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ORDERED LEFT PP MONOIDS

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ABSTRACT. An ordered monoid S in which every principal left ideal, regarded as an S-poset, is projective is called an ordered left PP monoid, for short, an ordered *lpp* monoid. In this paper, we introduce a new kind of ordered relations instead of Green relations adopted by J. B. Fountain, ordered lpp monoids are described by these ordered relations. Some similar results of c-rpp semigroups are duduced and proved, in particular, Fountain's results about C-*lpp* monoids are generalized to ordered monoids.

0. Introduction

A monoid in which every principal left ideal, regarded as an S-act, is projective is called a left PP monoid, for short, an *lpp* monoid. It is known that such kind of monoids are generalization of regular monoids. It is clear that such a monoid contains all left cancellative monoids. The left lpp monoids in the literature were studied by J. B. Fountain in 1977. In particular, Fountain paid special attention to the rpp monoids whose idempotents lie in the center of the monoid. He then called these rpp monoids C-rpp monoids. The most important theorem in this topics is that a C-rpp semigroup can be expressed as a strong semilattice of left cancellative monoids [5]. This theorem has been extended and generalized by many authors such as Y. Q. Guo, X. J. Guo, X. M. Ren and K. P. Shum ([9]–[21]). Some ordered semigroups are discussed ([13]–[12]). But there are no discuss about C-rpp monoids in ordered semigroup.

Green's relations $\mathcal{R}$ and $\mathcal{L}$ play an important role in the algebraic theory of semigroups. Generalized Green's relations $\mathcal{R}^*$ and $\mathcal{L}^*$ are investigated by many

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authors (see, [4], [16]). It is well known that a monoid S is an *lpp* monoid if and only if every $\mathcal{R}^*$-class of S contains at least one idempotent. A monoid S is called left [semi-] hereditary if all [finitely generated] left ideals of S are projective. Kilp had proved that commutative PP monoids are semilattices of cancellative monoids in [15]. Fountain generalized Kilp's result to case of right PP monoids with central idempotents and generalized Dorofeeva's result about semi-hereditary and hereditary monoids in [5].

A monoid S that is also a partially ordered set, in which the binary operation and the order relation are compatible, is called a partially ordered monoid, or pomonoid. Partially ordered acts over a pomonoid S, or S-posets, appear naturally in the study of mappings between posets. In [19], projective S-posets are discussed and ordered *lpp* monoids are introduced. We can see the class of ordered *lpp* monoids contains all ordered right cancellative monoids and all regular pomonoids. In [18], weakly po-flat and principally weakly po-flat S-posets over ordered *lpp* monoids are considered. The present paper is devoted to generalizing some results about *lpp* monoids to ordered *lpp* monoids.

Let S be a partially ordered monoid. A partially ordered set A is called a left S-poset if S acts on A in such a way that

(i) the action is monotonic in each of the variables,

(ii) for $s, t \in S$ and $a \in A$ we have $s(ta) = (st)a$,

(iii) $1a = a$ where 1 is the identity of S and sa denotes the result of action of s on a.

An S-morphism from S-poset A to S-poset C is a monotonic map that preserves S-action. The class of left S-posets and S-morphisms forms a category, which we denote by S-Pos (see [2]). In S-Pos, one should note that an S-isomorphism $f\colon A \to C$ is an order- and action-preserving bijection and $f^{-1}\colon C \to A$ is also a monotonic map. An S-poset P is called *projective* if for any S-surmorphism $\pi\colon A \to B$ and any S-morphism $\varphi\colon P \to B$ there exists an S-morphism $\psi\colon P \to A$ such that $\varphi = \pi\psi$. In [19], we proved that an S-poset P is projective if and only if $P \cong \coprod_{i\in I} Se_i$ where $e_i^2 = e_i \in S$, $i \in I$. A free S-poset is the coproduct $\coprod_{i\in I} S_i$, where each S_i is S-isomorphic to the S-poset S, a semifree S-poset is a coproduct of cyclic S-posets.

Let S be an ordered semigroup. A left ideal of S is a non-empty subset I of S such that $SI \subseteq I$. By an ordered left ideal of S we mean a non-empty subset I of S such that

(i) $SI \subseteq I$,

(ii) $a \le b \in I$ implies $a \in I$, for all $a, b \in S$.

We emphasize that in this paper, a left ideal of an ordered semigroup S need not be an ordered left ideal.

In order to characterize ordered left PP semigroups by $\underline{\mathcal{R}}^*$-classes of S, we first introduce the relations $\underline{\mathcal{R}}^*$ and $\underline{\mathcal{L}}^*$ on ordered semigroups. Some results similar to [4] are discussed. In the second section we discuss properties of ordered left PP monoids and a characterization of ordered left semi-hereditary and hereditary ordered monoids is given. In the third section we consider ordered left PP monoids with central idempotents, some results of [5] are generalized to ordered monoids. The results on left PP monoids can be also obtained as applications of the results in this paper.

1. The relations $\underline{\mathcal{R}}^*$, $\underline{\mathcal{L}}^*$ in ordered semigroups

Let S be an ordered semigroup, $x \in S$, $R(x)$ ($L(y)$) denotes the ordered right ideal (ordered left ideal) generated by x, that is,

$$R(x) = (xS^1] = \big\{s \in S : \ (\exists t \in S^1)(s \leq xt)\big\},$$
$$L(x) = (S^1x] = \big\{s \in S : \ (\exists t \in S^1)(s \leq tx)\big\}.$$

In [10], Niovi Kehayopulu introduced Green's relations $\mathcal{R}$ and $\mathcal{L}$ on an ordered semigroup similar to discrete case as follows:

$$\mathcal{R} := \big\{(x,y) \in S \times S : \ R(x) = R(y)\big\};$$
$$\mathcal{L} := \big\{(x,y) \in S \times S : \ L(x) = L(y)\big\}.$$

For the relation $\mathcal{R}$, we have

Lemma 1.1. *Let S be an ordered semigroup and let $a, b \in S$. Then the following statements are equivalent:*

(1) $(a,b) \in \mathcal{R}$;

(2) $(aS^1] = (bS^1]$;

(3) *There exist $x, y \in S^1$ such that $a \leq bx$, $b \leq ay$.*

Similar to $\mathcal{R}^*$, $\mathcal{L}^*$ in semigroups, we define $\underline{\mathcal{R}}^*$ and $\underline{\mathcal{L}}^*$ on ordered semigroup S (see also in [14]), where $a\underline{\mathcal{R}}^*b$ means that the right translations ρ_a and ρ_b have same ordered kernel, $a\underline{\mathcal{L}}^*b$ means that the left translations λ_a and λ_b have same ordered kernel. That is,

$$a\underline{\mathcal{R}}^*b \iff \big[(\forall x \in S^1)(\forall y \in S^1)(xa \leq ya \iff xb \leq yb)\big];$$
$$a\underline{\mathcal{L}}^*b \iff \big[(\forall x \in S^1)(\forall y \in S^1)(ax \leq ay \iff bx \leq by)\big].$$

One should note that $\underline{\mathcal{R}}^* \subseteq \mathcal{R}^*$ and $\underline{\mathcal{L}}^* \subseteq \mathcal{L}^*$, where $\mathcal{R}^* := \big\{(a,b) \in S \times S : (\forall x \in S^1)(\forall y \in S^1)(xa = ya \iff xb = yb)\big\}$ and $\mathcal{L}^* := \big\{(a,b) \in S \times S : (\forall x \in S^1)(\forall y \in S^1)(ax = ay \iff bx = by)\big\}$ are the relations in semigroups.

***Example* 1.2.**

(1) Let $\mathbb{N}$ be the set of all natural numbers, then $(\mathbb{N}, \times, \leq)$ becomes an ordered monoid if the multiplication is usual multiplication and ordering of the natural number is the usual one. Since $(\mathbb{N}, \times, \leq)$ is ordered cancellable, $\underline{\mathcal{R}}^* = \mathbb{N} \times \mathbb{N}$. By Lemma 1.1, $\mathcal{R} = \mathbb{N} \times \mathbb{N}$. Therefore $\mathcal{R} = \underline{\mathcal{R}}^*$ in this ordered monoid.

(2) Let $\overline{\mathbb{Z}^+}$ be the set of all nonpositive integral numbers, then $S = (\overline{\mathbb{Z}^+}, +, \leq)$ becomes an ordered monoid if the plus is usual plus and ordering is the usual one. Since S is ordered cancellable, $\underline{\mathcal{R}}^* = S \times S$, but $\mathcal{R} = 1_S$, in fact, if $(a,b) \in \mathcal{R}$, then there exist $x, y \in S$ such that $a \leq b+x$, $b \leq a+y$, thus $a \leq b+x \leq a+x+y$, by cancellation, $0 \leq x+y$, from the ordering we have $x = y = 0$, therefore $a = b$.

(3) Let $S = \{0, 1\}$ be an ordered semigroup, where the multiplication '+' is 'join' and the ordering '$\leq$' is usual ordering, that is,

$$1 = 0 + 1 = 1 + 1 = 1 + 0, \quad 0 = 0 + 0$$
$$\leq \; := \big\{(0,0), (0,1), (1,1)\big\}$$

Because $0 \leq 0 + 1$, $1 \leq 1 + 0$, therefore $0\mathcal{R}1$. Note that $1 + 1 \leq 1 + 0$, if $(1,0) \in \underline{\mathcal{R}}^*$ then $0+1 \leq 0+0$, it is a contradiction. Thus $\mathcal{R} \not\subseteq \underline{\mathcal{R}}^*$ in this ordered monoid.

Lemma 1.3. *Let S be an ordered semigroup and let $a, b \in S$. Then the following statements are equivalent:*

(1) $(a,b) \in \underline{\mathcal{R}}^*$;

(2) *There is an S^1-isomorphism (in S-*Pos*) $\varphi\colon S^1a \to S^1b$ with $\varphi(a) = b$.*

Proof.

(1) $\Longrightarrow$ (2). Suppose that $xa \leq ya \iff xb \leq yb$, for all $x, y \in S^1$. We can define a map $\varphi\colon S^1a \to S^1b$ such that $\varphi(sa) = sb$ for all $s \in S^1$. It is easy to see that the map φ is an S^1-isomorphism with $\varphi(a) = b$.

(2) $\Longrightarrow$ (1). Let $\varphi\colon S^1a \to S^1b$ be an isomorphism with $\varphi(a) = b$. If $xa \leq ya$, then $xb = x\varphi(a) = \varphi(xa) \leq \varphi(ya) = y\varphi(a) = yb$; If $xb \leq yb$, then $xa = x\varphi^{-1}(b) = \varphi^{-1}(xb) \leq \varphi^{-1}(yb) = y\varphi^{-1}(b) = ya$, thus $(a,b) \in \underline{\mathcal{R}}^*$. □

Lemma 1.4. ([19: Proposition 3.2]) *Let S be an ordered semigroup and $e \in E(S)$, $a \in S$. Then the following statements are equivalent:*

(1) $(e,a) \in \underline{\mathcal{R}}^*$;

(2) *There is an S^1-isomorphism (in S-*Pos*) $\varphi\colon S^1a \to S^1e$ with $\varphi(a) = e$;*

(3) *$a = ea$ and $sa \leq ta$ implies $se \leq te$ for $s, t \in S$.*

Proposition 1.5. *Let S be an ordered semigroup. Then $\underline{\mathcal{R}}^*$ (resp. $\underline{\mathcal{L}}^*$) is a left (resp. right) congruence on S.*

Proof. It is easy to see that $\underline{\mathcal{R}}^*$ is a equivalence relation on semigroup S. Let $(a, b) \in \underline{\mathcal{R}}^*$ and $s \in S$, then $(xs)a \leq (ys)a$ if and only if $(xs)b \leq (ys)b$ for all $x, y \in S$, that is, $x(sa) \leq y(sa)$ if and only if $x(sb) \leq y(sb)$, thus $(sa, sb) \in \underline{\mathcal{R}}^*$. □

Remark. In [14], the authors show that if S is a separative ordered semigroup, then $\underline{\mathcal{L}}^*$ is a semilattice congruence on S. Furthermore, if S is a completely separative ordered semigroup, then $\underline{\mathcal{L}}^*$ is the greatest strongly complete semilattice congruence on S such that $\underline{L}_a^*$ is cancellative for each $a \in S$.

We use $\underline{R}_a^*$ denote the $\underline{\mathcal{R}}^*$-class which contains element a of the ordered semigroup S. The corresponding notion will be used for the classes $\underline{\mathcal{L}}^*$. A right (left) ideal I of semigroup S is called a *right (left)* **-ideal* if $\underline{R}_a^* \subseteq I(\underline{L}_a^* \subseteq I)$ for all $a \in I$.

From the definition of left *-ideal and right *-ideal, we have:

Lemma 1.6. *If $\{I_\lambda : \lambda \in \Lambda\}$ is a set of left *-ideals (right *-ideals) of an ordered semigroup S, then*

(1) $\bigcap\{I_\lambda : \lambda \in \Lambda\}$ *is a left *-ideal (right *-ideal), if it is not empty;*

(2) $\bigcup\{I_\lambda : \lambda \in \Lambda\}$ *is a left *-ideal (right *-ideal).*

Let a be an element of an ordered semigroup S. In view of (1), there is a smallest left *-ideal $\underline{\mathcal{L}}^*(a)$ containing a and a smallest right *-ideal $\underline{\mathcal{R}}^*(a)$ containing a. We call $\underline{\mathcal{L}}^*(a)$ ($\underline{\mathcal{R}}^*(a)$) the principal left *-ideal (principal right *-ideal). Now we give some alternative characterizations of these *-ideals.

Similar to [4: Lemma 1.7], we have:

Lemma 1.7. *Let S be an ordered semigroup and $a \in S$. Then*

(1) $b \in \underline{\mathcal{L}}^*(a)$ *if and only if there exist elements $a_0, a_1, \dots, a_n \in S$, $x_1, \dots x_n \in S^1$, such that $a = a_0$, $b = a_n$, and $(a_i, x_i a_{i-1}) \in \underline{\mathcal{L}}^*$ for $i = 1, 2, \dots, n$.*

(2) $b \in \underline{\mathcal{R}}^*(a)$ *if and only if there exist elements $a_0, a_1, \dots, a_n \in S$, $x_1, \dots x_n \in S^1$, such that $a = a_0$, $b = a_n$, and $(a_i, a_{i-1} x_i) \in \underline{\mathcal{R}}^*$ for $i = 1, 2, \dots, n$.*

Corollary 1.8. *For elements a, b of an ordered semigroup S, we have*

(1) $a\underline{\mathcal{L}}^* b$ *if and only if* $\underline{\mathcal{L}}^*(a) = \underline{\mathcal{L}}^*(b)$;

(2) $a\underline{\mathcal{R}}^* b$ *if and only if* $\underline{\mathcal{R}}^*(a) = \underline{\mathcal{R}}^*(b)$.

2. Ordered left PP monoids

Similar to left PP monoids and left [semi-] hereditary semigroup, an ordered monoid S is called an *ordered lpp monoid* if all principal left ideals of S are projective, that is, the S-subposet Sx is projective for all $x \in S$ (Note, however, that Sx may not be an ordered ideal of the ordered monoid S.) and S is ordered left [semi-] hereditary if all [finitely generated] left ideals of the semigroup S are projective.

Lemma 2.1. ([19: Proposition 3.2]) *Let A be an S-poset and $a \in A$. Then the following statements are equivalent:*

(1) *Sa is projective;*

(2) *There exists an element $e \in E(S)$ such that $a = ea$, and $sa \leq ta$ implies $se \leq te$ for $s, t \in S$;*

(3) *$Sa \cong Se$ in S-Pos for some $e \in E(S)$.*

Proposition 2.2. *An ordered monoid S is an ordered lpp monoid if and only if each $\underline{\mathcal{R}}^*$-class contains an idempotent.*

P r o o f. It is clear by Lemma 1.4. □

Corollary 2.3. *An ordered monoid S is an ordered lpp monoid if and only if for each element a in S there is an idempotent e in S such that $\underline{\mathcal{R}}^*(a) = eS$.*

P r o o f. We observe that for an idempotent e in any pomonoid S the right ideal eS is a right *-ideal. For if $a \in eS$, then $a = ea$. For any element b in $\underline{R}^*_a$, since $\underline{\mathcal{R}}^* \subseteq \mathcal{R}^*$, we have $b = eb$. Thus $\underline{\mathcal{R}}^*(e) = eS$. By Proposition 2.2, an ordered monoid S is an ordered *lpp* monoid if and only if for each element a in S there is an idempotent e in S such that $e\underline{\mathcal{R}}^*a$, and by Corollary 1.8, we have the result. □

Definition 2.1. Let A is an S-poset, $a \in A, e^2 = e \in S$, a is called ordered right e-cancellable if $a = ea$ and $sa \leq ta$ implies $se \leq te$ for $s, t \in S$.

From Lemma 1.4, we can get that a is ordered right e-cancellable if and only if $a\underline{\mathcal{R}}^*e$.

Corollary 2.4. *S is an ordered lpp monoid if and only if for all $a \in S$ there exists an idempotent $e \in S$ such that a is ordered right e-cancellable.*

Let S be a regular ordered monoid. Then pomonoid S is an ordered *lpp* monoid. In fact, for any $s \in S$, there exists $t \in S$ such that $s = sts$. Put $e = st$, then $e^2 = e \in S$ and $s = es$. For $p, q \in S$, if $ps \leq qs$, then $pst \leq qst$, that is, $pe \leq qe$. Therefore S is an ordered *lpp* monoid.

Lemma 2.5. *Let A be an S-poset and B be an S-subposet of A, $e^2 = e \in S$. Then $B \cong Se$ if and only if B is generated by an ordered right e-cancellable element.*

P r o o f. Let $\phi\colon B \to Se$ be an S-isomorphism with $\phi^{-1}(e) = a$. Then $ea = e\phi^{-1}(e) = \phi^{-1}(e) = a$, if $sa \leq ta$, then $se = s\phi(a) = \phi(sa) \leq \phi(ta) = te$. The converse is obvious by Lemma 2.1. □

Lemma 2.6. ([18: Lemma 2.14]) *All finitely generated S-subposets [all S-subposets] of an S-poset A are semifree if and only if A satisfies the condition*

(c) *incomparable cyclic S-subposets of A are disjoint [and the A.C.C. for cyclic S-subposets of A].*

Corollary 2.7. *All finitely generated left ideals [all left ideals] of a monoid S are semifree if and only if S satisfies the condition*

(c) *incomparable principal left ideals of S are disjoint [and the A.C.C. for principal left ideals of S].*

We note that if an S-poset P is projective, then $P \cong \coprod_{i\in I} Se_i$ where $e_i^2 = e_i \in S$, $i \in I$ (see [19]), thus P is semifree. From Corollary 2.7, we have the following theorem.

Theorem 2.8. *An ordered monoid S is left semi-hereditary [hereditary] if and only if*

(1) *S is an ordered lpp monoid;*

(2) *incomparable principal left ideals of S are disjoint;*

[(3) *and the A.C.C. for principal left ideals of S*].

Corollary 2.9. *A commutative ordered monoid S is left semi-hereditary if and only if all principal left ideals of S form a chain and every principal left ideal of S is generated by an ordered right e-cancellable element for some $e^2 = e \in S$.*

Corollary 2.10. *A commutative ordered monoid S is left hereditary if and only if all left ideals of the monoid S are principal and each is generated by an ordered right e-cancellable element, for some $e^2 = e \in S$.*

3. Ordered left PP monoids with central idempotents

An ordered semigroup $(\Gamma, \cdot, \leq)$ is called an ordered semilattice if $(\Gamma, \cdot)$ is a semilattice. An ordered semilattice Γ is said to be natural if $\alpha \leq \beta \iff \alpha\beta = \alpha$ for all $\alpha, \beta \in \Gamma$. We note that an ordered semilattice may of course be not natural.

In this section we discuss ordered *lpp* monoids with central idempotents.

Lemma 3.1. *If idempotents of an ordered lpp monoid S commute, then for each element $a \in S$ there is a unique idempotent e such that a is ordered right e-cancellable.*

An ordered *lpp* monoid S is called ordered C-*lpp* monoid if all idempotents are central.

Note that if all idempotents of semigroup S commute, then they form a semilattice with respect to the natural ordering.

Let S be an ordered C-*lpp* monoid. For each idempotent e in S, put

$$S_e = \underline{R}^*_e = \{a \in S : \ a \ \text{is ordered right } e\text{-cancellable}\}.$$

Theorem 3.2. *Let S be an ordered C-lpp monoid and E the set of all idempotents of S. Then S is a strong semilattice of ordered right cancellative monoids S_e $(e \in E)$. Furthermore, if S is an ordered left semi-hereditary [hereditary] monoid, then*

1. *E is a chain with a maximum element [an inversely well-ordered chain, that is, E is a well-ordered chain and there exist a maximum element in every non-empty subset of E];*
2. *if E has a zero z, then S_z is left semi-hereditary [left hereditary] and if e is a non-zero element of E then S_e has its principal left ideals linearly ordered [is an ordered principal left ideal monoid].*

P r o o f. From Corollary 2.4, we see that $S = \bigcup_{e\in E} S_e$ where E is the set of all idempotents of S.

In view of Lemma 3.1, if $e, f \in E$ and $e \neq f$, then $S_e \cap S_f = \emptyset$.

Let $a \in S_e$ and $b \in S_f$. We have $ab = ea \cdot fb = efab$, if $sab \leq tab$, then $saf \leq taf$, so $sfa \leq tfa$, consequently $sfe \leq tfe$. It follows that $ab \in S_{ef}$ and similarly $ba \in S_{ef}$.

If $a, b \in S_e$, then $ab \in S_{ee} = S_e$, that is, S_e is an ordered subsemigroup of S with identity e. If $ba \leq ca$ for $a, b, c \in S_e$, then $be \leq ce$, that is, $b \leq c$.

For each pair of $e, f \in E$ such that $e \preceq f$, we define a map $\varphi_{f,e} \colon S_f \to S_e$ by $\varphi_{f,e}(a) = ae$, then it is easy to see that the map is well defined and an ordered monoid morphism.

From the above proof we have $\varphi_{e,e}$ is the identity mapping of S_e for every $e \in E$.

If $e, f, g \in E$ with $e \preceq f \preceq g$, then for $a \in S_g$, we have

$$\varphi_{f,e}\varphi_{g,f}(a) = \varphi_{f,e}(af) = afe = ae = \varphi_{g,e}(a).$$

Thus S is a strong semilattice of ordered right cancellative monoids S_e $(e \in E)$.

(1) Suppose that S is an ordered left semi-hereditary monoid. For $e, f \in E$, $ef \in Se \cap Sf$, by Theorem 2.8, Se and Sf are comparable, say $Se \subseteq Sf$, then $e = sf$ for some $s \in S$, so $ef = e$, that is, $e \preceq f$. We note that $e \preceq 1$ for any $e \in E$, hence E is a chain with the maximum element 1. If S is an ordered left hereditary monoid, then S satisfies the A.C.C. for principal left ideals of monoid S, it follows that E is an inverse well-ordered chain.

(2) The proof is similarly to [5]. □

Note that there is no idempotent other than e in S_e. In fact, if $f^2 = f \in S_e$, then $fe = e$ by e-cancellation. On the other hand, $ef = f$ since e is identity, thus $f = e$.

Let Γ be a natural ordered semilattice with identity. For every $\alpha \in \Gamma$, let S_α be an ordered monoid (let the identity be e_α) such that $S_\alpha \bigcap S_\beta = \emptyset$ if $\alpha \neq \beta$. Assume that for each pair of elements $\alpha, \beta \in \Gamma$ with $\alpha \preceq \beta$, there exists an ordered monoid morphism $\varphi_{\beta,\alpha}\colon S_\beta \to S_\alpha$ satisfying:

(1) $\varphi_{\alpha,\alpha} = 1_{S_\alpha}$ for each $\alpha \in \Gamma$;

(2) $\varphi_{\gamma,\beta}\varphi_{\beta,\alpha} = \varphi_{\gamma,\alpha}$ for every $\alpha, \beta, \gamma \in \Gamma$ such that $\alpha \preceq \beta \preceq \gamma$.

Let $S = \bigcup_{\alpha\in\Gamma} S_\alpha$, define a multiplication "$\cdot$" and an order "$\leq$" on it by the rule that: if $a_\alpha \in S_\alpha$, $b_\beta \in S_\beta$,

$$a_\alpha b_\beta = \varphi_{\alpha,\alpha\beta}(a_\alpha)\varphi_{\beta,\alpha\beta}(b_\beta),$$
$$a_\alpha \leq b_\beta \iff \alpha \preceq \beta \text{ and } a_\alpha \leq \varphi_{\beta,\alpha}(b_\beta) \text{ in } S_\alpha.$$

Then S is an ordered monoid, which is called strongly naturally ordered semilattice of its ordered submonoids S_α $(\alpha \in \Gamma)$ (see [1], [3]).

For the converse of Theorem 3.2, we have the following theorem.

Theorem 3.3. *If S is strongly naturally ordered semilattice of its ordered right cancellative submonoids S_α $(\alpha \in \Gamma)$, then S is an ordered C-lpp monoid. Moreover, if Γ has a zero ζ, let S_ζ be left semi-hereditary [left semi-hereditary] and if α is a non-zero element of Γ, let S_α have its principal left ideals linearly ordered [be a principal left ideal monoid]. Then S is an ordered left semi-hereditary [hereditary] monoid with central idempotents.*

P r o o f. Suppose that α_0 is the identity of semilattice Γ. Let $a \in S$, say $a \in S_\alpha$, then $ae_{\alpha_0} = \varphi_{\alpha,\alpha\alpha_0}(a_\alpha)\varphi_{\alpha_0,\alpha\alpha_0}(e_{\alpha_0}) = \varphi_{\alpha,\alpha}(a_\alpha)\varphi_{\alpha_0,\alpha}(e_{\alpha_0}) = a_\alpha e_\alpha = a_\alpha = a$, this implies S is monoid and identity is the identity e_{α_0} of monoid S_{α_0}.

Clearly, $E(S) = \{e_\alpha : \ \alpha \in \Gamma\}$. From [5], we know that idempotents are central.

Let $a \in S_\alpha$, then $a = e_\alpha a$. Now for every two elements $s, t \in S$, say $s \in S_\beta$, $t \in S_\gamma$, with $sa \leq ta$, by the order we have $\beta\alpha \preceq \gamma\alpha$ and $sa \leq \varphi_{\gamma\alpha,\beta\alpha}(ta)$. Hence

$$\begin{aligned}\varphi_{\beta,\beta\alpha}(s)\varphi_{\alpha,\beta\alpha}(a) = sa \leq \varphi_{\gamma\alpha,\beta\alpha}(ta) &= \varphi_{\gamma\alpha,\beta\alpha}(\varphi_{\gamma,\gamma\alpha}(t)\varphi_{\alpha,\gamma\alpha}(a))\\ &= \varphi_{\gamma\alpha,\beta\alpha}(\varphi_{\gamma,\gamma\alpha}(t))\varphi_{\gamma\alpha,\beta\alpha}(\varphi_{\alpha,\alpha\gamma}(a)) = \varphi_{\gamma,\beta\alpha}(t)\varphi_{\alpha,\beta\alpha}(a),\end{aligned}$$

that is, $\varphi_{\beta,\beta\alpha}(s)\varphi_{\alpha,\beta\alpha}(a) \leq \varphi_{\gamma,\beta\alpha}(t)\ \varphi_{\alpha,\beta\alpha}(a)$. Since $S_{\beta\alpha}$ is ordered right cancellable, this gives $\varphi_{\beta,\beta\alpha}(s) \leq \varphi_{\gamma,\beta\alpha}(t)$, thus

$$\varphi_{\beta,\beta\alpha}(s)\varphi_{\alpha,\beta\alpha}(e_\alpha) \leq \varphi_{\gamma,\beta\alpha}(t)\varphi_{\alpha,\beta\alpha}(e_\alpha),$$

that is, $se_\alpha \leq te_\alpha$. Therefore a is ordered right e_α-cancellable and consequently S is an ordered *C-lpp* monoid. The residue of proof is similarly to [5]. □

Observe that if every $\underline{\mathcal{R}}^*$-class has an idempotent, then so does every $\mathcal{R}^*$-class. Hence the monoids in Theorem 3.2 and 3.3 are left abundant with central idempotents.

From Theorem 3.2 and Theorem 3.3, and noting that $\varphi_{f,e}(b) = be$ for $e \preceq f$ in Theorem 3.2, we have

Theorem 3.4. *The following conditions are equivalent on an ordered monoid S:*

(1) *S is an ordered lpp [left semi-hereditary (hereditary)] monoid with central idempotents satisfying:*

$$\big(\forall a \in S_e\big)\big(\forall b \in S_f\big)\big(a \leq b \iff e \preceq f \text{ and } a \leq be\big),$$

where $S_e = \{a \in S : a$ is ordered right e-cancellable$\}$;

(2) *there exists natural ordered semilattice Γ such that S is a strong natural ordered semilattice of ordered right cancellable monoids S_α $(\alpha \in \Gamma)$;*

[(2) *Γ is a chain with a maximum element (an inverse well-ordered chain), and if Γ has a zero z, then S_z be left semi-hereditary (left hereditary) and if e is a non-zero element of Γ then S_e has its principal left ideals linearly ordered (is an ordered principal left ideal monoid)*].

Let S be a *C-lpp* monoid, note that we also have the monoid S_e to work with, and therefore we can define an ordered relation on S as following

$$\big(\forall a \in S_e\big)\big(\forall b \in S_f\big)\big(a \leq b \iff e = ef \text{ and } a = be\big),$$

where $S_e = \{a \in S : a$ is right e-cancellable$\}$. It is easy to see that monoid S become an ordered monoid and $e \leq f$ if and only if $e \preceq f$ for $e, f \in E(S)$. Let $a \in S_e$, $b \in S_f$, $c \in S_g$ for $e, f, g \in E(S)$. If $ac \leq bc$, then $eg = egfg$ and $ac = bceg$ since $ac \in S_{eg}$, $bc \in S_{fg}$. By g-cancellable and $ac = bceg$, we have $ag = beg$, thus $ag = bgeg$ and $eg \preceq fg$, therefore $ag \leq bg$, hence c is ordered right g-cancellable. Thus S is an ordered *C-lpp* monoid and ordered right e-cancellable coincides with right e-cancellable. From the above discussion, we have:

Corollary 3.5. *The monoid S is a C-lpp monoid if and only if S is a strong semilattice of right cancellable monoids.*

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