Research Article

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Space-Time Estimates of Mild Solutions of a Class of Higher-Order Semilinear Parabolic Equations in $L^p$

Abstract: We establish the well-posedness of boundary value problems for a family of nonlinear higher-order parabolic equations which comprises some models of epitaxial growth and thin film theory. In order to achieve this result, we provide a unified framework for constructing local mild solutions in $C^0([0, T]; L^p(\Omega))$ by introducing appropriate time-weighted Lebesgue norms inspired by a priori estimates of solutions. This framework allows us to obtain global existence of solutions under the proviso that initial data are reasonably small.

Keywords: Epitaxy, Thin-film Equation, Scaling invariance, $L_p$ – $L_q$ Estimates, Analytic Semigroup, Kato’s Method, Mild Solution

MSC: 35A01, 35A02

DOI 10.2478/msds-2014-0003
Received January 16, 2014; accepted March 21, 2014.

1 Introduction

Epitaxy deals with the growth of a thin film on a substrate in such a way that the thin film inherits some desired properties of the underlying substrate. Atoms of the thin film commonly refer to as adatoms are deposited through methods including pulse laser, molecular beam and chemical vapor. Epitaxial growth is amongst the most cost effective ways of producing modern nanomaterials and semiconductors. There are several lengths and times scale involved in epitaxial growth which make its mathematical modeling and numerical simulation difficult. There are three main approaches to modeling epitaxy: atomistic [11, 16], continuum [17] and hybrid [15]. In the atomistic approaches adatoms dynamics is depicted using kinematic Monte Carlo methods. Atomistic approaches are computationally costly but they are useful for capturing small scale features of epitaxial growth. Continuum models are expressed in terms of partial differential equations which encode mainly conservation laws. They are good for describing macroscopic features of epitaxial growth at reasonable computational cost. Hybrid approaches strike a compromise between atomistic and continuum approaches. In this paper, we are chiefly interested in continuum models in which the dynamical quantity of interest is the thin film height. Under the assumption that relaxation processes occurring at the surface of the thin film maintain the deposited volume and fluctuations in the deposition flux are negligible, the height evolution in a frame comoving with the thin film interface can be expressed as [10]

$$\frac{\partial h(y, t)}{\partial t} = -\nabla \cdot J,$$

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where $J = (J_1, J_2)$ is the surface current vector, $y = (y_1, y_2)$ is a point on the two-dimensional base plane and $t$ is time.

In this work we consider a subclass of Eq. (1) which has the following form:

$$u_t + (-\Delta)^m u = \nabla \cdot f(\nabla u),$$  
(2)

where $u$ may be interpreted as the scaled film height and Eq. (2) is supplemented with boundary conditions

$$u(0) = \varphi(x), \quad x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n,$$  
(3)

or

$$u(0) = \varphi(x), \quad x \in \Omega,$$

$$\frac{\partial u}{\partial \nu} \bigg|_{\partial \Omega} = 0, \quad j \leq m - 1,$$  
(4)

where $m$ is an integer greater or equal to 2, $\Omega \subset \mathbb{R}^n$ is a bounded domain with $C^{2m}$ boundary $\partial \Omega$, $\nu$ denotes the unit outward normal to $\partial \Omega$, $\nabla = (\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n})$, $\Delta = \sum_{i=1}^{n} \partial_{x_i}^2$, and $\partial u/\partial \nu = \nu \cdot \nabla u$.

Throughout this paper we assume that $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, $f(0) = 0$ and for some $\alpha > 1$ and $C > 0$, $f$ satisfies the growth condition

$$|f'(\xi_1) - f'(\xi_2)| \leq C(|\xi_1|^{\alpha-1} + |\xi_2|^{\alpha-1})|\xi_1 - \xi_2|, \quad \forall \xi_1, \xi_2 \in \mathbb{R}^n,$$  
(5)

where the prime is a shorthand notation for the gradient and $| \cdot |$ is the Euclidian norm on $\mathbb{R}^n$. A typical example of a function satisfying the growth condition (5) is $f(\xi) = |\xi|^\alpha \xi$.

Our result relies on the scaling invariance of (2): If $u(x, t)$ solves Eq. (2) in $\mathbb{R}^n$, then for each $\lambda > 0$,

$$u_\lambda(x, t) = \lambda^{\frac{2m-1}{\alpha - 1}} u(\lambda x, \lambda^2 t)$$  
(6)

also solves Eq. (2). The scaling identity (6) implies that

$$\|u(t, \cdot)\|_{L^p(\mathbb{R}^n)} = \lambda^{\frac{2m-1}{\alpha - 1} - \frac{n}{p}} \|u_\lambda(t, \cdot)\|_{L^p(\mathbb{R}^n)}.$$

Thus, one may expect under certain restrictions on the parameter regime, well-posedness results in homogeneous Sobolev-spaces $L^p(\Omega)$ where $p = na/(2(m-1) - \alpha)$.

In the spirit of Amman[1], Giga [9] and Miao [20], we rewrite Eqs. (2)-(3) and Eqs. (2)-(4) as a semilinear evolution equation of the form

$$u_t + Au = \nabla \cdot f(\nabla u), \quad x \in \Omega \text{ or } \mathbb{R}^n, \quad t \in [0, T),$$

$$u(0) = \varphi(x),$$

where

$$A := (-\Delta)^m \quad \text{and} \quad D(A) = W^{2m,p}(\Omega) \quad \text{if} \quad \Omega = \mathbb{R}^n,$$

or

$$D(A) = \bigcap_{0 < p \leq \infty} W^{2m,p}(\Omega) = \left\{ u \in W^{2m,p}(\Omega), \quad \frac{\partial u}{\partial \nu} \bigg|_{\partial \Omega} = 0, \quad j \leq m - 1 \right\}$$

whenever $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain. It can be verified that $A$ generates and analytic semigroup [1, 5, 27, 28]. This allows us to reformulate the problems (2)-(3) and (2)-(4) as the integral equation

$$u(t) = e^{-tA} \varphi + \int_{0}^{t} e^{-(t-s)A} \nabla \cdot f(\nabla u(s)) \, ds.$$  
(7)
Following Kato and Fujita [7, 12, 13], Giga et al. [8, 9] and Wiegner [31], we construct a solution in $L^{na/(2(m-1)-a)}(\Omega)$ to the integral Eq. (7) by successive approximations (see Eqs. (12)-(13)). These approximations are such that the sequences $\{K^1_j\}$ and $\{K^2_j\}$ defined by

$$K^1_j := \sup_{0 \leq t \leq T} t^{\frac{m-a}{2}} \|\nabla u(t)\|_{\Omega}$$

and

$$K^2_j := \sup_{0 \leq t \leq T} t^{\frac{5}{2}} \|\nabla^2 u(t)\|_{\Omega},$$

where $\{u(t)\}$ is the sequence of approximants, are bounded for some $\delta \in (0, 1)$. In order to establish the later result, we show that $K^1_j$ and $K^2_j$ satisfy respectively the recursive inequalities

$$K^1_{j+1} \leq K_0 + c (K^1_j)^{a} \cdot K^2_j$$

and

$$K^2_{j+1} \leq K_0 + c (K^1_j)^{a} \cdot K^2_j$$

(see Lemma 3.2).

By setting $R_j := \max \{K^1_j, K^2_j\}$, it follows that $R_j$ satisfies the recursive relation $R_{j+1} \leq R_0 + c R^{1-a}_j$. Thus, if $u(0) = \varphi$ has small $L^{na/(2(m-1)-a)}$-norm, then these recursive relations are uniformly bounded i.e. there is $R > 0$ such that for all $j \geq 0$, $R_j \leq R$ (see Corollary 3.1). This estimate allows us to use a standard argument to show that there is a uniformly converging sequence $\{u_j\}$, whose limit is a solution to the problem in $L^{na/(2(m-1)-a)}(\Omega)$ (see Lemma 3.3). The main contribution of this paper is summarized in the following theorem.

**Theorem 1.1.** Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain with $C^{2m}$ boundary, $1 < a < 2(m-1)$, $p = na/(2(m-1)-a)$ and $u(0) = \varphi \in L^p(\Omega)$. Then, the following statements are true.

(i) There exists $T > 0$ and a unique mild solution $u \in C^0([0, T]; L^p(\Omega))$ of Eq. (2)-(3) and Eq. (2)-(4) such that, for all $\delta \in (0, 1)$,

$$\sup_{0 \leq t \leq T} \left\{ t^{\frac{m-a}{2}} \|\nabla u(t)\|_{\Omega} + t^{\frac{5}{2}} \|\nabla^2 u(t)\|_{L^p} \right\} < \infty.$$  

(ii) If $\|\varphi\|_p$ is sufficiently small, the mild solution extends to a global one $u \in C^0([0, \infty); L^p(\Omega))$.

The remainder of this paper is organized as follows. In Section 2, we recall the general abstract $L_p - L_q$ basic estimates which are employed to prove bilinear estimates. These estimates turn out to be crucial in establishing solutions of Eqs. (2)-(3) and Eqs. (2)-(4). In Section 3, we first give some necessary nonlinear space-time estimates from which we infer local existence and uniqueness results in $L^p$-spaces. Also, we establish the existence of global solutions for sufficiently small initial data. In the last section, we summarize our findings while indicating future extensions and possible hurdles one may face.

## 2 A priori estimates

In order to solve the nonlinear problems (2)-(3), we first give an estimate of the nonlinear term $\nabla \cdot (f(\nabla u))$. For later convenience, we express $\nabla \cdot (f(\nabla u)) = \nabla^2 u \cdot f'(\nabla u)$ as a product $U \cdot V$ with $U = \nabla^2 u$ and $V = f'(\nabla u)$, where $\nabla^2 u$ denotes the collection of all second-order derivatives of $u$. We will need the following fundamental result of Weisler [30].

**Lemma 2.1** (Weisler [30]). Let $\Omega$ be a bounded domain of $\mathbb{R}^n$ with boundary $\partial \Omega$ of class $C^{2m}$. Let $A = (-\Delta)^m$ with domain $D(A)$ continuously embedded in $W^{2m,2}(\Omega)$ and codomain $L^4(\Omega)$. Assume further that $A$ generates an analytic semigroup $e^{-tA}$. Then, $e^{-tA} : L^p(\Omega) \to L^q(\Omega)$ is a bounded linear operator whenever $1 < p \leq q < \infty$ and $t > 0$. Additionally, for any $T > 0$, there is a constant $C > 0$ such that for any nonnegative integer $j < 2m$, the following inequality

$$\|\nabla^j e^{-tA}\|_{L^q(L^p, L^q)} \leq C(p, q, T) t^{\frac{j}{2} - \frac{j(1 - \frac{1}{p})}{2}}$$

is satisfied for all $t \in (0, T]$.
Now, let us fix our notations. We define the generalized admissible weighted Lebesgue norms by:

$$
\|u\|_{\alpha,p} := \sup_{0 \leq t \leq T} t^{\alpha} \|u(t)\|_p
$$

for $T > 0$, $\sigma > 0$ and the beta function by $\beta(x, y) = \int_0^1 (1 - t)^{x-1} t^{y-1} dt$ for all $x, y > 0$. Taking into account the constraint $|f'(\xi)| \leq c|\xi|^\sigma$, we have the following nonlinear estimates.

**Proposition 2.1.** Let $m \in \mathbb{N}$, $m \geq 2$. Suppose that $\max\{2(m-1)/(n+1), m/(m-1)\} < \alpha < 2(m-1)$ and $\delta \in (0, 1)$. Let

$$
\mathcal{B}(U, V)(t) := \int_0^t e^{-t-s}A F(U(s), V(s)) ds,
$$

where $F$ is continuous with

$$
|F(Z_1, Z_2)| \leq c |Z_1| |Z_2| \quad \text{for all} \quad Z_1 \in \mathbb{R}^m, \ Z_2 \in \mathbb{R}^n.
$$

Then, there are constants $c_1$ and $c_2$ independent of $U$ and $V$ such that for $0 \leq k \leq 2m$,

**Proof.** We first deduce from the assumption on $F$ and Hölder’s inequality that

$$
\|F(U(s), V(s))\|_{\frac{m+1}{m-1}, \frac{p}{s+1}} \leq c \left\| U(s) \right\|_{\frac{m+1}{m-1}, \frac{p}{s+1}} \left\| V(s) \right\|_{\frac{m+1}{m-1}, \frac{p}{s+1}} \leq c s^{-\frac{m}{m-1}} M,
$$

where $M := \left\| U \right\|_{\frac{m+1}{m-1}, \frac{p}{s+1}} \left\| V \right\|_{\frac{m+1}{m-1}, \frac{p}{s+1}}$. Using Lemma 2.1 with $p = an/(2(m-1) - \alpha + 2\delta \alpha)$ and $q = an/(2\delta)$, we infer that

$$
\left\| \nabla e^{-tA} \right\|_{\mathcal{L}(L^p, L^q)} \leq c t^{-\frac{m-1+\delta(\alpha-1)}{m}}.
$$

where $c$ is a constant whose value may change from line to line,

$$
\left\| \int_0^t \nabla e^{-t-s}A F(U(s), V(s)) ds \right\|_q \leq c M t^{-\frac{m-1+\delta(\alpha-1)}{m}} \int_0^t s^{-\frac{m}{m-1}} ds = c M \beta \left(1 - \frac{m-1+\delta(\alpha-1)}{ma}, \delta \right) t^{-\frac{m-1+\delta}{m}}
$$

and the inequalities $(m-1+\delta(\alpha-1))/ma < (m-1+\delta)/ma < (m+\delta)/m < 1$ hold for $\delta \in (0, 1)$. These inequalities ensure the integrability of the beta function $\beta \left(1 - \frac{m-1+\delta(\alpha-1)}{ma}, \delta \right)$, and lead us to Eq. (10).

Next, choosing $p$ as above and $q = an/(2(m-1) - \alpha)$, we obtain from Lemma 2.1

$$
\left\| e^{-tA} \right\|_{\mathcal{L}(L^p, L^q)} \leq c t^{-\delta/m} \quad \text{and} \quad \left\| \nabla^k e^{-tA} \right\|_{\mathcal{L}(L^p, L^q)} \leq c t^{-\delta/(2\delta+k)(2m)}.
$$

The latter estimate leads to the following one

$$
\left\| \int_0^t \nabla^k e^{-t-s}A F(U(s), V(s)) ds \right\|_q \leq c M t^{-\frac{m}{m+1}} \int_0^t s^{-\frac{m}{m+1}} ds = c M \beta \left(1 - \frac{\delta}{m} - \frac{k}{2m}, \delta \right) t^{-\frac{m}{m}}.
$$

Hence, the inequality (11) holds and the proof is completed. \qed
3 Well-posedness of our problem

This section is devoted to proving Theorem 1.1. The key step of the proof is to deduce a priori estimates for $K^1_j$, $K^2_j$ and $R_j$. We first state the following auxiliary lemma [19]. Its proof is done by induction.

**Lemma 3.1.** Let $\alpha$, $\lambda > 0$ and $\{b_m\}$ be a nonnegative sequence such that for all $m \in \mathbb{N}$

\[ b_m \leq b_0 + \lambda b^{1+\alpha}_{m-1}. \]

Assume further that $2\lambda(2b_0)^\alpha < 1$. Then, for all $m \in \mathbb{N}$,

\[ b_m \leq \frac{b_0}{1 - \lambda(2b_0)^\alpha}. \]

The classical approach to prove the existence and uniqueness of local solutions of Eq. (7) is the method of successive approximation. To solve the integral Eq. (7), we define the Picard iterates

\[ u_{j+1}(t) = u_0(t) + G(u_j)(t), \quad t \geq 0, \ j \geq 0, \tag{12} \]

with

\[ u_0(t) = e^{-tA} \phi \quad \text{and} \quad G(u)(t) = \int_0^t e^{-(t-s)A} \nabla \cdot f(\nabla u(s)) ds, \tag{13} \]

where $\phi \in L^{\infty}/(2(m-1)-\alpha)(\Omega)$.

Let us fix $0 < \delta < 1$ and $T > 0$. We have the following Lemma.

**Lemma 3.2.** There are constants $c_0$, $c_1$ and $c_2$ solely depending on $\alpha$, $\delta$ and $n$ such that

\[ K^1_{j+1} \leq K^3_0 + c_1 (K^1_j)^\alpha \cdot K^2_j, \tag{14} \]

\[ K^2_{j+1} \leq K^3_0 + c_2 (K^2_j)^\alpha \cdot K^2_j \tag{15} \]

for every $j \in \mathbb{N}$. Furthermore, we have the recursive inequalities

\[ R_j \leq R_0 + c R^{1+\alpha}_j, \quad \text{with} \quad R_0 \leq c_0 \| \phi \|_{\frac{n+\alpha}{n-1+\alpha}}, \ j \geq 0 \quad \text{and} \quad c = c_1 c_2. \tag{16} \]

**Proof.** We start by deriving a priori estimates for $K^1_j$ and $K^2_j$. Owing to the fact that $f'(|\xi|)$ behaves like $|\xi|^a$, we have $\|f'(\nabla u)\|_{\frac{n+\alpha}{n-1+\alpha}, \frac{n+\alpha}{\gamma}} \leq c \|\nabla u\|_{\frac{n+\alpha}{n-1+\alpha}, \frac{n+\alpha}{\gamma}}$. Applying Proposition 2.1 with $U = \nabla^2 u_j$ and $V = f'(\nabla u_j)$ yields

\[ \| \nabla u_{j+1}(t) \|_{\frac{n+\alpha}{\gamma}, \frac{n+\alpha}{\gamma}} \leq \| \nabla u_0(t) \|_{\frac{n+\alpha}{\gamma}, \frac{n+\alpha}{\gamma}} + \int_0^t \| \nabla \left[ e^{-(t-s)A} \nabla f(\nabla u_j) \right] \|_{\frac{n+\alpha}{\gamma}, \frac{n+\alpha}{\gamma}} ds \]

and

\[ \| \nabla u_{j+1}(t) \|_{\frac{n-1+\alpha}{\gamma}, \frac{n-1+\alpha}{\gamma}} \leq \| \nabla u_0(t) \|_{\frac{n-1+\alpha}{\gamma}, \frac{n-1+\alpha}{\gamma}} + c_1 \| U \|_{\frac{n-1+\alpha}{n-1+\alpha}, \frac{n+\alpha}{\gamma}} \| V \|_{\frac{n-1+\alpha}{n-1+\alpha}, \frac{n+\alpha}{\gamma}} \leq \| \nabla u_0(t) \|_{\frac{n-1+\alpha}{\gamma}, \frac{n-1+\alpha}{\gamma}} + c c_1 \| \nabla^2 u_j \|_{\frac{n-1+\alpha}{n-1+\alpha}, \frac{n+\alpha}{\gamma}} \| \nabla u_j \|_{\frac{n+\alpha}{n+\alpha}, \frac{n+\alpha}{\gamma}}. \]

Therefore,

\[ K^1_{j+1} \leq K^3_0 + c (K^1_j)^\alpha \cdot K^2_j \]
which implies Eq. (14). Similarly, we derive an a priori estimate for $K_j^2$ as follows:

$$
\left\| \nabla^2 u_{j+1}(t) \right\|_{\frac{m}{2(m-1)-\alpha}} \leq \left\| \nabla^2 u_0(t) \right\|_{\frac{m}{2(m-1)-\alpha}} + \int_0^t \left\| \nabla^2 \left[ e^{-(t-s)A} \nabla^2 u_j \cdot f' ( \nabla u_j ) \right] \right\|_{\frac{m}{2(m-1)-\alpha}} ds
$$

and

$$
\left\| \nabla^2 u_{j+1}(t) \right\|_{\frac{m}{2(m-1)-\alpha}} \leq \left\| \nabla^2 u_0(t) \right\|_{\frac{m}{2(m-1)-\alpha}} + c_2 \left\| U \right\|_{\frac{m}{2(m-1)-\alpha}} \left\| V \right\|_{\frac{2 \alpha - 1 - \alpha}{\alpha - 1}} \left\| \nabla^2 u_j \right\|_{\frac{m}{2(m-1)-\alpha}} \left\| \nabla u_j \right\|_{\frac{1}{\alpha - 1}} + c_2 c \left\| \nabla^2 u_j \right\|_{\frac{m}{2(m-1)-\alpha}} \left\| \nabla u_j \right\|_{\frac{1}{\alpha - 1}}.
$$

The last inequality implies that

$$
K_{j+1}^2 \leq K_0^2 + c(K_j^2) \cdot K_j^2
$$

as desired. Next, setting $R_j := \max \{ K_j^2, K_j^2 \}$, the iterative estimates (14) and (15) yield $R_{j+1} \leq R_j + cR_{j+1}^\alpha$.

It remains to prove that $R_0 \leq c_0 \left\| \varphi \right\|_{\frac{m}{2(m-1)-\alpha}}$. To this end, let $t \in (0, T)$. Then, Lemma 2.1 with $p = an/(2(m-1)-\alpha)$ and $q = an/(2\delta)$ leads to

$$
\left\| \nabla (e^{-tA} \varphi) \right\|_{\frac{m}{2(m-1)-\alpha}} \leq cT^{-\frac{\alpha}{4}} \left[ \frac{(2(m-1)-\alpha)^{\frac{\alpha}{2}} - 1}{\alpha - 1} \right] \left\| \varphi \right\|_{\frac{m}{2(m-1)-\alpha}} = cT^{-\frac{\alpha}{4}} \left\| \varphi \right\|_{\frac{m}{2(m-1)-\alpha}}.
$$

Likewise, for $p = q = an/(2(m-1)-\alpha)$, we obtain from Lemma 2.1

$$
\left\| \nabla^2 (e^{-tA} \varphi) \right\|_{\frac{m}{2(m-1)-\alpha}} \leq T^{-\frac{\alpha}{4}} \left\| \varphi \right\|_{\frac{m}{2(m-1)-\alpha}}
$$

which is equivalent to Eq. (16). This completes the proof of Lemma 3.2.

Lemma 3.1 and Lemma 3.2 immediately imply an a priori estimate for the approximate solution $u_j$. By choosing the norm $\left\| \varphi \right\|_{\frac{m}{2(m-1)-\alpha}}$ sufficiently small, we obtain the following result.

**Corollary 3.1.** There exists a constant $c > 0$ such that

$$
R_j \leq 2R_0 \text{ for all } j \geq 1, \text{ provided } \left\| \varphi \right\|_{\frac{m}{2(m-1)-\alpha}} \leq c.
$$

**Proof.** The inequality (17) is an immediate consequence of Lemma 3.1.

To show the convergence of $\{ u_j \}$, we consider the successive difference of $u_j$ defined by

$$
w_0 = u_0, \quad w_j = u_j - u_{j-1}, \quad j \geq 1
$$

and we introduce the corresponding $K$-like quantities

$$
\tilde{K}_j^1 := \sup_{0 \leq s \leq T} \int_0^{\frac{m}{2(m-1)-\alpha}} \left\| \nabla w_j(t) \right\|_{\frac{m}{2(m-1)-\alpha}} dt, \quad \tilde{K}_j^2 := \sup_{0 \leq s \leq T} \int_0^{\frac{m}{2(m-1)-\alpha}} \left\| \nabla^2 w_j(t) \right\|_{\frac{m}{2(m-1)-\alpha}} dt, \quad \tilde{R}_j := \max \{ \tilde{K}_j^1, \tilde{K}_j^2 \}.
$$

Also, we set

$$
M_j := \sup_{0 \leq s \leq T} \left\| w_j(t) \right\|_{\frac{m}{2(m-1)-\alpha}}.
$$

By direct calculations, we obtain the identity

$$
w_j(t) = G(u_j)(t) - G(u_{j-1})(t)
= \int_0^t e^{-(t-s)A} \left[ \nabla^2 u_j(s) \cdot f'(\nabla u_j(s)) - \nabla^2 u_{j-1}(s) \cdot f'(\nabla u_{j-1}(s)) \right] ds
= \int_0^t e^{-(t-s)A} \nabla^2 w_j(s) \cdot f'(\nabla u_j(s)) ds + \int_0^t e^{-(t-s)A} \nabla^2 u_{j-1}(s) \left[ f'(\nabla u_j(s)) - f'(\nabla u_{j-1}(s)) \right] ds. \quad (18)
$$

Using Eq. (18), we infer the following Lemma.

□
Lemma 3.3. There is a constant $c > 0$ such that for $\|\varphi\|_{\frac{m+1-d}{m}, \frac{\alpha}{m}, T} \leq c$ (with $c$ as in Corollary 3.1)

$$\tilde{R}_{j+1} \leq c \tilde{R}_j R_0^\alpha,$$  \hspace{1cm} (19)

$$M_{j+1} \leq c \tilde{R}_j R_0^\alpha,$$  \hspace{1cm} (20)

$$M_j \leq c (R_0^0)^j$$ for every $j \in \mathbb{N}$.  \hspace{1cm} (21)

Proof. Based on Eq. (18), we split the estimate of $w_j$ into two parts. Taking into account the fact that $f'(\xi)$ behaves like $|\xi|^\alpha$, we then have

$$\|f'(\nabla u)\|_{\frac{m+1-d}{m}, \frac{\alpha}{m}, }, \frac{\alpha}{m}, T} \leq c \|\nabla u\|_{\frac{m+1-d}{m}, \frac{\alpha}{m}, T}.$$

We next derive an a priori estimate for

$$\tilde{K}_j^1 := \sup_{0 \leq t \leq T} t^{\frac{m-1}{m}} \|\nabla w_j(t)\|_{\frac{m}{m}, \frac{\alpha}{m}, T} = \|\nabla w_j\|_{\frac{m-1}{m}, \frac{\alpha}{m}, T}.$$

Applying Proposition 2.1 (see estimate (10)) with $U = \nabla^2 w_j$ and $V = f'(\nabla u_j)$ produces

$$\left\| \int_0^t \nabla \left[ e^{-(t-s)A} \nabla^2 w_j \cdot f'(\nabla u_j) \right] ds \right\| \leq c \left\| \nabla^2 w_j \right\|_{\frac{m-1}{m}, \frac{\alpha}{m}, T} \left\| \nabla u_j \right\|_{\frac{m-1}{m}, \frac{\alpha}{m}, T} \leq c \tilde{R}_j R^\alpha \leq c \tilde{R}_j R_0^\alpha.$$

The growth condition on $f'$ and Hölder's inequality lead to the inequalities

$$\left\| f'(\nabla u_j) - f'(\nabla u_{j-1}) \right\|_{\frac{m}{m}, \frac{\alpha}{m}, T} \leq \left\| \nabla w_j \right\|_{\frac{m}{m}, \frac{\alpha}{m}, T} \left( |\nabla u_j|^{a-1} + |\nabla u_{j-1}|^{a-1} \right) \leq \left\| \nabla w_j \right\|_{\frac{m}{m}, \frac{\alpha}{m}, T} \left( |\nabla u_j|^{a-1} + |\nabla u_{j-1}|^{a-1} \right)$$

and

$$\left\| f'(\nabla u_j) - f'(\nabla u_{j-1}) \right\|_{\frac{m-1}{m}, \frac{\alpha}{m}, T} \leq \left[ (K_j^1)^{a-1} + (K_{j-1}^1)^{a-1} \right] \left\| \nabla w_j \right\|_{\frac{m-1}{m}, \frac{\alpha}{m}, T} \leq \left[ R_j^{a-1} + R_{j-1}^{a-1} \right] \tilde{K}_j \leq c \tilde{R}_j R_0^{a-1}.$$

Likewise, Proposition 2.1 with $U = \nabla^2 u_{j-1}$ and $V = f'(\nabla u_j) - f'(\nabla u_{j-1})$ yields

$$\left\| \int_0^t \nabla \left[ e^{-(t-s)A} \nabla^2 u_{j-1}(s) \left( f'(\nabla u_j(s)) - f'(\nabla u_{j-1}(s)) \right) \right] ds \right\| \leq c_1 \left\| \nabla^2 u_{j-1} \right\|_{\frac{m}{m}, \frac{\alpha}{m}, T} \left\| f'(\nabla u_j) - f'(\nabla u_{j-1}) \right\|_{\frac{m-1}{m}, \frac{\alpha}{m}, T} \leq c \tilde{R}_j R_0^\alpha$$

and ultimately

$$\tilde{K}_{j+1}^1 = \left\| \nabla w_{j+1} \right\|_{\frac{m-1}{m}, \frac{\alpha}{m}, T} \leq c \tilde{R}_j R_0^\alpha.$$

By the same modus operandi as above (to estimate (11), take $k = 2$), the a priori estimate for $\tilde{K}_j^2$ yields

$$\tilde{K}_{j+1}^2 = \sup_{0 \leq t \leq T} t^{\frac{m}{m}} \left\| \nabla^2 w_j(t) \right\|_{\frac{m}{m}, \frac{\alpha}{m}, T} \leq c \tilde{R}_j R_0^\alpha.$$

By setting $\tilde{R}_j := \max \{ \tilde{K}_j^1, \tilde{K}_j^2 \}$, one obtains Eq. (19).

Finally, the same procedure can be used to estimate $M_j$ and we obtain

$$M_{j+1} := \sup_{0 \leq t \leq T} \left\| w_{j+1}(t) \right\|_{\frac{m}{m}, \frac{\alpha}{m}, T} \leq c \tilde{R}_j R_0^\alpha$$

and thus Eq. (20). The estimate (21) comes from Eq. (19) and Eq. (20). This completes the proof of Lemma 3.3. \qed
We can now state the conditions under which the sequence \( \{u_j\} \) converges.

**Proposition 3.1.** For all \( \delta \in (0, 1) \), there is a constant \( \epsilon > 0 \) such that if

\[
R_0 = \max \left\{ \sup_{0 \leq t \leq T} t^{\frac{m-1}{m}} \| \nabla u_0(t) \|_{L^m}, \sup_{0 \leq t \leq T} t^\frac{1}{p} \| \nabla^2 u_0(t) \|_{L^{\frac{m}{m-2}}/2} \right\} < \epsilon,
\]

then the sequence \( \{u_j\} \subset C^0([0, T]; L^{\frac{m}{m-2}}(\Omega)) \) converges uniformly.

The proof of Proposition 3.1 is similar to that performed in the papers [24, 25]. Equipped with Proposition 3.1, we are now able to sketch the proof of the main theorem of this paper.

**Proof.** Sketch of the proof of Theorem 1.1. The proof will be done in three steps which are enumerated and described below.

**Step 1 (Well-definition of \( u_j \)).** A priori estimates of Lemma 3.2 show that \( u_j(t) \) is well-defined for \( j \geq 0 \) an element of \( L^p(\Omega) \).

**Step 2 (Existence and uniqueness).** Using Weierstraß-M Test, Lemma 3.3 implies that the sequence \( M_j \) is summable provided \( R_0 < p_0 := \min(e, c^{1/\delta}) \). Indeed, from Proposition 3.1, it follows that \( u_j = u_0 + \sum_{k=1}^j w_k \) defines a Cauchy sequence in \( C^0([0, T]; L^{\frac{m}{m-2}}(\Omega)) \) which converges to some solution \( u \in C^0([0, T]; L^{\frac{m}{m-2}}(\Omega)) \) of the integral equation

\[
u(t) = e^{-t\lambda} \varphi + \int_0^t e^{-(t-s)\lambda} \nabla \cdot f(\nabla u(s))ds.
\]

The continuity in time for \( t \in [0, T] \) follows from standard results on Lebesgue's integral [26] whereas the estimate (8) follows from (16).

**Step 3 (Global solution).** Corollary 3.1 implies that \( \| u_j(t) \|_{L^p} \) is bounded in \( j \) provided that \( \| \varphi \|_{L^p} \) is sufficiently small for \( T \). Since the estimate (16) includes no \( T \) explicitly, the solution can be extended to a global one. \( \square \)

### 4 Conclusion and Future Outlook

In recent years there have been numerous papers dealing with higher-order parabolic equations and systems. These papers include the comprehensive work by Giga [8, 9], Caristi et al. [4], Miao [19–23] and King et al. [14]. These authors mainly used functional analytic methods, in the spirit of semigroup theory. However, with the exception of King et al. [14], they do not allow nonlinearities depending on the divergence of the solution.

As demonstrated in the papers [24, 25], when \( m = 2 \), the well-posedness relies upon scaling invariance of the problem: Suppose that \( u(x, t) \) is a smooth solution of Eq. (2) in \( \mathbb{R}^n \), then for every \( \lambda > 0 \),

\[
u_{\lambda}(x, t) = \lambda^{\frac{2}{p}-1} u(\lambda x, \lambda^2 t)
\]

also solves Eq. (2) and for \( k \in \mathbb{N} \),

\[
\| \nabla^k u_{\lambda}(\cdot, t) \|_{L^p(\mathbb{R}^n)} = \lambda^{\frac{2}{p}(k+1)-\frac{n}{2}} \| \nabla^k u(\cdot, \lambda^2 t) \|_{L^p(\mathbb{R}^n)}.
\]

Therefore we expect, under appropriate restrictions on the parameter regime, well-posedness results in homogeneous Sobolev-spaces

\[
\mathcal{D}^{k,p}(\mathbb{R}^n) := \left\{ u \in C^0_0(\mathbb{R}^n) : \| \nabla^k u \|_{L^p(\mathbb{R}^n)} < \infty \right\}.
\]

It would be useful to know the well-posedness of Eq. (2) in the case \( \alpha \geq 2 \) [14]. One might then wonder how to construct the admissible weighted Lebesgue norms in order to show the existence of solutions. We
posit that the exponent of weighted Lebesgue norms should be chosen judiciously such that some integrals involving the semigroup converge. Therefore, we must find condition on \( k \) which ensures the integrability of the beta function intervening in our estimate. In the case \( m = 2 \), we have found that \( \frac{k}{2} + \frac{1}{2} < 1 \). This means that well-posedness can only be expected for \( k = 0 \) or \( k = 1 \). The case \( k = 0 \) was studied in [24]. Note that Melcher [18] studied in the whole space \( \Omega = \mathbb{R}^n \) the case \( k = 1 \) which works for \( \alpha > \frac{n}{2} \). The admissible weighted Lebesgue norms are defined by:

\[
K_j^1 := \sup_{0 < t < T} t^{\frac{\delta}{2}} \| \nabla u_j(t) \|_{\Omega}, \quad K_j^2 := \sup_{0 < t < T} t^{\delta} \| \nabla^2 u_j(t) \|_{\Omega^2} \quad \text{for some } 0 < \delta < 1
\]

and

\[
R_j := \max \left\{ K_j^1, K_j^2 \right\}, \quad M_j := \sup_{0 < t < T} \| \nabla w_j(t) \|_{\Omega^2} \quad \text{where } w_j = u_j - u_{j-1}.
\]

For bounded domain \( \Omega \), a formal integration of Eq. (2) implies that \( \int_D u(x, t) \, dx = \int_D \varphi(x) \, dx \). This suggests the introduction of a new dependent variable \( v(x, t) = u(x, t) - \varphi(x) \) such that \( \int_D v(x, t) \, dx = 0 \). We will consider the same weighted Lebesgue norms above and due to Poincaré’s inequality, we expect well-posedness in the Sobolev space

\[
\mathcal{D}^{1,p}(\Omega) := \left\{ u \in W^{1,p}(\Omega), \int_D u(x, t) \, dx = 0 \right\}
\]

endowed with the norm \( \| u \|_{\mathcal{D}^{1,p}(\Omega)} = \| \nabla u \|_{L^p(\Omega)} \). However, as pointed out, by Melcher [18], local and global solutions can only be obtained if the gradient of initial data is small. But, for the computation of numerical solutions to an initial-boundary value problems, it is commonly known that the gradient of initial data is more complicated to control. We made some progress towards filling this gap by stating and proving that local solutions can only be obtained if the gradient of initial data is small. But, for the computation of numerical solutions, it is commonly known that the gradient of initial data is required. This was indeed the main motivation for us to treat the boundary value problem in different function spaces. Of course, Melcher’s framework has been crucial to deal with our case because the main estimates (18), Proposition 1) turned out to be useful for our purposes only after some technical modifications.

A further natural question is whether the solution constructed above is classical. Such regularity results must be based on interpreting the nonlinear term \( \nabla \cdot (f(\nabla u)) \).

It may also be asked what happens for the semilinear fractional power dissipative equation

\[
u_t + (-\Delta)^{\beta} u = \nabla \cdot f(\nabla u)
\]

supplemented with suitable boundary conditions with real number \( \beta > 0 \).

Beyond this work, several aspects of nonlinear parabolic equations which have not received a rigorous mathematical treatment can be solved using our method. We suspect that our results may be extended to elliptic operators of order \( 2m \) with nonlinear part \( F(u) = f(u, \nabla u, \cdots, \nabla^m u) \) with the growth condition on \( f \) judiciously specified in order to arrive at admissible weighted Lebesgue norms. This task will be undertaken in a forthcoming paper.

Acknowledgement: We are grateful to Professors Wiegner and Melcher for pointing out that the scaling symmetry of the equation studied in this paper may guide the construction of admissible weighted Lebesgue norms. The work of CWS was partly supported by the NSF grant DMS-1214359.

References


