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Controllability of the Semilinear Heat Equation with Impulses and Delay on the State

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Abstract: In this paper we prove the interior approximate controllability of the following Semilinear Heat Equation with Impulses and Delay

$$\begin{cases} \dfrac{\partial z(t,x)}{\partial t} &= \Delta z(t,x) + 1_\omega u(t,x) + f(t, z(t-r,x), u(t,x)), \text{in} \quad (0,\tau]\times\Omega, t \neq t_k, \\ z(s,x) &= 0, \quad \text{on} \quad (0,\tau)\times\partial\Omega, \\ z(s,x) &= \phi(s,x), \quad s\in[-r,0], x\in\Omega, \\ z(t_k^+,x) &= z(t_k^-,x) + I_k(t_k, z(t_k,x), u(t_k,x)), \quad k = 1,2,3,\dots,p, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N (N \geq 1)$, $\phi : [-r, 0]\times\Omega \to \boldsymbol{R}$ is a continuous function, ω is an open nonempty subset of Ω, 1_ω denotes the characteristic function of the set ω and the distributed control u belongs to $L_2([0,\tau]; L_2(\Omega;))$. Here $r \geq 0$ is the delay and the nonlinear functions $f, I_k : [0,\tau]\times\boldsymbol{R}\times\boldsymbol{R} \to \boldsymbol{R}$ are smooth enough, such that

$$|f(t,z,u)| \quad \leq a_0|z| + c_0, \quad u \in \mathbb{R}, z \in \mathbb{R}.$$

Under this condition we prove the following statement: For all open nonempty subset ω of Ω the system is approximately controllable on $[0,\tau]$, for all $\tau > 0$.

Keywords: interior approximate controllability,semilinear heat equation with impulses and delay, strongly continuous semigroup

MSC: primary: 93B05; secondary: 93C10

1 Introduction

For many control systems in real life, impulses and delays are intrinsic phenomena that do not modify their controllability. So we conjecture that, under certain conditions, perturbations of the system caused by abrupt changes and delays do not affect certain properties such as controllability.

There are many practical examples of impulsive control systems, a chemical reactor system, a financial system with two state variables, the amount of money in a market and the savings rate of a central bank; and the growth of a population diffusing throughout its habitat modeled by a reaction-diffusion equation. One may easily visualize situations in these examples where abrupt changes such as harvesting, disasters and instantaneous stocking may occur. These problems are modeled by impulsive differential equations, and for more information see the monographs, Lakshmikantham [12] and Samoilenko and Perestyuk [18].

The controllability of Impulsive Evolution Equations has been studied recently for several authors, but most them study the exact controllability only, to mention: D.N. Chalishajar([6]), studied the exact controllability of impulsive partial neutral functional differential equations with infinite delay and S. Selvi, M. Mallika

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Arjunan([22]) studied the exact controllability for impulsive differential systems with finite delay.
To our knowledge, there are a few works on approximate controllability of impulsive semilinear evolution equations, to mention: Lizhen Chen and Gang Li([9]) studied the Approximate controllability of impulsive differential equations with nonlocal conditions, using measure of noncompactness and Monch's fixed point Theorem, and assuming that the nonlinear term $f(t, z)$ does not depend on the control variable.
Recently, in [17, 19–21] the approximate controllability of semilinear evolution equations with impulses has been studied applying Rothe's Fixed Point Theorem. The existence of solutions for Impulsive Evolution Equations with Delays has been studied by N. Abada, M. Benchohra and H. Hammouche [1] and R.Shikharchard Jain and M. Baburao Dhakne in [23].

In this paper, avoiding the use of Fixed Point Theorems, we will apply the new technique presented in A.E. Bashirov and Noushin Ghahramanlou([3]), A.E. Bashirov and Noushin Ghahramanlou([4]) and A.E. Bashirov, N. Mahmudov, N. Semi and H. Etikan([5]) to prove the interior approximate controllability of the following Heat Equation with impulses and delay on the state variable

$$\begin{cases} \dfrac{\partial z(t,x)}{\partial t} &= \Delta z(t,x) + 1_\omega u(t,x) + f(t, z(t-r,x), u(t,x)), \, t \neq t_k, \\ z(s,x) &= 0, \quad \text{on} \quad (0,\tau)\times\partial\Omega, \\ z(s,x) &= \phi(s,x), \quad s \in [-r,0], x\in\Omega, \\ z(t_k^+,x) &= z(t_k^-,x) + I_k(t_k, z(t_k,x), u(t_k,x)), \quad k = 1,2,3,\dots,p, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N (N \geq 1)$, $\phi : [-r, 0]\times\Omega \to \boldsymbol{R}$ is a continuous function, ω is an open nonempty subset of Ω, 1_ω denotes the characteristic function of the set ω, the distributed control u belongs to $L_2([0,\tau]; L_2(\Omega))$ and the standard notation $z_t(x)$ defines a function from $[-r, 0]$ to $\boldsymbol{R}$ by $z_t(x)(s) = z(t + s, x)$, $-r \leq s \leq 0$. Here $r \geq 0$ is the delay and the nonlinear functions $f, I_k : [0,\tau]\times\boldsymbol{R}\times\boldsymbol{R} \to \boldsymbol{R}$ are smooth enough, such that

$$|f(t,z,u)| \leq a_0|z| + b_0, \quad u\in\mathbb{R}, z\in\mathbb{R}, \tag{1.2}$$

where $a_0, b_0 \geq 0$.
The particular case that really gave the motivation for the realization of this work was the following linear control system with impulses and delay governed by the heat equation

$$\begin{cases} \dfrac{\partial z(t,x)}{\partial t} &= \Delta z(t,x) + 1_\omega u(t,x) + z(t-r,x), \, t \neq t_k, \\ z(s,x) &= 0, \quad \text{on} \quad (0,\tau)\times\partial\Omega, \\ z(s,x) &= \phi(s,x), \quad s \in [-r,0], x\in\Omega, \\ z(t_k^+,x) &= z(t_k^-,x) + I_k(t_k, z(t_k,x), u(t_k,x)), \quad k = 1,2,3,\dots,p, \end{cases} \tag{1.3}$$

We shall denote by C the space of continuous functions:

$$C = \{\phi : [-r, 0] \to L_2(\Omega) = Z : \phi \quad \text{is continuous}\},$$

endowed with the norm

$$\|\phi\| = \sup_{-r\leq s\leq 0} \|\phi(s)\|_{L_2(\Omega)}, \text{ and } \phi(s)(x) = \phi(s,x), x\in\Omega.$$

Definition 1.1. (**Approximate Controllability**) *The system (1.1) is said to be approximately controllable on* $[0,\tau]$ *if for every* $\phi \in C$ *and* $z_1 \in Z = U = L_2(\Omega)$, $\varepsilon > 0$ *there exists* $u \in L^2(0,\tau;U)$ *such that the mild solution* $z(t)$ *of (1.1) corresponding to u verifies:*

$$z(0) = \phi(0) \text{ and } \|z(\tau) - z_1\|_Z < \varepsilon.$$

where

$$\|z(\tau) - z_1\|_Z = \left(\int_\Omega |z(\tau,x) - z_1(x)|^2 dx\right)^{\frac{1}{2}}$$

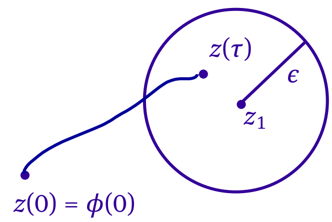


Fig.1

Definition 1.2. *(**Controllability to Trajectories**) The system (1.1) is said to be controllable to trajectories on* $[0, \tau]$ *if for every* $\phi, \hat{\phi} \in C$ *and* $\hat{u} \in L^2([0, \tau]; U)$ *there exists* $u \in L^2([0, \tau]; U)$ *such that the mild solution* $z(t)$ *of (1.1) corresponding to* u *and* ϕ *verifies:*

$$z(\tau, \phi, u) = z(\tau, \hat{\phi}, \hat{u}), \quad (Fig.2).$$

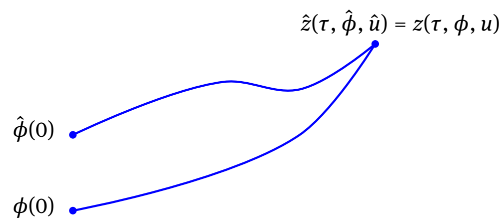


Fig.2

Definition 1.3. *(**Null Controllability**) The system (1.1) is said to be null controllable on* $[0, \tau]$ *if for every* $\phi \in C$ *there exists* $L^2([0, \tau]; U)$ *such that the mild solution* $z(t)$ *of (1.1) corresponding to* u *verifies:*

$$z(0) = \phi(0) \ \ and \ \ z(\tau) = 0, \quad (Fig.3).$$

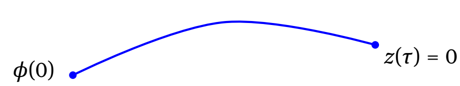


Fig.3

Remark 1.1. *It is clear that exact controllability of the system(1.1) implies approximate controllability, null controllability and controllability to trajectories of the system. But, it is well known (D. Barcenas, Hugo Leiva and Z. Sivoli [7], 2005)(K. Balachandran and J. H. Kim [8], 2006) that due to the diffusion effect or the compactness of the semigroup generated by* $-\Delta$*, the heat equation is not exactly controllable.*
We observe also that in the linear case controllability to trajectories and null controllability are equivalent. Nevertheless, the approximate controllability and the null controllability are in general independent. In this paper we will focus on the study of the approximate controllability of the system(1.1).

This problem has been opened for quite some time, but could not be solved with classical techniques. Now, using a characterization dense range linear operator from [13], the approximate controllability of the linear heat equation

$$\begin{cases} \dfrac{\partial z(t,x)}{\partial t} = \Delta z(t,x) + 1_{\omega} u(t,x) & \text{in} \quad (0,\tau] \times \Omega, \\ z = 0, & \text{on} \quad (0,\tau) \times \partial\Omega, \\ z(0,x) = z^0(x), & x \in \Omega, \end{cases} \tag{1.4}$$

on $[0,\tau]$ for all $\tau > 0$ and the ideas presented by A.E. Bashirov and Noushin Ghahramanlou([3]), A.E. Bashirov and Noushin Ghahramanlou([4]) and A.E. Bashirov, N. Mahmudov, N. Semi and H. Etikan([5]) the problem can finally be solved.

2 Abstract Formulation of the Problem.

In this section we choose a Hilbert space where system (1.1) can be written as an abstract differential equation; to this end, we consider the following notations:
Let us consider the Hilbert space $Z = L_2(\Omega)$ and $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_j \longrightarrow \infty$ the eigenvalues of $-\Delta$, each one with finite multiplicity γ_j equal to the dimension of the corresponding eigenspace. Then we have the following well known properties
(i) There exists a complete orthonormal set $\{\phi_{j,k}\}$ of eigenvectors of $A = -\Delta$.
(ii) For all $z \in D(A)$ we have

$$Az = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} < \xi, \phi_{j,k} > \phi_{j,k} = \sum_{j=1}^{\infty} \lambda_j E_j z, \tag{2.5}$$

where $< \cdot, \cdot >$ is the inner product in Z and

$$E_n z = \sum_{k=1}^{\gamma_j} < z, \phi_{j,k} > \phi_{j,k}. \tag{2.6}$$

So, $\{E_j\}$ is a family of complete orthogonal projections in Z and $z = \sum_{j=1}^{\infty} E_j z, \quad z \in Z$.
(iii) $-A$ generates an analytic semigroup $\{T(t)\}$ given by

$$T(t)z = \sum_{j=1}^{\infty} e^{-\lambda_j t} E_j z \text{ and } \|T(t)\| \le e^{-\lambda_1 t}, \quad t \ge 0. \tag{2.7}$$

Consequently, systems (1.1) can be written as an abstract functional differential equations with memory in Z:

$$\begin{cases} z' = -Az + B_{\omega} u + f^e(t, z_t(-r), u(s)), \quad z \in Z \quad t \ge 0, \\ z(s) = \phi(s), \quad s \in [-r, 0], \\ z(t_k^+) = z(t_k^-) + I_k^e(t_k, z(t_k), u(t_k)), \quad k = 1, 2, 3, \ldots, p, \end{cases} \tag{2.8}$$

where $u \in L^2([0,\tau]; U)$, $U = Z$, $B_{\omega} : U \longrightarrow Z$, $B_{\omega} u = 1_{\omega} u$ is a bounded linear operator, $z_t \in C([-r,0]; Z)$ and is defined by $z_t(s) = z(t+s)$, $-r \le s \le 0$ and the functions $I_k^e : [0,\tau] \times Z \times U \to Z$, $f^e : [0,\tau] \times C \times U \to Z$ are defined for $k = 1, 2, \ldots, p$ by

$$I_k^e(t, z, u)(x) = I_k(t, z(x), u(x)), \quad f^e(t, \phi, u) = f(t, \phi(-r, \cdot), u(\cdot)),$$

or

$$f^e(t, \phi(-r, \cdot), u(\cdot))(x) = f(t, \phi(-r, x), u(x)) \ \forall x \in \Omega.$$

The following result follows from condition (1.2)

Proposition 2.1. *Under the conditions (1.2)the function f^e satisfies:*

$$\|f^e(t, \phi, u)\|_Z \le \hat{a}_0 \|\phi(-r)\|_Z + \hat{b}_0. \tag{2.9}$$

3 Controllability of the Linear Equation

In this section we shall present some characterization of the approximate controllability of the corresponding linear heat equations. To this end, we note that, for all $z_0 \in Z$ and $u \in L_2(0, \tau; U)$ the initial value problem

$$\begin{cases} z' = -Az + B_\omega u(t), & z \in Z, \\ z(t_0) = z_0, \end{cases} \tag{3.10}$$

admits only one mild solution given by

$$z(t) = z(t, t_0, z_0, u) = T(t)z_0 + \int_{t_0}^{t} T(t-s)B_\omega u(s)ds, \quad t \in [t_0, \tau], \quad 0 \le t_0 \le \tau. \tag{3.11}$$

Definition 3.1. *For system (3.10) we define the following concept: The controllability maps* $G_{\tau\delta} : L^2(\tau - \delta, \tau; U)) \to Z$, $G_\delta : L^2(0, \delta; U) \to Z$ *defined by*

$$G_{\tau\delta}u = \int_{\tau-\delta}^{\tau} T(\tau - s)B_\omega u(s)ds, \quad u \in L^2(\tau - \delta, \tau; U), \tag{3.12}$$

$$G_\delta v = \int_0^\delta T(s)B_\omega v(s)ds, \quad v \in L^2(0, \delta; U), \tag{3.13}$$

satisfy the following relation:

$$G_{\tau\delta}u = \int_{\tau-\delta}^{\tau} T(\tau - s)B_\omega u(s)ds = \int_0^\delta T(s)B_\omega u(\tau - s)ds = G_\delta u(\tau - \cdot). \tag{3.14}$$

The adjoint of these operators $G^*_{\tau\delta} : Z \to L^2(\tau - \delta, \tau; U)$, $G^*_\delta : Z \to L^2(0, \delta; U)$ *are given by*

$$(G^*_{\tau\delta}z)(t) = B^*_\omega T^*(\tau - t)z, \quad t \in [\tau - \delta, \tau].$$

$$(G^*_\delta x)(t) = B^*_\omega T^*(t)z, \quad t \in [0, \delta].$$

The Gramian controllability operators are given by:

$$Q_\delta = G_\delta G^*_\delta = \int_0^\delta T(t)B_\omega B^*_\omega T^*(t)dt \tag{3.15}$$

$$Q_{\tau\delta} = G_{\tau\delta}G^*_{\tau\delta} = \int_{\tau-\delta}^{\tau} T(\tau - t)B_\omega B^*_\omega T^*(\tau - t)dt = Q_\delta. \tag{3.16}$$

The following lemma holds in general for a linear bounded operator $G : W \to Z$ between Hilbert spaces W and Z(see [10], [11], [13]).

Lemma 3.1. *The following statements are equivalent to the approximate controllability of the linear system (3.10) on* $[\tau - \delta, \tau]$.

a) $\overline{\text{Rang}(G_{\tau\delta})} = Z$.
b) $Ker(G^*_{\tau\delta}) = \{0\}$.
c) $\langle Q_{\tau\delta}z, z\rangle > 0, z \neq 0 \quad in \quad Z$.

d) $\lim_{\alpha\to 0^+} \alpha(\alpha I + Q_{\tau\delta})^{-1}z = 0$.

e) *For all* $z \in Z$ *we have* $G_{\tau\delta}u_\alpha = z - \alpha(\alpha I + Q_{\tau\delta})^{-1}z$, *where*

$$u_\alpha = G^*_{\tau\delta}(\alpha I + Q_{\tau\delta})^{-1}z, \quad \alpha \in (0,1].$$

So, $\lim_{\alpha\to 0} G_{\tau\delta}u_\alpha = z$ *and the error* $E_{\tau\delta}z$ *of this approximation is given by the formula*

$$E_{\tau\delta}z = \alpha(\alpha I + Q_{\tau\delta})^{-1}z, \quad \alpha \in (0,1].$$

f) *Moreover, if we consider for each* $v \in L^2(\tau-\delta,\tau;U))$ *the sequence of controls given by*

$$u_\alpha = G^*_{\tau\delta}(\alpha I + Q_{\tau\delta})^{-1}z + (v - G^*_{\tau\delta}(\alpha I + Q_{\tau\delta})^{-1}G_{\tau\delta}v), \quad \alpha \in (0,1],$$

we get that:

$$G_{\tau\delta}u_\alpha = z - \alpha(\alpha I + Q_{T\delta})^{-1}(z + G_{\tau\delta}v)$$

and

$$\lim_{\alpha\to 0} G_{\tau\delta}u_\alpha = z,$$

with the error $E_{\tau\delta}z$ *of this approximation is given by*

$$E_{\tau\delta}z = \alpha(\alpha I + Q_{\tau\delta})^{-1}(z + G_{\tau\delta}v), \quad \alpha \in (0,1].$$

Remark 3.1. *The foregoing Lemma implies that the family of linear operators* $\Gamma_{\alpha\tau\delta} : Z \to W$, *defined for* $0 < \alpha \le 1$ *by*

$$\Gamma_{\alpha\tau\delta}z = G^*_{\tau\delta}(\alpha I + Q_{\tau\delta})^{-1}z, \tag{3.17}$$

is an approximate inverse for the right of the operator W, *in the sense that*

$$\lim_{\alpha\to 0} G_{\tau\delta}\Gamma_{\alpha\tau\delta} = I, \tag{3.18}$$

in the strong topology.

Lemma 3.2. $Q_\delta > 0$ *if, and only if, the linear system (3.10) is approximately controllable on* $[\tau-\delta,\tau]$. *Moreover, given an initial state* y_0 *and a final state* z_1 *we can find a sequence of controls* $\{u^\delta_\alpha\}_{0<\alpha\le 1} \subset L^2(\tau-\delta,\tau;U)$

$$u_\alpha = G^*_{\tau\delta}(\alpha I + G_{\tau\delta}G^*_{\tau\delta})^{-1}(z_1 - T(\tau)y_0), \quad \alpha \in (0,1],$$

such that the solutions $y(t) = y(t,\tau-\delta,y_0,u^\delta_\alpha)$ *of the initial value problem*

$$\begin{cases} y' = Ay + Bu_\alpha(t), & y \in Z, \quad t > 0, \\ y(\tau-\delta) = y_0, \end{cases} \tag{3.19}$$

satisfies

$$\lim_{\alpha\to 0^+} y(\tau,\tau-\delta,y_0,u_\alpha) = z_1.$$

e.i.,

$$\lim_{\alpha\to 0^+} y(\tau) = \lim_{\alpha\to 0^+}\left\{T(\delta)y_0 + \int_{\tau-\delta}^{\tau} T(\tau-s)Bu_\alpha(s)ds\right\} = z_1.$$

4 Controllability of the Equation with Impulses and Delay

In this section we shall prove the main result of this paper, the interior approximate controllability of the semilinear heat equation with impulses and delay (1.1), which is equivalent to prove the approximate controllability of the system (2.8). To this end, for all $\phi \in C$ and $u \in L^2(0, \tau; U)$ the initial value problem

$$\begin{cases} z' = -Az + B_\omega u + f^e(t, z_t(-r), u(s)), \quad z \in Z \quad t \geq 0, \\ z(s) = \phi(s), \quad s \in [-r, 0], \\ z(t_k^+) = z(t_k^-) + I_k^e(t_k, z(t_k), u(t_k)), k = 1, 2, 3, \dots, p. \end{cases} \tag{4.20}$$

admits only one mild solution given by

$$\begin{aligned} z^u(t) =& T(t)\phi(0) + \int_0^t T(t-s)B_\omega u(s)ds + \int_0^t T(t-s)f^e(s, z(s-r), u(s))ds \\ &+ \sum_{0<t_k<t} T(t-t_k)I_k^e(t_k, z(t_k), u(t_k)), \quad t \in [0, \tau]. \end{aligned} \tag{4.21}$$

Now, we are ready to present and prove the main result of this paper, which is the interior approximate controllability of the semilinear heat equation with impulses and delay (1.1).

Theorem 4.1. *Under the condition (1.2) the semilinear heat equation with impulses and delay (1.1) is approximately controllable on* $[0, \tau]$.

Proof. Given $\phi \in C$, a final state z^1 and $\epsilon > 0$, we want to find a control $u_\alpha^\delta \in L^2(0, \tau; U)$ steering the system from $\phi(0)$ to an ϵ-neighborhood of z^1 on time τ. Specifically, if $z^{\delta,\alpha}$ is the corresponding solution of (1.1), then

$$\lim_{\alpha \to 0} \left\{ T(\tau)\phi(0) + \int_0^\tau T(\tau-s)B_\omega u_\alpha^\delta(s)ds + \int_0^\tau T(\tau-s)f^e(s, z^{\delta,\alpha}(s-r), u_\alpha^\delta(s))ds \right.$$
$$\left. + \sum_{0<t_k<\tau} T(\tau-t_k)I_k^e(t_k, z(t_k), u_\alpha^\delta(t_k)) \right\} = z_1.$$

Consider any $u \in L^2(0, \tau; U)$ and the corresponding solution $z(t) = z(t, 0, \phi, u)$ of the initial value problem (4.20). For $\alpha \in (0, 1]$ we define the control $u_\alpha^\delta \in L^2(0, \tau; U)$ as follows

$$u_\alpha^\delta(t) = \begin{cases} u(t), & \text{if} \quad 0 \leq t \leq \tau - \delta, \\ u_\alpha(t), & \text{if} \quad \tau - \delta < t \leq \tau. \end{cases}$$

where

$$u_\alpha(t) = B_\omega^* T^*(\tau - t)(\alpha I + G_{\tau\delta}G_{\tau\delta}^*)^{-1}(z_1 - T(\delta)z(\tau - \delta)), \quad \tau - \delta < t \leq \tau.$$

Now, assume that $0 < \delta < \tau - t_p$. Then the corresponding solution $z^{\delta,\alpha}(t) = z(t, 0, \phi, u_\alpha^\delta)$ of the initial value problem (4.20) at time τ can be written as follows:

$$\begin{aligned}
z^{\delta,\alpha}(\tau) =& T(\tau)\phi(0) + \int_0^\tau T(\tau-s)B_\omega u_\alpha^\delta(s)ds + \int_0^\tau T(\tau-s)f^e(s, z^{\delta,\alpha}(s-r), u_\alpha^\delta(s))ds + \sum_{0<t_k<\tau} T(\tau-t_k)I_k^e(t_k, z(t_k), u_\alpha^\delta(t_k))\\
=& T(\delta)\Bigg\{ T(\tau-\delta)\phi(0) + \int_0^{\tau-\delta} T(\tau-\delta-s)B_\omega u_\alpha^\delta(s)ds\\
&+ \int_0^{\tau-\delta} T(\tau-\delta-s)f^e(s, z^{\delta,\alpha}(s-r), u_\alpha^\delta(s))ds + \sum_{0<t_k<\tau-\delta} T(\tau-\delta-t_k)I_k^e(z^{\delta,\alpha}(t_k), u_\alpha^\delta(t_k))\Bigg\}\\
&+ \int_{\tau-\delta}^{\tau} T(\tau-s)B_\omega u_\alpha^\delta(s)ds + \int_{\tau-\delta}^{\tau} T(\tau-s)f^e(s, z^{\delta,\alpha}(s-r), u_\alpha^\delta(s))ds\\
=& T(\delta)z(\tau-\delta) + \int_{\tau-\delta}^{\tau} T(\tau-s)B_\omega u_\alpha(s)ds + \int_{\tau-\delta}^{\tau} T(\tau-s)f^e(s, z^{\delta,\alpha}(s-r), u_\alpha(s))ds.
\end{aligned}$$

So,

$$z^{\delta,\alpha}(\tau) = T(\delta)z(\tau-\delta) + \int_{\tau-\delta}^{\tau} T(\tau-s)B_\omega u_\alpha(s)ds + \int_{\tau-\delta}^{\tau} T(\tau-s)f^e(s, z^{\delta,\alpha}(s-r), u_\alpha(s))ds.$$

The corresponding solution $y^{\delta,\alpha}(t) = y(t, \tau-\delta, z(\tau-\delta), u_\alpha)$ of the initial value problem (3.19) at time τ is given by:

$$y^{\delta,\alpha}(\tau) = T(\delta)z(\tau-\delta) + \int_{\tau-\delta}^{\tau} T(\tau-s)B_\omega u_\alpha(s)ds.$$

Therefore, using proposition 2.9 we obtain

$$\|z^{\delta,\alpha}(\tau) - y^{\delta,\alpha}(\tau)\| \leq \int_{\tau-\delta}^{\tau} \|T(\tau-s)\|\{\hat{a}_0\|z^{\delta,\alpha}(s-r)\| + \hat{b}_0\}ds.$$

if we take $0 < \delta < r$ and $\tau-\delta \leq s \leq \tau$, then $s-r \leq \tau-r < \tau-\delta$ and

$$z^{\delta,\alpha}(s-r) = z(s-r).$$

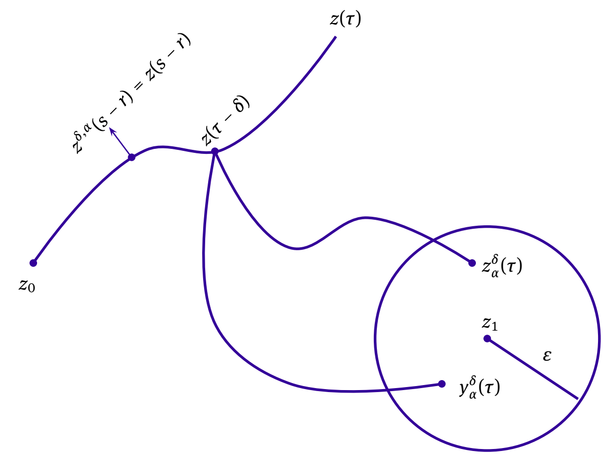
Therefore, there exists $\delta > 0$ small enough such that

$$\|z^{\delta,\alpha}(\tau) - y^{\delta,\alpha}(\tau)\| \leq \int_{\tau-\delta}^{\tau} \|T(\tau-s)\|\{\hat{a}_0\|z(s-r)\| + \hat{b}_0\}ds < \frac{\epsilon}{2}.$$

Hence,

$$\begin{aligned}
\|z^{\delta,\alpha}(\tau) - z^1\| &\leq \int_{\tau-\delta}^{\tau} \|T(\tau-s)\|\{\hat{a}_0\|z^{\delta,\alpha}(s-r)\| + \hat{b}_0\}ds + \|y^{\delta,\alpha}(\tau) - z^1\|\\
&\leq \int_{\tau-\delta}^{\tau} \|T(\tau-s)\|\{\hat{a}_0\|z(s-r)\| + \hat{b}_0\}ds + \|y^{\delta,\alpha}(\tau) - z_1\|\\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon.
\end{aligned}$$

Geometrically, the proof goes as follows:



This completes the proof of the Theorem. □

5 Final Remark

Our technique is simple and can be apply to those system with impulses and delay in the nonlinear perturbation like some control system governed by diffusion processes. For example, The Benjamin -Bona-Mohany Equation impulses and delay, the strongly damped wave equations impulses and delay, beam equations impulses and delay, etc.

Open Problem 5.1. *The approximate controllability of the linear part of the Benjamin Bona-Mohany(BBM) equation was prove in [14]. This result was used to study the controllability of the nolinear BBM equations in [15], Which can serve as a basis for studying the BBM equation under the influence of impulses and delay*

$$\begin{cases} z_t - a\Delta z_t - b\Delta z = 1_\omega u(t,x) + f(t, z(t-r,x), u(t,x)), \quad in \quad (0,\tau)\times\Omega, \\ z(t,x) = 0, \quad on \quad (0,\tau)\times\partial\Omega, \\ z(s,x) = \phi(s,x), \quad s\in[-r,0], x\in\Omega, \\ z(t_k^+,x) = z(t_k^-,x) + I_k(t, z(t_k,x), u(t_k,x)), \quad k = 1,2,3,\dots,p, \end{cases}$$

where $a \geq 0$ and $b > 0$ are constants, Ω is a domain in $\mathbf{R}^N$, ω is an open nonempty subset of Ω, 1_ω denotes the characteristic function of the set ω, the distributed control $u \in L_2(0,\tau; L_2(\Omega))$, $\phi : [-r,0]\times\Omega \to \mathbf{R}$ is a continuous function and $f(t,z,u)$ is a nonlinear perturbation.

Open Problem 5.2. *We believe that this technique can be applied to prove the interior controllability of the strongly damped wave equation under the influence of impulses and delays*

$$\begin{cases} \dfrac{\partial^2 w(t,x)}{\partial t^2} + \eta(-\Delta)^{1/2}\dfrac{\partial w(t,x)}{\partial t} + \gamma(-\Delta)w = 1_\omega u(t,x) \\ +f(t, z(t-r,x), u(t,x)), \quad in \quad (0,\tau)\times\Omega, \\ w = 0, \quad on \quad (0,\tau)\times\partial\Omega, \\ w(s,x) = \phi(s,x), \quad w_t(s,x) = \psi(s,x), s\in[-r,0], x\in\Omega, \\ w_t(t_k^+,x) = w_t(t_k^-,x) + I_k(t, w(t_k,x), w_t(t_k,x), u(t_k,x)), \quad k = 1,2,3,\dots,p, \end{cases}$$

in the space $Z_{1/2} = D((-\Delta)^{1/2}) \times L_2(\Omega)$, where Ω is a bounded domain in $\mathbf{R}^n$, ω is an open nonempty subset of Ω, 1_ω denotes the characteristic function of the set ω, the distributed control $u \in L_2(0,\tau; L_2(\Omega))$,

$\psi, \phi : [-r, 0] \times \Omega \to \boldsymbol{R}$ *are continuous functions and* η, γ *are positive numbers.*

Open Problem 5.3. *Another example where this technique may be applied is a partial differential equations modeling the structural damped vibrations of a string or a beam under the influence of impulses and delays:*

$$\begin{cases} \dfrac{\partial^2 y(t,x)}{\partial t^2} - 2\beta\Delta \dfrac{\partial y(t,x)}{\partial t} + \Delta^2 y = 1_\omega u(t,x) \\ +f(t, y(t-r,x), u(t,x)), \quad on \quad (0,\tau)\times\Omega, \\ y = \Delta y = 0, \quad on \quad (0,\tau)\times\partial\Omega, \\ y(s,x) = \phi(s,x), \quad y_t(s,x) = \psi(s,x), \quad s \in [-r,0], x \in \Omega, \\ y_t(t_k^+, x) = y_t(t_k^-, x) + I_k(t, y(t_k,x), y_t(t_k,x), u(t_k,x)), \quad k = 1,2,3,\dots,p, \end{cases}$$

where Ω *is a bounded domain in* $\boldsymbol{R}^n$*,* ω *is an open nonempty subset of* Ω*,* 1_ω *denotes the characteristic function of the set* ω*, the distributed control* $u \in L_2(0,\tau; L_2(\Omega))$*,* $\psi, \phi : [-r,0]\times\Omega \to \boldsymbol{R}$ *are continuous functions and* $y_0, y_1 \in L_2(\Omega)$*.*

Moreover, our result can be formulated in a more general setting. Indeed, we can consider the following semilinear evolution equation in a general Hilbert space Z under the influence of impulses and delays

$$\begin{cases} \acute{z} = -Az + Bu(t) + F(t, z(t-r), u(t)), \quad z \in Z, \quad t \in (0,\tau], \\ z(s) = \phi(s), \quad s \in [-r,0], \\ z(t_k^+) = z(t_k^-) + I_k^e(t_k, z(t_k), u(t_k)), \quad k = 1,2,3,\dots,p, \end{cases} \tag{5.22}$$

where, $A : D(A) \subset Z \to Z$ is an unbounded linear operator in Z with the following spectral decomposition:

$$Az = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} < z, \phi_{j,k} > \phi_{j,k},$$

with the eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots < \cdots \lambda_n \to \infty$ of A having finite multiplicity γ_j equal to the dimension of the corresponding eigenspaces, and $\{\phi_{j,k}\}$ is a complete orthonormal set of eigenfunctions of A. The operator $-A$ generates a strongly continuous compact semigroup $\{T_A(t)\}_{t\geq 0}$ given by

$$T_A(t)z = \sum_{j=1}^{\infty} e^{-\lambda_j t} \sum_{k=1}^{\gamma_j} < z, \phi_{j,k} > \phi_{j,k}.$$

The control $u \in L_2(0,\tau; U)$, with $U = Z$, $B : Z \to Z$ is a linear and bounded operator(linear and continuous) and the functions $I_k^e : [0,\tau]\times Z \times U \to Z$, $F : [0,\tau]\times C \times U \to Z$ are smooth enough and the following estimate holds:

$$\|F(t,\phi,u)\|_Z \leq a_0\|\phi(-r)\|_Z + c_0. \tag{5.23}$$

In this case the characteristic function set is a particular operator B, and the following theorem is a generalization of Theorem 4.1.

Theorem 5.1. *If vectors* $B^*\phi_{j,k}$ *are linearly independent in* Z*, then the system (5.22) is approximately controllable on* $[0,\tau]$*.*

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