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POLYNOMIALS WITH COEFFICIENTS FROM A FINITE SET

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ABSTRACT. This paper focuses on the problem concerning the location and the number of zeros of those polynomials when their coefficients are restricted with special conditions. The problem of the number of the zeros of reciprocal Littlewood polynomials on the unit circle $\mathbb{T}$ is discussed, the interest on bounds for the number of the zeros of reciprocal polynomials on the unit circle arose after 1950 when Erdös began introducing problems on zeros of various types of polynomials. Our main result is the problem of finding the number of zeros of complex polynomials in an open disk.

1. Introduction

Locating zeros of polynomials with special conditions for the coefficients, in particular, the location and the number of zeros of those polynomials having coefficients in a finite subset of the complex plane $\mathbb{C}$ has applications in many areas of applied mathematics, including linear control systems, electrical networks, root approximation, and signal processing, so there is a need for obtaining better results in these subjects. A review on the location of zeros of polynomials can be found in [1], [11], [14], [15], and [17]. Also, a problem is of finding the number of the zeros of reciprocal Littlewood polynomials on the unit circle. Lakatos and Losonczy [10] considered reciprocal polynomials which all their zeros are on the unit circle. Erdélyi [7] proved that every reciprocal Littlewood polynomial has at least one zero on the unit circle, but recently Mukunda [13: Theorem 3] improved this result for odd degree reciprocal Littlewood polynomials by proving

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that every reciprocal Littlewood polynomial of odd degree $n \geq 3$ has at least three zeros on $\mathbb{T}$, and also he determined all Pisot numbers whose minimal polynomials are Littlewood polynomials. Salem [16] showed that the set of all Pisot numbers is closed.

A *Pisot number* is a real algebraic integer greater than 1, all of whose conjugates lie inside the open unit disk.

If $p(z)$ is a polynomial in $\mathbb{R}[z]$ of degree n, then we define $p^*(z) = z^n p(z^{-1})$, and we say that $p(z)$ is *reciprocal* if $p(z) = \pm p^*(z)$.

In what follows, $\mathbb{C}[z]$ denotes the set of all complex polynomials with complex coefficients and we consider two subsets $\mathcal{U}_n$ and $\mathcal{L}_n$ of $\mathbb{C}[z]$ of all degree n polynomials h,

$$h(z) = \sum_{k=0}^{n} a_k z^k,$$

so that if $h \in \mathcal{U}_n$, then $|a_k| = 1$ for all k, and if $h \in \mathcal{L}_n$, then $a_k = \pm 1$ for all k. The members of $\mathcal{U}_n$ and $\mathcal{L}_n$ are called *Unimodular* and *Littlewood* (or sometimes real unimodular) polynomials, respectively.

Next, let $p \in \mathbb{C}[z]$ be a polynomial of degree n as

$$p(z) = \sum_{k=0}^{n} a_k z^k. \tag{1.1}$$

In this work, we will obtain bounds for the zeros of the following three types of polynomials in (1.1).

Type 1: $|a_0| = 1$ and $|a_k| \leq 1$ for every $k \in \{1, 2, \ldots, n\}$.

Type 2: $|a_n| = 1$ and $|a_k| \leq 1$ for every $k \in \{0, 1, \ldots, n-1\}$.

Type 3: $|a_0| = |a_n| = 1$ and $|a_k| \leq 1$ for every $k \in \{1, 2, \ldots, n-1\}$.

Throughout the paper $D(z_0, r)$ denotes the open disk in the complex plane centered at z_0 with radius $r > 0$.

2. On the location of zeros of polynomials

There are many new results about the location of zeros and bounds for the zeros of polynomials with restricted coefficients. In this section, we state some results about the location of zeros and the number of zeros of some types of polynomials. Borwein, Erdélyi, and Littmann [4] proved that any polynomial of type 3 has at least $8\sqrt{n}\log n$ zeros in disk with center on the unit circle and radius $33\pi\frac{\log n}{\sqrt{n}}$. The first theorem proved in [3], provides upper bounds for the number of real zeros of those polynomials that their coefficients are restricted in the closed unit circle.

Theorem 2.1.

(i) *There is an absolute constant $c_1 > 0$ such that every polynomial p of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j, \qquad a_j \in \mathbb{C}, \quad |a_0| = 1, \quad |a_j| \leq 1,$$

has at most $c_1\sqrt{n}$ zeros in $[-1, 1]$.

(ii) *There is an absolute constant $c_2 > 0$ such that every polynomial p of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j, \qquad a_j \in \mathbb{C}, \quad |a_n| = 1, \quad |a_j| \leq 1,$$

has at most $c_2\sqrt{n}$ zeros in $\mathbb{R}\backslash(-1, 1)$.

(iii) *There is an absolute constant $c_3 > 0$ such that every polynomial p of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j, \qquad a_j \in \mathbb{C}, \quad |a_0| = |a_n| = 1, \quad |a_j| \leq 1,$$

has at most $c_3\sqrt{n}$ real zeros.

The next theorem which is due to Erdélyi [7], give an upper bound for the number of zeros of polynomials of type 1.

Theorem 2.2. *Let $\alpha \in (0, 1)$. Every polynomial of the form*

$$p(x) = \sum_{j=0}^{n} a_j x^j, \qquad a_j \in \mathbb{C}, \quad |a_0| = 1, \quad |a_j| \leq 1,$$

has at most $(2/\alpha)\log(1/\alpha)$ zeros in the open disk $D(0, 1 - \alpha)$.

Now we provide a classical bound due to Caushy [5] for the zeros of polynomials with complex coefficients. Some other classical bounds can be found in [11], [12], and [14].

Theorem 2.3 (Cauchy's bound). *Let $p(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + z^n$ be a monic polynomial with complex coefficients. Then all the zeros of $p(z)$ lie in the open disk $D(0, 1 + A)$, where $A = \max|a_j|$, for every $j = 0, \ldots, n - 1$.*

The following theorem and lemma which are respectively proved in [9] and [13] have key role in the proof of the theorem (Theorem 3.1) about the number of zeros of reciprocal Littlewood polynomials on the unit circle.

Theorem 2.4. *Let γ_n be a Pisot number of degree n whose minimal polynomial is of the form*

$$p_n(z) = z^d + \sum_{k=0}^{d-1} a_k z^k,$$

where $a_k = \pm N$ $(k = 0, \dots, d-1)$ for positive integer $N \geq 2$. Then

$$p_n(z) = z^n - N \sum_{k=0}^{n-1} z^k,$$

and the sequence $\{\gamma_n\}$ is strictly increasing and converges to $N+1$.

Lemma 2.1. *Suppose $p(z)$ is a polynomial in $\mathbb{C}[z]$, m is a positive integer and w is a complex number on $\mathbb{T}$. Then the number of roots of $R_m(z) = wz^m p(z) \pm p^*(z)$ in the closed unit disk is greater than or equal to the number of roots of $S_m(z) = z^m p(z)$ in the same region.*

Notice that $p(z)$ and $p^*(z)$ have the same zeros on the unit circle, and that these are also zeros of both $R_m(z)$ and $S_m(z)$.

3. The main results

In this section, we state and prove some results concerning bounds for the zeros of some classes of complex polynomials, among them the unimodular polynomials, and also we find bounds for the number of zeros of some types of polynomials which lie in an open disk. The first theorem of this section is an important result about the number of the zeros of reciprocal Littlewood polynomials on the unit circle. The next lemma provides a bound for the number of zeros of some complex reciprocal Littlewood polynomial on the unit circle. These can be found in [6].

Lemma 3.1. *The reciprocal polynomial $p(z) = z^{2n} + z^{2n-1} + \dots + z^{n+1} - z^n + z^{n-1} + \dots + z + 1$ has at least $(2n-8)/3$ zeros on the unit circle.*

Proof. Let $q(z) := (z-1)p(z) = z^{2n+1} - 2z^{n+1} + 2z^n - 1$. Fix $z \in \mathbb{T}$ and let $t \in \mathbb{R}$ be so that $z = \mathrm{e}^{\mathrm{i}t}$. Then by Euler's identity $q(\mathrm{e}^{\mathrm{i}t}) = 2\mathrm{i}\psi(t)\mathrm{e}^{\frac{2n+1}{2}t\mathrm{i}}$, where $\psi(t) = \sin\frac{2n+1}{2}t - 2\sin\frac{t}{2}$. Since $\psi(2\pi - t) = \psi(t)$, so the function $\psi(t)$ has the same number of zeros in the intervals $\left[0, \frac{\pi}{3}\right]$ and $\left[\frac{5\pi}{3}, 2\pi\right]$, but $\psi(t)$ has at least $(n-4)/3$ zeros in the interval $\left(0, \frac{\pi}{3}\right]$, because it changes its sign in the interval

$$\left[\frac{2k\pi + \pi}{2n+1}, \frac{2(k+1)\pi + \pi}{2n+1}\right] \subset \left(0, \frac{\pi}{3}\right],$$

for $k = 0, 1, \ldots, \left[\frac{n-4}{3}\right]$. Therefore, $\psi(t)$ has at least $(2n-8)/3$ zeros in the interval $(0, 2\pi)$, that is, the function $q(\mathrm{e}^{\mathrm{i}t})$ has at least $(2n-8)/3$ zeros in the interval $(0, 2\pi)$. Thus the polynomial $p(z)$ has at least $(2n-8)/3$ zeros on $\mathbb{T}$. □

Theorem 3.1. *Every reciprocal Littlewood polynomial of odd degree $m \geq 3$ has at least three zeros on the unit circle. Every reciprocal Littlewood polynomial of even degree $m \geq 14$ has at least four zeros on the unit circle.*

Proof. Let $R(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n + a_{n-1} z^{n+1} + \cdots + a_0 z^{2n}$ be a reciprocal Littlewood polynomial of degree $2n$, and so $R(z)$ has $2n$ zeros. Note that $1/z_0$ is a zero of $R(z)$, whenever z_0 is a zero of $R(z)$. Since $R(\pm 1)$ is an odd integer, so both $R(1)$ and $R(-1)$ are nonzero, and $R(z)$ has at least one zero on the unit circle (see [7]), hence it must have the even number of zeros on the unit circle. If this the number is exactly two, then $R(z)$ has $n-1$ zeros inside the open unit circle and we write $R(z) = a_n/2\big(z^n p^*(z) + p(z)\big)$, where

$$p(z) = z^n + 2a_n \sum_{k=0}^{n-1} a_k z^k.$$

Now by Lemma 2.1 and that $R(z)$ has $n-1$ zeros inside the open unit circle we conclude that $z^n p^*(z)$ has $n+1$ zeros inside the closed unit circle, that is, $p^*(z)$ has only one zero in the closed unit circle. That means $p(z)$ has only one zero γ outside the open unit circle, so γ is real and $|\gamma|$ is a Pisot number. If we put $q_n(z) = z^n - 2z^{n-1} - \cdots - 2z - 2$, then by Theorem 2.4, for $N = 2$, we have $p(z) = q_n(z)$ if $\gamma > 1$, and in this case

$$R(z) = -a_n(z^{2n} + z^{2n-1} + \cdots + z^{n+1} - z^n + z^{n-1} + \cdots + z + 1)$$

and $p(z) = (-1)^n q_n(-z)$ if $\gamma < -1$, also in this case,

$$R(z) = (-1)^{n+1} a_n \big[(-z)^{2n} + (-z)^{2n-1} + \cdots + (-z)^{n+1} - (-z)^n + (-z)^{n-1} + \cdots + (-z) + 1\big].$$

Therefore, if degree $R(z) \geq 4$, then it has at least four zeros on the unit circle, except possibly when $R(z)$ is of the form $r(z) = z^{2n} + z^{2n-1} + \cdots + z^{n+1} - z^n + z^{n-1} + \cdots + z + 1$. On the other hand, by Lemma 3.1 for $n \geq 10$, we conclude $r(z)$ has at least $(2n-8)/3 \geq 4$ zeros on $\mathbb{T}$. Finally, a computational argument with computer for $n \in \{7, 8, 9\}$, shows that the polynomial $r(z)$ has at least four zeros on the unit circle and this completes the proof. □

Note that bounds on degree m of polynomials in Theorem 3.1 are sharp, for example, the polynomial $S(z) = z + 2$ has no zeros on $\mathbb{T}$, and the polynomial $R(z) = z^{12} + z^{11} + z^{10} + z^9 + z^8 + z^7 - z^6 + z^5 + z^4 + z^3 + z^2 + z + 1$ has only two zeros on $\mathbb{T}$. The next theorem shows that a bound sharper than Cauchy's bound can be obtain if the length of all consecutive coefficients in a monic polynomial are small enough. For example, the polynomial $p(z) = 1 + z + 2z^2 + 2z^3 + z^4$.

Theorem 3.2. *Let $p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n$ be a polynomial in $\mathbb{C}[z]$. If z_0 is any zero of $p(z)$, then $|z_0| < 1 + A$, where $A = \max\{|a_0|, |a_1 - a_0|, \ldots, |1 - a_{n-1}|\}$.*

P r o o f. We may assume that $|z_0| > 1$, for otherwise there is nothing to prove. Now consider the polynomial $q(z) := (1 - z)p(z)$. If $|z| > 1$, then we obtain

$$\begin{aligned} |q(z)| &\geq |z|^{n+1} - \{|a_0| + |a_1 - a_0||z| + \cdots + |1 - a_{n-1}||z|^n\} \\ &\geq |z|^{n+1} - A \sum_{i=0}^{n} |z|^i \\ &= |z|^{n+1} - A \frac{|z|^{n+1} - 1}{|z| - 1} \\ &> |z|^{n+1} - A \frac{|z|^{n+1}}{|z| - 1} \\ &= \frac{1}{|z| - 1} \left\{ |z|^{n+2} - |z|^{n+1}(1 + A) \right\}. \end{aligned}$$

We can write

$$|z|^{n+2} - |z|^{n+1}(1 + A) = |z|^{n+1}[|z| - (1 + A)].$$

Hence, we conclude that $|q(z)| > 0$, whenever $|z| \geq 1 + A$. Therefore, all the zeros of $q(z)$ lie in the disk $D(0, 1 + A)$. Since $p(z)$ and $q(z)$ have the same zeros, thus $|z_0| < 1 + A$. □

Corollary 3.2.1. *Let $p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n$ be a polynomial in $\mathbb{C}[z]$ such that $|a_i| \leq |a_0|$, for every $i = 1, \ldots, n - 1$. Then all the zeros of $p(z)$ lie in the open disk $D(0, 1 + r)$, where $r = \max\{1 + |a_0|, 2|a_0|\}$.*

Theorem 3.3. *Suppose $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ is a degree $n \geq 1$ polynomial in $\mathbb{C}[z]$ such that for every $i = 0, \ldots, n - 1$, $|a_i| \leq |a_n|$. If z_0 is any zero of $p(z)$, then $|z_0| < 2$.*

P r o o f. First, we see that

$$\begin{aligned} |p(z)| &\geq |a_n||z|^n - \left\{ |a_0| + |a_1||z| + \cdots + |a_{n-1}||z|^{n-1} \right\} \\ &= |a_n||z|^n \left\{ 1 - \left\{ \frac{|a_0|}{|a_n|} \frac{1}{|z|^n} + \cdots + \frac{|a_{n-1}|}{|a_n|} \frac{1}{|z|} \right\} \right\}. \end{aligned}$$

It is clear that if $|z_0| \leq 1$, then there is nothing to prove, so without loss of generality we assume $|z| > 1$. Because of $|a_i| \leq |a_n|$ for all i, by above we get,

$$\begin{aligned} |p(z)| &\geq |a_n||z|^n\left\{1-\sum_{i=1}^{n}\frac{1}{|z|^i}\right\} > |a_n||z|^n\left\{1-\sum_{i=1}^{\infty}\frac{1}{|z|^i}\right\} \\ &= |a_n||z|^n\left\{2-\frac{1}{1-\frac{1}{|z|}}\right\} = |a_n||z|^n\left\{\frac{|z|-2}{|z|-1}\right\}. \end{aligned}$$

Thus $|p(z)| > 0$, whenever $|z| \geq 2$ and the proof is complete. □

Corollary 3.3.1. *If p is a polynomial of type 3, then every zero of p lie in the annulus*

$$\frac{1}{2} < |z| < 2.$$

P r o o f. The proof is based on the fact that if $p(z)$ is of type 3, then $q(z) = z^n p(1/z)$ is also of type 3. □

Remark 1. Since every unimodular polynomials is of type 3, so the above corollary satisfies for the unimodular polynomials.

Proposition 3.4. *Suppose p is a degree $n \geq 1$ polynomial of type 2. If z_0 is any zero of p, then $|z_0| \leq r$, where $r = \max\{1, s\} < 3$ and $s \neq 1$ is the positive zero of the equation*

$$q(z) = z^{n+2} - 3z^{n+1} + z + 1.$$

P r o o f. Consider the polynomial $(1-z)p(z)$, then we have

$$\begin{aligned} |(1-z)p(z)| &= \left|a_0 + (a_1-a_0)z + \cdots + (a_n - a_{n-1})z^n - a_n z^{n+1}\right| \\ &\geq |a_n||z|^{n+1} - \{|a_0| + |a_1-a_0||z| + \cdots + |a_n-a_{n-1}||z|^n\} \\ &\geq |z|^{n+1} + 1 - 2\{1 + |z| + \cdots + |z|^n\} \\ &= |z|^{n+1} + 1 - 2\frac{|z|^{n+1}-1}{|z|-1} = \frac{|z|^{n+2} - 3|z|^{n+1} + |z| + 1}{|z|-1}. \end{aligned}$$

Applying Descarte's rules of sign, $q(z)$ has only two positive zeros, say 1 and s. Clearly, $q(s) = 0$ implies that $s < 3$. Since $q(1) = 0$ and sign$\{q(0)\}$ is positive, therefore $|q(z)| > 0$ if $|z| > r$ and so $|(1-z)p(z)| > 0$ if $|z| > r$, this completes the proof. □

Finally, we will use a well-known theorem of complex analysis; namely, "Jensen's Formula" it provides bounds for the number of zeros of polynomials in a given open disk.

Lemma 3.2 (Jensen's Formula). *Suppose that h is a nonnegative integer and that*

$$f(z)=\sum_{k=h}^{\infty}c_k(z-z_0)^k,\qquad c_h\neq 0,$$

is analytic on the closure of the disk $D(z_0,r)$ and also assume that the zeros of f in $D(z_0,r)\backslash\{z_0\}=\big\{z\in\mathbb{C}:\ 0<|z-z_0|<r\big\}$ are $z_1,z_2,\ldots,z_m$, where each zero is listed as many times as its multiplicity. Then

$$\log|c_h|+h\log r+\sum_{k=1}^{m}\log\frac{r}{|z_k-z_0|}=\frac{1}{2\pi}\int_0^{2\pi}\log|f(z_0+re^{i\theta})|\,d\theta.$$

Theorem 3.5. *Let p be a polynomial of type 1 and $0<\alpha<1$. Let $z_0\in\mathbb{C}$ be so that $D(z_0,1-\alpha)\subset D(0,1)$. Then p has less than $\frac{2}{\alpha}\log\big(\frac{2}{\alpha}c(\alpha)\big)$ zeros in the open disk $D(z_0,1-\alpha)$, where $c(\alpha)$ is a positive constant depending only on α.*

Proof. The inequality $\frac{|z|}{1-|z|}<1$ for $|z|<\frac{1}{2}$ implies that $|p(z)|>1-\frac{|z|}{1-|z|}>0$. If $\alpha\geq\frac{1}{2}+|z_0|$ and $z\in D(z_0,1-\alpha)$, then $|z|<\frac{1}{2}$. Hence, p does not have any zeros in $D(z_0,1-\alpha)$, whenever $\alpha\geq\frac{1}{2}+|z_0|$ (or $\alpha\geq\frac{1}{2}$), thus in this case the conclusion of the theorem is true, so assume that $0<\alpha<\frac{1}{2}$. Since $|z_0|+1-\alpha<1$, we obtain

$$\begin{aligned}|p(z_0)| &\geq |a_0|-\sum_{i=1}^{n}|a_i||z_0|^i\\ &\geq 2-\sum_{i=0}^{\infty}|z_0|^i\\ &> 2-\sum_{i=0}^{\infty}\alpha^i\\ &= 2-\frac{1}{1-\alpha}=\frac{1-2\alpha}{1-\alpha}.\end{aligned}$$

If $z\in D(0,1)$, then we have the growth condition

$$|p(z)|=\Big|\sum_{i=0}^{n}a_iz^i\Big|\leq\sum_{i=0}^{\infty}|z|^i=\frac{1}{1-|z|}.$$

Let $z_1,z_2,\ldots,z_s$ be the zeros of p in $D(z_0,1-\frac{\alpha}{2})\setminus\{z_0\}$ and each zero is listed as many times as its multiplicity, then by Jensen's formula we have

$$\log|p(z_0)|+\sum_{i=1}^{s}\log\frac{1-\frac{\alpha}{2}}{|z_i-z_0|}=\frac{1}{2\pi}\int_0^{2\pi}\log\Big|p\Big(z_0+\Big(1-\frac{\alpha}{2}\Big)e^{i\theta}\Big)\Big|\,d\theta.\qquad(3.1)$$

From the growth condition, we conclude that

$$\frac{1}{2\pi}\int_0^{2\pi}\log\Big|p\Big(z_0+\Big(1-\frac{\alpha}{2}\Big)e^{i\theta}\Big)\Big|\,d\theta\le\log\frac{2}{\alpha}. \tag{3.2}$$

In other words, if $z_1, z_2, \dots, z_m$ $(m \le s)$ are the zeros of p in $D(z_0, 1-\alpha)\backslash\{z_0\}$, then

$$\begin{aligned} m\log\frac{1-\frac{\alpha}{2}}{1-\alpha} &= \sum_{i=1}^{m}\Big(\log\Big(1-\frac{\alpha}{2}\Big)-\log(1-\alpha)\Big)\\ &< \sum_{i=1}^{m}\Big(\log\Big(1-\frac{\alpha}{2}\Big)-\log|z_i-z_0|\Big)\\ &< \sum_{i=1}^{s}\Big(\log\Big(1-\frac{\alpha}{2}\Big)-\log|z_i-z_0|\Big)\\ &= \sum_{i=1}^{s}\log\frac{1-\frac{\alpha}{2}}{|z_i-z_0|}. \end{aligned}$$

Combining this with inequalities (3.1) and (3.2), we obtain

$$\log|p(z_0)|+m\log\frac{1-\frac{\alpha}{2}}{1-\alpha}<\log\frac{2}{\alpha}. \tag{3.3}$$

By the power series expansion of the function $\log(1-x)$, where $|x|<1$, we get

$$\log\frac{1-\frac{\alpha}{2}}{1-\alpha}\ge\frac{\alpha}{2}. \tag{3.4}$$

Now $|p(z_0)| \ge \frac{1-2\alpha}{1-\alpha}$, and the inequalities (3.3), (3.4) imply that

$$m\frac{\alpha}{2}<\log\frac{2}{\alpha}-\log\frac{1-2\alpha}{1-\alpha}=\log\Big(\frac{2}{\alpha}\frac{1-\alpha}{1-2\alpha}\Big),$$

therefore, $m < \frac{2}{\alpha}\log\big(\frac{2}{\alpha}c(\alpha)\big)$, where $c(\alpha) = \frac{1-\alpha}{1-2\alpha}$. □

If p is a unimodular polynomial of degree n and $0 < r < 1$, then p has less than $\frac{2\log(1-\sqrt{r})}{\log r}$ zeros in the open disk $D(0, r)$. This is due to the following theorem.

Theorem 3.6. *Suppose that p is a polynomial of type 1 and $0 < r < 1$. Then p has less than $c(r)$ zeros in the open disk $D(0, r)$, where $c(r)$ is a positive constant depending only on r.*

P r o o f. Let p be a polynomial of the form given in the theorem and $z_1, z_2, \ldots, z_m$ be the zeros of p in $D(0,r) = \{z \in \mathbb{C} : \ |z| < r\}$ (note that: $p(0) = a_0 \neq 0$). Assume that $r < R < 1$ and applying Jensen's formula on $D(0,R)$ for p with $h = 0$ and $|a_0| = 1$, then we have

$$\sum_{j=1}^{s} \log R - \sum_{j=1}^{s} \log |z_j| = \frac{1}{2\pi} \int_0^{2\pi} \log |p(R\mathrm{e}^{\mathrm{i}\theta})| \, \mathrm{d}\theta,$$

where $z_1, z_2, \ldots, z_s$ are the zeros of p in $D(0,R)$ ($s \geq m$). The left-hand side of the above relation can be written as

$$\begin{aligned} \sum_{j=1}^{s} \log R - \sum_{j=1}^{s} \log |z_j| &= \sum_{j=1}^{m} (\log R - \log |z_j|) + \sum_{j=m+1}^{s} (\log R - \log |z_j|) \\ &= \frac{1}{2\pi} \int_0^{2\pi} \log |p(R\mathrm{e}^{\mathrm{i}\theta})| \, \mathrm{d}\theta. \end{aligned}$$

Since both terms in the middle part of this relation are positive, thus

$$\sum_{j=1}^{m} \big(\log R - \log |z_j|\big) < \frac{1}{2\pi} \int_0^{2\pi} \log |p(R\mathrm{e}^{\mathrm{i}\theta})| \, \mathrm{d}\theta.$$

But $|z_j| < r$ for all $j = 1, \ldots, m$ implies that

$$\log R - \log |z_j| > \log R - \log r,$$

and then

$$m(\log R - \log r) < \sum_{j=1}^{m} \big(\log R - \log |z_j|\big) < \frac{1}{2\pi} \int_0^{2\pi} \log |p(R\mathrm{e}^{\mathrm{i}\theta})| \, \mathrm{d}\theta. \tag{3.5}$$

We now try to evaluate $|p(R\mathrm{e}^{\mathrm{i}\theta})|$.

If $p(z) = \sum\limits_{j=0}^{n} a_j z^j$ and $z \in D(0,1)$, then by the growth condition, we obtain

$$\log |p(R\mathrm{e}^{\mathrm{i}\theta})| \leq -\log(1 - R),$$

finally, if we choose $R = \sqrt{r}$, then

$$\frac{1}{2\pi} \int_0^{2\pi} \log |p(\sqrt{r}\mathrm{e}^{\mathrm{i}\theta})| \, \mathrm{d}\theta \leq -\log\big(1 - \sqrt{r}\,\big).$$

This inequality along with (3.5), imply that $m < c(r)$, where $c(r) = \frac{2\log(1-\sqrt{r}\,)}{\log r}$.

□

Open problems

Two interesting open problems related to the reciprocal polynomials are:

1) What is the minimum number of zeros of modulus 1 of a reciprocal polynomial with coefficients in the set $\{0, 1\}$?
2) What is the minimum number of zeros of modulus 1 of a reciprocal polynomial with coefficients in the set $\{-1, 1\}$?

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