences $(*)$ and (†) (and the right-hand sides of $(**)$ and (‡)) are equivalent for P. This finishes the proof of (1) and (2).

By (1), for an ideal J such that $P \cap I = P \cap J$, the ideal J is P-Banach if and only if J is a P_I-regular. The ideal $J = s(P_I \triangle I)$ is the least P_I-regular ideal containing I and hence $s(P_I \triangle I)$ is the least P-Banach ideal containing I and disjoint from P_I. □

If we do not assume that P is eligible, then we have the following:

Corollary 6.6. *Let I be an ideal in a Boolean algebra A and let $P \subseteq A^+$.*

(1) *If I is P-regular, then I is P-Banach.*

(2) $\mathrm{dec}(P_I) \triangle I \subseteq \mathrm{Baire}(P, I) \subseteq \mathrm{dec}(P_I \triangle I)$.

(3) *$s(P_I \triangle I)$ is a P_I-regular and a P-Banach ideal.*

In [6] the ideal I of meager sets for a category base $(X, \mathcal{C})$ is defined as the σ-ideal generated by $s(\mathcal{C})$ and the family of Baire sets is defined to be $\mathrm{Baire}(\mathcal{C}, I)$ in $\mathcal{P}(X)$.

Definition 6.7. Let $\overline{A}$ denote the Boolean completion of A. Let $I(P)$ be the set of those $x \in A$ that there exist $x_n \in s_{\overline{A}}(P)$ for $n \in \omega$ such that $x = \bigvee\{x_n : n \in \omega\}$ (computed in $\overline{A}$). An element $x \in I(P)$ is called an element of *first category*. We write $I_A(P)$ instead of $I(P)$ if the algebra A is not obvious from the context. By definition, $I_A(P) = I_{\overline{A}}(P) \cap A$.

Remark 6.8. This definition extends the definition of meager elements from [5] for noncomplete Boolean algebras. We can see that $s(P) \subseteq I(P)$ and, if $s_{\overline{A}}(P) \subseteq s_{\overline{A}}(Q)$, then $I(P) \subseteq I(Q)$. Clearly, $x \in I(P)$ if and only if $x \in A$ and there exist open dense subsets $D_n \subseteq P$ for $n \in \omega$ such that $x = \bigvee\big\{y \leq x : (\exists n \in \omega)(D_n \perp y)\big\}$. The property "$x$ is an element of first category" is absolute for regular subalgebras: If $P \subseteq A \subseteq_r A'$, then $I_A(P) = I_{A'}(P) \cap A$. In particular, if A is ω_1-complete, then the ideal $I(P)$ is ω_1-complete.

Lemma 6.9. *If P is a stable subset of a Boolean algebra A and $I(P) \cap P = \emptyset$, then* $\mathrm{cmpl}(s(P)) \geq \omega_1$ *if and only if* $\mathrm{cmpl}(\mathrm{dec}(P)) \geq \omega_1$.

P r o o f. The lemma is a consequence of Theorem 2.10 (2) because $\mathrm{cov}(s(P), P) \geq \omega_1$ holds by the hypotheses. □

Theorem 6.10. *Let I be an ideal in A, let $P \subseteq A^+$ have the strong reduction property in A, and for $x \in A$ let $J_x = \{y \in A : \ y \leq x\}$.*

(1) *There is $x \in \mathrm{dec}(P)$ such that $s(P_{I(P)}) = s(P) \triangle J_x$, $\mathrm{dec}(P_{I(P)}) = \mathrm{dec}(P) \triangle J_x$, and $I(P_{I(P)}) = I(P) \triangle J_x$.*

(2) *$P_{I(P_{I(P)})} = P_{I(P)}$ and $P_{I(P_{I(P)})} \triangle I(P_{I(P)}) \sim P_{I(P)} \triangle I(P)$.*

(3) $\mathrm{Baire}(P, I(P_{I(P)})) = \mathrm{Baire}(P, I(P))$.

(4) *If $I(P)$ is P-Banach, then $I(P_{I(P)}) = I(P)$.*

P r o o f. (1) and (4) By Lemma 6.3 (7) there is $x \in \mathrm{dec}(P)$ such that $s(P_{I(P)}) = s(P) \triangle J_x$ and $\mathrm{dec}(P_{I(P)}) = \mathrm{dec}(P) \triangle J_x$. The inclusion $I(P) \triangle J_x \subseteq I(P_{I(P)})$ holds because $s_{\overline{A}}(P) \subseteq s_{\overline{A}}(P_{I(P)})$, hence $I(P) \subseteq I(P_{I(P)})$, and $x \in s(P_{I(P)}) \subseteq I(P_{I(P)})$. For the inverse inclusion let us assume that $z \in I(P_{I(P)})$. Then $z = \bigvee_{n\in\omega} z_n$ (in $\overline{A}$) for some $z_n \in s_{\overline{A}}(P_{I(P)}) = s_{\overline{A}}(P) \triangle J_x$, i.e., $z_n - x \in s_{\overline{A}}(P)$. Then $z - x = \bigvee_{n\in\omega} (z_n - x) \in I(P)$ and so $z \in I(P) \triangle J_x$. If $I(P)$ is P-Banach, then by Lemma 6.3, $x \in I(P)$ and $I(P) \triangle J_x = I(P)$.

(2)–(3) We consider the x from part (1). There is a dense set $D \subseteq P_{I(P)}$ such that $D \perp x$ because $x \in s(P_{I(P)})$. Then $P_{I(P)} \cap (I(P) \triangle J_x) = \emptyset$ and, by (1), $P_{I(P_{I(P)})} = P_{I(P)\triangle J_x} = P_{I(P)}$ and $P_{I(P)} \triangle I(P_{I(P)}) = P_{I(P)} \triangle I(P) \triangle J_x \sim P_{I(P)} \triangle I(P)$. By Theorem 6.5, $\mathrm{Baire}(P, I(P_{I(P)})) = \mathrm{dec}(P_{I(P_{I(P)})} \triangle I(P_{I(P)})) = \mathrm{dec}(P_{I(P)} \triangle I(P)) = \mathrm{Baire}(P, I(P))$. □

Theorem 6.11. ([5]) *If $P \subseteq A^+$ has the orthogonalization property, then $I(P)$ is P-Banach, $I(P_{I(P)}) = I(P)$, and $I(P)$ is $P_{I(P)}$-regular.*

P r o o f. Let $\overline{A}$ denote the Boolean completion of A. Since P has the orthogonalization property there is a separable $Q \subseteq (\overline{A})^+$ with the disjoint refinement property such that $P \sim Q$ and then $I_{\overline{A}}(P) = I_{\overline{A}}(Q)$. By Lemma 6.4 and by the absoluteness mentioned in Remark 6.2 and Remark 6.8, without loss of generality we can assume that A is complete and P is separable and has the disjoint refinement property.

Let $x \in A$ and let D be an open dense subset of P such that $u \wedge x \in I(P)$ for all $u \in D$. Let $E \subseteq D$ be a P-maximal antichain in A. For every $u \in E$, since $u \wedge x \in I(P)$, there is a sequence $\{x_n(u) : \ n \in \omega\} \subseteq s(P)$ such that $u \wedge x = \bigvee_{n\in\omega} x_n(u)$. As E is a P-maximal antichain $\bigvee_{u\in E} x_n(u) \in s(P)$ for all $n \in \omega$

and $-\bigvee E \in s(P)$. Therefore, $x \le -\bigvee E \vee \bigvee_{u\in E} u \wedge x = -\bigvee E \vee \bigvee_{n\in\omega}\Big(\bigvee_{u\in E} x_n(u)\Big)$ $\in I(P)$. This proves that $I(P)$ is P-Banach. Since $s(P_{I(P)})$ is the least P-Banach ideal containing $P \cap I(P)$, $s(P_{I(P)}) \subseteq I(P)$ and consequently $I(P_{I(P)}) \subseteq I(P)$. $I(P)$ is $P_{I(P)}$-regular by Theorem 6.5. □

Definition 6.12 (The Banach-Mazur game). ([7]) Let $P \subseteq A^+$ and $x \in A$. Players I and II in the game $\Gamma(P,x)$ construct an infinite sequence $\langle u_n : n \in \omega\rangle$ of elements of P such that $u_{n+1} \le u_n$ for all n. Player I chooses u_n for even n (starting with u_0) and player II chooses u_n for odd n. Each of the players when choosing u_n knows only $\langle u_k : k < n\rangle$. Player I wins if there is $z \in A$ such that $z \le x$ and $z \le u_n$ for all n; otherwise II wins. Let $I^{\mathrm{BM}}(P) = \big\{x \in A :$ II has a winning strategy in $\Gamma(P,x)\big\}$.

Clearly, $I^{\mathrm{BM}}(P)$ is absolute for regular subalgebras. One of the benefits of the game definition of first category is the P-Banach property.

Lemma 6.13. *Let A be a Boolean algebra and let $P \subseteq A^+$.*

(1) $I^{\mathrm{BM}}(P)$ *is P-Banach.*
(2) $I(P) \subseteq I^{\mathrm{BM}}(P) = I^{\mathrm{BM}}(P_{I^{\mathrm{BM}}(P)})$.
(3) *If $P \sim Q$, then* $I^{\mathrm{BM}}(P) = I^{\mathrm{BM}}(Q)$.
(4) *If each decreasing sequence of elements of P has a lower bound in A^+, then* $I(P) \cap P = I^{\mathrm{BM}}(P) \cap P = \emptyset$.
(5) $\mathrm{cmpl}(I^{\mathrm{BM}}(P)) \ge \omega_1$.
(6) $I^{\mathrm{BM}}(P_{I^{\mathrm{BM}}(P)}) = I^{\mathrm{BM}}(P_{I^{\mathrm{BM}}(P)} \vartriangle I^{\mathrm{BM}}(P))$ *provided that P is potentially eligible.*

Proof.

(1) If $x \in A$ and $x \wedge u \in I^{\mathrm{BM}}(P)$ for a dense set D of $u \in P$, then player II wins the game $\Gamma(P,x)$, if in his first move he chooses $u \in D$ and later he plays by his winning strategy for the game $\Gamma(P, x\wedge u)$. Hence $x \in I^{\mathrm{BM}}(P)$.

(2) Let $x \in I(P)$ and let $x = \bigvee_{n\in\omega} x_n$ with $x_n \in s_{\overline{A}}(P)$. Let $D_n \subseteq P$ be dense subsets such that $x_n \perp D_n$ for $n \in \omega$. Then the strategy of player II by which in his nth move he chooses $u_{2n+1} \in D_n$ is a winning strategy in $\Gamma(P,x)$. Therefore $x \in I^{\mathrm{BM}}(P)$ and $I(P) \subseteq I^{\mathrm{BM}}(P)$.

Any winning strategy of player II in the game $\Gamma(P,x)$ is also the winning strategy in $\Gamma(P_{I^{\mathrm{BM}}(P)}, x)$ because $P_{I^{\mathrm{BM}}(P)}$ is an open subset of P. Therefore $I^{\mathrm{BM}}(P) \subseteq I^{\mathrm{BM}}(P_{I^{\mathrm{BM}}(P)})$. We prove the inverse inclusion. If $x \in I^{\mathrm{BM}}(P_{I^{\mathrm{BM}}(P)})$, then player II wins $\Gamma(P,x)$, if he chooses $u_1 \in P_{I^{\mathrm{BM}}(P)} \cup (P \cap I^{\mathrm{BM}}(P))$ and then, if $u_1 \in P_{I^{\mathrm{BM}}(P)}$, he follows his winning strategy for the game $\Gamma(P_{I^{\mathrm{BM}}(P)}, x)$, and if $u_1 \in P \cap I^{\mathrm{BM}}(P)$, he follows his winning strategy for $\Gamma(P, x \wedge u_1)$.

(3) and (4) are obvious.

(5) Assume that $x = \bigvee_n x_n$ and player II has winning strategies in all games $\Gamma(P, x_n)$ for all $n \in \omega$. Let $\langle N_n : n \in \omega \rangle$ be a partition of ω into infinite sets. The player II wins in $\Gamma(P, x)$ if his move u_{2k+1} when $k \in N_n$ will follow the winning strategy in $\Gamma(P, x_n)$ with the information $\langle u_0 \rangle^\frown \langle u_{2i-1}, u_{2i} : 0 < i \leq k$ and $i \in N_n \rangle$.

(6) Since $I^{\mathrm{BM}}(P) = I^{\mathrm{BM}}(P_{I^{\mathrm{BM}}(P)})$ without loss of generality we can assume that $I^{\mathrm{BM}}(P) \cap P = \emptyset$. By (3) and by absoluteness of definition of $I^{\mathrm{BM}}(P)$ for regular subalgebras, without loss of generality we can assume that P is separable. Therefore in the proof we assume that P is separable and $I^{\mathrm{BM}}(P) \cap P = \emptyset$ and we prove $I^{\mathrm{BM}}(P) = I^{\mathrm{BM}}(P \triangle I^{\mathrm{BM}}(P))$.

If $x \in I^{\mathrm{BM}}(P)$, then player II wins the game $\Gamma(P \triangle I^{\mathrm{BM}}(P), x)$ using this strategy: Assume $\langle v_k : k \leq 2n \rangle$ is a decreasing sequence of moves of the the players where $v_k = (u_k - x_k) \vee (y_k - u_k)$ for some $u_k \in P$ and $x_k, y_k \in I^{\mathrm{BM}}(P)$. At the beginning of his $(2n+1)$th move, using separability of P, player II finds $u'_{2n} \in P$ such that $u'_{2n} \leq u_{2n-1}$ and $u'_{2n} \leq u_{2n}$, and then he uses his winning strategy in $\Gamma(P, x)$ with the information $\langle u'_0, u_1, \ldots, u_{2n-1}, u'_{2n} \rangle$ to find $u_{2n+1} \leq u_{2n}$ and his answer in $\Gamma(P \triangle I^{\mathrm{BM}}(P), x)$ will be $v_{2n+1} = u_{2n+1} - x_{2n}$. Therefore $x \in I^{\mathrm{BM}}(P \triangle I^{\mathrm{BM}}(P))$

Assume that $x \in I^{\mathrm{BM}}(P \triangle I^{\mathrm{BM}}(P))$. Let $\langle N_n : n \in \omega \rangle$ be a fixed partition of ω into infinite sets and we define a winning strategy for player II in $\Gamma(P, x)$. We let player I perform a play $\sigma = \langle u_k : k \in \omega \rangle$ in $\Gamma(P, x)$ while player II performs a play $\tau = \langle v_k : k \in \omega \rangle$ in $\Gamma(P \triangle I^{\mathrm{BM}}(P), x)$ where $v_k = (u''_k - x_k) \vee (y_k - u''_k)$ for some $x_k, y_k \in I^{\mathrm{BM}}(P)$ and $u''_k \in P$. We translate moves of player I from σ to τ so that $v_0 = u''_0 = u_0$, $x_0 = y_0 = 0$ and $v_{2k} = u_{2k} - x_{2k-1}$ for $0 < k \leq n$, i.e., $u''_{2k} = u_{2k}$, $x_{2k} = x_{2k-1}$, $y_{2k} = 0$. Player II makes his move $v_{2k+1} = (u''_{2k+1} - x_{2k+1}) \vee (y_{2k+1} - u''_{2k+1})$ using his winning strategy in $\Gamma(P \triangle I^{\mathrm{BM}}(P))$ with the information $\tau \restriction (2n+1)$. Using separability of P we find $u'_{2k+1} \in P$ such that $u'_{2k+1} \leq u''_{2k+1}$ and $u'_{2k+1} \leq u''_{2k}$. If $k \in N_n$ and $k \geq n$, let u_{2k+1} be the move of player II by his winning strategy for $\Gamma(P, x_{2n+1})$ with the information $\langle u_{2i-1}, u'_{2i+1} : n < i \leq k$ and $i \in N_n \rangle$; otherwise let $u_{2k+1} = u'_{2k+1}$. Assume that $z \in A$ is such that $z \leq u_n$ for all $n \in \omega$. Then $z \leq u_{2n+2} \leq v_{2n+2} \vee x_{2n+1}$ for all $n \in \omega$. We have $z \wedge x_{2n+1} = 0$ because player II wins the play $\langle u_{2i-1}, u'_{2i+1} : i > n$ and $i \in N_n \rangle$ in $\Gamma(P, x_{2n+1})$ and $z \leq u_{2i-1}$ for all i. Therefore $z \leq v_{2n+2}$ for all $n \in \omega$ and then $z \wedge x = 0$ because player II wins the play τ in $\Gamma(P \triangle I^{\mathrm{BM}}(P), x)$. Therefore $x \in I^{\mathrm{BM}}(P)$. □

Theorem 6.14. *Let A be a Boolean algebra and let $P \subseteq A^+$.*

(1) $s(P_{I(P)} \triangle I(P)) \subseteq s(P_{I^{\mathrm{BM}}(P)} \triangle I^{\mathrm{BM}}(P))$ *and* $\mathrm{dec}(P_{I(P)} \triangle I(P)) \subseteq \mathrm{dec}(P_{I^{\mathrm{BM}}(P)} \triangle I^{\mathrm{BM}}(P))$.

(2) *If P is eligible, then* $s(P_{I(P)} \triangle I(P)) \subseteq I^{\mathrm{BM}}(P) = s(P_{I^{\mathrm{BM}}(P)} \triangle I^{\mathrm{BM}}(P))$ *and* $\operatorname{dec}(P_{I(P)} \triangle I(P)) \subseteq \operatorname{Baire}(P, I^{\mathrm{BM}}(P) = \operatorname{dec}(P_{I^{\mathrm{BM}}(P)} \triangle I^{\mathrm{BM}}(P))$.

(3) *If P is eligible, then* $\operatorname{cmpl}(\operatorname{Baire}(P, I^{\mathrm{BM}}(P))) \geq \omega_1$.

(4) *If P has the strong reduction property in A, then*
  (a) $\operatorname{Baire}(P, I^{\mathrm{BM}}(P)) = \operatorname{dec}(P_{I^{\mathrm{BM}}(P)}) \triangle I^{\mathrm{BM}}(P)$,
  $\operatorname{Baire}(P, I(P)) = \operatorname{dec}(P_{I(P)}) \triangle s(P_{I(P)} \triangle I(P))$, *and*
  $\operatorname{Baire}(P, I(P)) = \operatorname{dec}(P_{I(P)}) \triangle I(P)$, *if P has also the orthogonalization property.*
  (b) $\operatorname{Baire}(P, I^{\mathrm{BM}}(P))/I^{\mathrm{BM}}(P) \simeq \operatorname{dec}(P_{I^{\mathrm{BM}}(P)})/s(P_{I^{\mathrm{BM}}(P)})$ *and*
  $\operatorname{Baire}(P, I(P))/s(P_{I(P)} \triangle I(P)) \simeq \operatorname{dec}(P_{I(P)})/s(P_{I(P)})$
  are complete Boolean algebras and regular subalgebras of $A/I^{\mathrm{BM}}(P)$, $A/s(P_{I^{\mathrm{BM}}(P)})$, $A/s(P_{I(P)} \triangle I(P))$, $A/s(P_{I(P)})$, *respectively.*

Proof.

(1) By Lemma 6.13, $I(P) \subseteq I^{\mathrm{BM}}(P)$. If $I \subseteq J$ are any ideals in A, then $s(P_I \triangle I) \subseteq s(P_J \triangle I) \subseteq s(P_J \triangle J)$ and $\operatorname{dec}(P_I \triangle I) \subseteq \operatorname{dec}(P_J \triangle I) \subseteq \operatorname{dec}(P_J \triangle J)$ because P_J is an open subset of P_I, hence $P_J \triangle I$ is an open subset of $P_I \triangle I$, and Lemma 4.5, because $P_J \triangle J = (P_J \triangle I) \triangle J$.

(2) By Lemma 6.13 (1), $I^{\mathrm{BM}}(P)$ is P-Banach. Therefore the assertion follows by (1) and Theorem 6.5.

(3) We write J instead of $I^{\mathrm{BM}}(P)$. By Lemma 6.13 (5), $\operatorname{cmpl}(J) \geq \omega_1$, and by (2), $\operatorname{Baire}(P, J) = \operatorname{dec}(P_J \triangle J)$ and $s(P_J \triangle J) = J$. Therefore it is enough to verify the assumptions of Lemma 6.9 for the base $Q = P_J \triangle J$. The set Q is stable by Lemma 4.18 and, by Lemma 6.13 (2) and (6), $I(Q) \cap Q \subseteq I^{\mathrm{BM}}(Q) \cap Q = J \cap Q = \emptyset$.

(4) By Lemma 5.4 (7), P is eligible. Hence all assertions follow by Theorem 6.5 (2) and Theorem 5.11, because by (2), $s(P_{I^{\mathrm{BM}}(P)} \triangle I^{\mathrm{BM}}(P)) = I^{\mathrm{BM}}(P)$, and by Theorem 6.11, $s(P_{I(P)} \triangle I(P)) = I(P)$, if P has the orthogonalization property. □

Let us recall that A is ω-distributive, if for every sequence $\langle D_n : n \in \omega \rangle$ of open dense subsets A, the intersection $\bigcap_{n \in \omega} D_n$ is a dense subset of A.

Theorem 6.15. *If P has the orthogonalization property and A is ω-distributive, then* $I(P) = I^{\mathrm{BM}}(P)$.

Proof. Without loss of generality we can assume that A is complete and P has the disjoint refinement property because all notions in theorem are absolute for regular subalgebras and $I(P) = I(Q)$ and $I^{\mathrm{BM}}(P) = I^{\mathrm{BM}}(Q)$ whenever $Q \sim P$.

Let f be a winning strategy of II in $\Gamma(P, x)$. There are a tree $S \subseteq {}^{<\omega}P$, a sequence of P-maximal antichains $E_{n+1} \subseteq P$ for $n \in \omega$ and a mapping φ which maps each E_{n+1} onto $S \cap {}^{2n+2}P$ such that the following conditions hold:

(1) Each $s \in S$ is a decreasing sequence in P and $s(2k+1) = f(s{\restriction}(2k+1))$ for all $2k+1 < |s|$, i.e., s is the result of a play in which player II uses his winning strategy f.

(2) E_{n+2} refines E_{n+1} and if $v \in E_{n+1}$, $w \in E_{n+2}$, and $w \le v$, then $\varphi(v) = \varphi(w){\restriction}(2n+2)$ and $\varphi(v)(2n+1) \ge v \ge \varphi(w)(2n+2) \ge \varphi(w)(2n+3) \ge w$.

We define by induction on $n \in \omega$ the levels $S_{2n} = S \cap {}^{2n}P$ of the tree S, the sets E_{n+1}, and the mappings $\varphi{\restriction}E_{n+1} \colon E_{n+1} \to S_{2n+2}$. Let $S_0 = \{\emptyset\}$. Assume that S_{2n} has been defined and we define E_{n+1}, S_{2n+2}, and $\varphi{\restriction}E_{n+1} \colon E_{n+1} \to S_{2n+2}$. Let D_n be the set of all $v \in P$ for which there exist $s_v \in S_{2n}$ and $u_v^{\mathrm{II}} \le u_v^{\mathrm{I}} \le s_v$ in P such that $v \le u_v^{\mathrm{II}}$ and $u_v^{\mathrm{II}} = f(s_v{}^\frown\langle u_v^{\mathrm{I}}\rangle)$. For each $v \in D_n$ let us fix such s_v, u_v^{I}, and u_v^{II}. The set D_n is an open dense subset of P and hence there is a P-maximal antichain $E_{n+1} \subseteq D_n$ in A. Now we define $\varphi(v) = s_v{}^\frown\langle u_v^{\mathrm{I}}, u_v^{\mathrm{II}}\rangle$ for $v \in E_{n+1}$ and $S_{2n+2} = \bigl\{\varphi(v) :\ v \in E_{n+1}\bigr\}$.

To obtain a contradiction let us assume that $x \notin I(P)$. Let $x_n = -\bigvee E_{n+1}$. Since $\bigvee_{n\in\omega} x_n \in I(P)$, there exists $y \le x$ in A^+ such that $y \wedge x_n = 0$ for all $n \in \omega$. By ω-distributivity of A there is a nonzero $z \le y$ such that for every $n \in \omega$ there is $v_{n+1} \in E_{n+1}$ such that $z \le v_{n+1}$. Then $v_{n+2} \le v_{n+1}$ for all n and hence $\bigcup_{n\in\omega} \varphi(v_{n+1})$ is an infinite branch $\langle u_n : n \in \omega\rangle$ in s, which is a result of an infinite play in which player II used the strategy f. Since $z \le x$ and $z \le v_{n+1} \le u_{2n+1}$ for all n, we have obtained a contradiction with the assumption that $x \in I^{\mathrm{BM}}(P)$. Therefore $I(P) = I^{\mathrm{BM}}(P)$. □

7. Suslin operation

The result of the Suslin operation $\mathcal{A}$ for a system of sets B_s, $s \in {}^{<\omega}\omega$, is defined by $\mathcal{A}\{B_s :\ s \in {}^{<\omega}\omega\} = \bigcup_{x\in{}^\omega\omega} \bigcap_{n\in\omega} B_{x\restriction n}$.

Definition 7.1. Let A be a Boolean algebra. We say that a set $B \subseteq A$ is a *covering subset of* A if for every $x \in A$ the set $\{u \in B :\ x \le u\}$ has a least element which we call a *covering of* x in B.

Easily we can check that, if B is a covering subalgebra of A, then $B \subseteq_{rc} A$. For the inverse implication some amount of the completeness of A is necessary to assume. For example:

(a) If $B \subseteq A$ is a complete algebra, then B is a covering subalgebra if and only if B is a regular subalgebra of A.

(b) If A is complete, then $B \subseteq A$ is a covering subalgebra if and only if B is A-complete.

The Marczewski Theorem can be stated in this form:

Theorem 7.2. *Let B be a system of subsets of a set X and let J be a σ-ideal on X such that $B \triangle J$ is a σ-field and B/J is a covering subset of $\mathcal{P}(X)/J$. Then $B \triangle J$ is closed under the Suslin operation $\mathcal{A}$.*

P r o o f. Let $A = \mathcal{A}\{B_s : s \in {}^{<\omega}\omega\}$ where $B_s \in B \triangle J$ and $B_{s^\frown n} \subseteq B_s$. Denote $A_s = \bigcup_{x \in {}^\omega\omega} \bigcap_{n\in\omega} B_{s^\frown x\restriction n}$. Then $A = A_\emptyset$ and $A_s \subseteq B_s$. Choose $M_s \in B \triangle J$ such that $A_s \subseteq M_s \subseteq B_s$ and $[M_s]_J$ is a covering of $[A_s]_J$. If $x \in M_\emptyset$ is such that $x \notin \bigcup_{s\in{}^{<\omega}\omega} \big(M_s \setminus \bigcup_{n\in\omega} M_{s^\frown n}\big)$, then inductively we can define $y \in {}^\omega\omega$ such that $x \in \bigcap_{n\in\omega} M_{y\restriction n} \subseteq A_\emptyset$. Therefore $M_\emptyset \setminus \bigcup_{s\in{}^{<\omega}\omega} \big(M_s \setminus \bigcup_{n\in\omega} M_{s^\frown n}\big) \subseteq A_\emptyset$ and consequently $M_\emptyset \setminus A_\emptyset \subseteq \bigcup_{s\in{}^{<\omega}\omega} \big(M_s \setminus \bigcup_{n\in\omega} M_{s^\frown n}\big)$. As $A_s = \bigcup_n A_{s^\frown n} \subseteq \bigcup_n M_{s^\frown n}$ and $[M_s]_J$ is a covering of $[A_s]_J$ in B/J it follows that $[M_s]_J \leq [\bigcup_{n\in\omega} M_{s^\frown n}]_J$, i.e., $M_s \setminus \bigcup_{n\in\omega} M_{s^\frown n} \in J$ for all s. Therefore $M_\emptyset \setminus A_\emptyset \in J$ and hence $A_\emptyset \in B \triangle J$. $\square$

Let us note that $\mathrm{Baire}(B, J) = \mathrm{Baire}(B \triangle J, J) = B \triangle J$ for B and J in Theorem 7.2. The first equality holds by definition. For the second equality notice that $B/J \subseteq_{rc} A/J$. By Corollary 4.17, $\mathrm{dec}((B \triangle J) \setminus J) = B \triangle J$ and $s((B\triangle J)\setminus J) = J$. Then $\mathrm{Baire}(B\triangle J, J) = \mathrm{Baire}((B\triangle J)\setminus J, J) = \mathrm{dec}((B\triangle J)\setminus J)$ because $\mathrm{Baire}(P, s(P)) = \mathrm{dec}(P)$.

Corollary 7.3. *Let $P \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$. If P has the strong reduction property in $\mathcal{P}(X)$ and I is a P-Banach σ-ideal in $\mathcal{P}(X)$, then* $\mathrm{Baire}(P, I)$ *is closed under the Suslin operation $\mathcal{A}$. In particular,* $\mathrm{Baire}(P, I^{\mathrm{BM}}(P))$ *is closed under the operation $\mathcal{A}$.*

P r o o f. P is eligible because P has the strong reduction property. Therefore by Theorem 6.5, I is P_I-regular and $\mathrm{Baire}(P, I) = \mathrm{dec}(P_I \triangle I)$. P_I has the strong reduction property, too, because it is an open subset of P. By Corollary 5.12, $\mathrm{dec}(P_I \triangle I) = \mathrm{dec}(P_I) \triangle I$, $\mathrm{dec}(P_I)/I \subseteq_{rc} \mathcal{P}(X)/I$, and $\mathrm{dec}(P_I)/I$ is a complete algebra, hence a covering subalgebra. By Marczewski Theorem, $\mathrm{dec}(P_I) \triangle I$ is closed under the Suslin operation $\mathcal{A}$. $\square$

8. Examples

In the literature one can find a lot of important examples of bases of measurability. We do not make any overview of these bases and we just recall several basic examples. Some of the following examples correspond to classical tree-like forcing notions. The reader can find some basic facts about them in [2].

1. A *closure algebra*, is by definition, a Boolean algebra A with an operation $\mathrm{C}\colon A \to A$ such that $\mathrm{C}(u \vee v) = \mathrm{C}\,u \vee \mathrm{C}\,v$, $u \le \mathrm{C}\,u$, $\mathrm{C}\,\mathrm{C}\,u = \mathrm{C}\,u$, $\mathrm{C}\,0 = 0$ (see [8: §41]). An element u of A is called closed if $u = \mathrm{C}\,u$; u is open if $-u$ is closed. Let $P_1 \subseteq A$ be the set of all nonzero open elements. Then P_1 is π-complete and hence it is separable and has the disjoint refinement property. $s(P_1)$ is the set of nowhere dense elements and $\mathrm{dec}(P_1) = (P_1 \cup \{0\}) \triangle s(P_1)$.

2. Let $(X, \mathcal{O})$ be a topological space. Let $A_2 = \mathcal{P}(X)$ and let $P_2 = \mathcal{O} \setminus \{\emptyset\}$.

(a) P_2 is π-complete. Therefore, P_2 is separable and it is κ-resolute for all κ. In particular P_2 has the disjoint refinement property and (X, P_2) is a category base (in the sense of [6]; see Lemma 5.4). P_2 has the strong reduction property as well.

(b) $s(P_2)$ is the family of nowhere dense subsets of X and $\mathrm{dec}(P_2) = \mathcal{O} \triangle s(P_2)$ (Lemma 4.2).

(c) $I(P_2)$ is the σ-ideal of meager sets in X, $I(P_2) = I^{\mathrm{BM}}(P_2)$ (see proof of [7: Theorem 6.1] or Theorem 6.15).

(d) $I(P_2)$ is a P_2-Banach and $(P_2)_{I(P_2)}$-regular σ-ideal and

$$\mathrm{Baire}(P_2, I(P_2)) = \mathrm{dec}(P_2) \triangle I(P_2) = \mathcal{O} \triangle I(P_2)$$

is a σ-algebra of sets with the Baire property (Theorem 6.5, Theorem 6.14, Corollary 7.3, Corollary 5.12). It is closed under the Suslin operation $\mathcal{A}$.

3. Let B be an infinite σ-complete Boolean algebra, $X = \mathrm{St}(B)$, $A_3 = \mathcal{P}(X)$, and let P_3 be the family of perfect subsets of X. Then $\bigcup P_3$ is the closed set of non-principal ultrafilters.

Lemma 8.1.

(1) *Every infinite closed subset of* $\mathrm{St}(B)$ *contains a perfect subset.*

(2) *Every perfect set in* $\mathrm{St}(B)$ *contains* 2^{2^ω} *many disjoint perfect subsets.*

(3) *If* $\langle u_n : n \in \omega \rangle$ *is a decreasing sequence of perfect subsets of* $\mathrm{St}(B)$, *then* $u = \bigcap_{n\in\omega} u_n$ *is perfect, i.e.,* P_3 *is* ω*-closed.*

Using these facts we can easily verify that P_3 is separable and $\mathrm{dec}(P_3)$ contains all closed subsets of $\mathrm{St}(B)$. $s(P_3) = I(P_3)$ is a P_3-regular σ-ideal (use Lemma 8.1 (1) and (3)) and hence $\mathrm{Baire}(P_3, I(P_3)) = \mathrm{dec}(P_3)$ (Lemma 4.10 and Theorem 6.5). P_3 is 2^{2^ω}-resolute (Lemma 8.1 (1) and (2)). If $|\mathrm{St}(B)| = 2^{2^\omega}$, then $\left(\bigcup P_3, P_3\right)$ is a category base. Hence P_3 has the disjoint refinement property, $I^{\mathrm{BM}}(P_3) = s(P_3)$, and $\mathrm{dec}(P_3)$ is closed under the Suslin operation $\mathcal{A}$ (Theorem 6.15 and Corollary 7.3).

4. Let $A_4 = \mathcal{P}(X)$ for an uncountable Polish space X, and let P_4 be the family of perfect subsets of X, i.e., nonempty closed sets without isolated points. It is known that $\left(\bigcup P_4, P_4\right)$ is a category base. Hence P_4 is 2^ω-resolute, has the disjoint refinement property, and has the strong reduction property in A_4. One can verify that $s(P_4) = I(P_4) = I^{\mathrm{BM}}(P_4) = s^0$ is a P_4-regular σ-ideal, called Marczewski's ideal, and $\operatorname{dec}(P_4)$ is a σ-algebra closed under the Suslin operation $\mathcal{A}$.

5. Let $A_5 = \mathcal{P}({}^\omega\omega)$. Let us recall that a tree $p \subseteq {}^{<\omega}\omega$ is a *Laver tree*, i.e, $p \in \mathbb{L}$, if p has a stem s such that for every $t \in p$, either $t \subseteq s$, or $s \subseteq t$ and $\{n : t^\frown n \in p\}$ is infinite; p is a *Miller tree*, i.e., $p \in \mathbb{M}$, if for every $s \in p$ there is $t \in p$ such that $s \subseteq t$ and $\{n : t^\frown n \in p\}$ is infinite. For a tree $p \subseteq {}^{<\omega}\omega$ let $[p] = \big\{x \in {}^\omega 2 : (\forall n)(x{\restriction}n \in p)\big\}$. Let $P_{\mathbb{L}} = \big\{[p] : p \in \mathbb{L}\big\}$ and $P_{\mathbb{M}} = \big\{[p] : p \in \mathbb{M}\big\}$ in the Boolean algebra A_5. The ideals $s(P_{\mathbb{L}})$ and $s(P_{\mathbb{M}})$ are known as l^0 and m^0, respectively, and they are σ-ideals. Therefore $\operatorname{Baire}(P_{\mathbb{L}}, I(P_{\mathbb{L}})) = \operatorname{dec}(P_{\mathbb{L}})$ and $\operatorname{Baire}(P_{\mathbb{M}}, I(P_{\mathbb{M}})) = \operatorname{dec}(P_{\mathbb{M}})$. By [4: Lemma 2.5], if the least cardinality of an unbounded family of functions is $\mathfrak{b} = 2^\omega$, then $P_{\mathbb{L}}$ and $P_{\mathbb{M}}$ satisfy the disjoint refinement property and therefore $\operatorname{dec}(P_{\mathbb{L}})$ and $\operatorname{dec}(P_{\mathbb{M}})$ are closed under the Suslin operation $\mathcal{A}$ (see [5: Corollary 4.7]).

6. Let $A_6 = \mathcal{P}({}^\omega 2)$. For a partial function p from ω to $\{0,1\}$ let $[p] = \{x \in {}^\omega 2 : p \subseteq x\}$. Let $P_6 = \big\{[p] : p$ is a partial function from ω to $\{0,1\}$ and $\omega \setminus \operatorname{dom}(p)$ is infinite$\big\}$ and consider P_6 as a subset of the Boolean algebra A_6. P_6 corresponds to *Silver's forcing*. The ideal $s(P_6)$ is known as v^0 and it is a σ-ideal. If $[p]$ and $[q]$ from P_6 are incompatible, then $[p] \cap [q]$ is finite. It follows that P_6 has the disjoint refinement property and $\operatorname{dec}(P_6)$ is closed under the Suslin operation $\mathcal{A}$.

7. Let $A_7 = \mathcal{P}(\mathbb{R})$ and P_7 be the system of perfect sets of positive Lebesgue measure (compare with Remark 2.6 (3)). Then $(\mathbb{R}, P_7)$ is a category base, $\operatorname{dec}(P_7)$ is the σ-algebra of Lebesgue measurable sets and $s(P_7)$ is the σ-ideal of null sets.

8. Let $A_8 = \mathcal{P}(\mathbb{R})$ and P_8 be the system of G_δ non-meager sets. Then $(\mathbb{R}, P_8)$ is a category base, $\operatorname{dec}(P_8)$ is the σ-algebra of sets with the Baire property and $s(P_8)$ is the σ-ideal of meager sets.

9. Let $A_9 = \mathcal{P}(\mathbb{R})$ and $P_9 = P_7 \cup P_8$. Then $\operatorname{dec}(P_9) = \operatorname{dec}(P_7) \cap \operatorname{dec}(P_8)$ and $s(P_9) = s(P_7) \cap s(P_8)$ (by Lemma 2.3 (6) due to a decomposition of $\mathbb{R}$ into a meager set and a measure zero set).

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Mathematical Institute
Slovak Academy of Sciences
Grešákova 6
SK–040 01 Košice
SLOVAK REPUBLIC
Institute of Computer Science
Faculty of Science
P. J. Šafárik University
Jesenná 5
SK–041 54 Košice
SLOVAK REPUBLIC
E-mail: repicky@saske.sk
miroslav.repicky@upjs.sk