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Functional envelope of a non-autonomous discrete system

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Abstract: Let $(X, F = \{f_n\}_{n=0}^{\infty})$ be a non-autonomous discrete system by a compact metric space X and continuous maps $f_n : X \to X$, $n = 0, 1, \ldots$. We introduce functional envelope $(S(X), G = \{G_n\}_{n=0}^{\infty})$, of $(X, F = \{f_n\}_{n=0}^{\infty})$, where $S(X)$ is the space of all continuous self maps of X and the map $G_n : S(X) \to S(X)$ is defined by $G_n(\varphi) = F_n \circ \varphi$, $F_n = f_n \circ f_{n-1} \circ \ldots \circ f_1 \circ f_0$. The paper mainly deals with the connection between the properties of a system and the properties of its functional envelope.

Keywords: non-autonomous discrete system, functional envelope, recurrent point

MSC: 37B20, 54H20

1 Introduction

Let (X, d) be a compact metric space and $F = \{f_n\}_{n=0}^{\infty}$ be a sequence of continuous self maps on X, where f_0 is the identity map on X. The pair $(X, F = \{f_n\}_{n=0}^{\infty})$ is called a non-autonomous discrete system. In fact a natural discrete analogue of a non-autonomous differential equation $\frac{dx}{dt} = f(x, t)$, is a different equation of the form $x_{n+1} = f_n(x_n)$. If $x_0 \in X$ and $x_n = f_n(x_{n-1})$, then the sequence $\mathcal{O}_F(x_0) = \{x_n\}_{n=0}^{\infty}$ is called the orbit of x_0 under non-autonomous discrete system $F = \{f_n\}_{n=0}^{\infty}$ i.e. if $F_n = f_n \circ f_{n-1} \circ \ldots f_1 \circ f_0$, then $\mathcal{O}_F(x_0) = \{F_n(x_0)\}_{n=0}^{\infty}$.

In the recent past, lots of studies have been done regarding dynamical properties in non-autonomous discrete dynamical systems. Kolyada and Snoha introduced Non-autonomous discrete systems in [4]. In [3], Kolyada et al. discussed minimality of non-autonomous dynamical systems. In [2], [5], authors studied ω-limit sets in non-autonomous discrete systems. Note that if $f_n = f$ for every $n \in \mathbb{N}$, then (X, f) is a classical discrete system also it is called autonomous discrete system and it is a discrete analogue of the autonomous differential equation $\frac{dx}{dt} = f(x)$. Dynamics of a non-autonomous discrete system is very different from the autonomous case, for example in any classical dynamical system, orbit of any periodic point forms an invariant set, but this is not true in non-autonomous discrete system, see [3]. Also since every continuous map $f : [0, 1] \to [0, 1]$ has at least a fixed point, thus there are no minimal systems on $[0, 1]$ ((X, f) is minimal if every point in X has dense orbit) but there is a minimal non-autonomous discrete system on $[0, 1]$, see Example 2.4. Topological version of Poincarè recurrence theorem state that in autonomous discrete system (X, f) on a compact metric space, almost every point is recurrent. We give an example to show that there is a non-autonomous discrete system on compact metric space without recurrent point, see Example 4.2.

Auslander et al in [1] , introduced functional envelope $(S(X), F_f)$ of (X, f), that $S(X)$ is the space of all continuous maps $\varphi : X \to X$ with compact- open topology and $F_f : S(X) \to S(X)$ is defined by $F_f(\varphi) = f \circ \varphi$. They studied some relations between the dynamical properties of a dynamical system (X, f) given by a compact metric space and a continuous map $f : X \to X$, and dynamical properties of its functional envelope.

In this paper, we introduce functional envelope ,$(S(X), G = \{G_n\}_{n=0}^{\infty})$, of a non-autonomous discrete system $(X, F = \{f_n\}_{n=0}^{\infty})$, and study relation between the dynamical properties of dynamical system $(X, F =$

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$\{f_n\}_{n=0}^{\infty}$) given by a compact metric space and continuous maps $f_n : X \to X$, and dynamical properties of its functional envelope. In section 2, we state some relation, between transitive points, recurrent points, minimal points and non-wandering points of non-autonomous discrete system $(X, F = \{f_n\}_{n=0}^{\infty})$ and its functional envelope $(S(X), G = \{G_n\}_{n=0}^{\infty})$. Example 2.4 shows that there is a non-autonomous discrete system $(X, F = \{f_n\}_{n=0}^{\infty})$ on every compact metric space X, such that every point of X is a transitive point, that is every point of X has a dense orbit, but its functional envelope has no transitive point. In the Theorem 2.5, we show that the set of transitive points in functional envelope of every non-autonomous discrete system on an arbitaray compact manifold is the empty set.

Recall that $S(X)$, the space of all continuous maps $\varphi : X \to X$, with compact- open topology is not compact in general, so it may be happen that the orbit closure $\mathcal{O}_G(\varphi) \subseteq S(X)$, for $\varphi \in S(X)$, is a non-compact set. In section 3 , Theorem 3.4 shows that compactness of orbit closure $\mathcal{O}_G(\varphi) \subseteq S(X)$, for $\varphi \in S(X)$ is a sufficient condition for equicontinuity of the family $\{\varphi, F_1 \circ \varphi, \dots, F_n \circ \varphi, \dots\}$ on compact metric space (X, d). Later in Corollary 3.8, we prove that in linear non-autonomous discrete system $(X, F = \{f_n\}_{n=0}^{\infty})$, if $F = \{f_n\}_{n=0}^{\infty}$ has a dense orbit or constant maps belong to the closure of the orbit $\mathcal{O}_G(\varphi) = \{\varphi, F_1(\varphi), F_2(\varphi), \dots\}$, then $\overline{\mathcal{O}_G(\varphi)}$ is not a compact set. Next we introduce k-th iteration system, $(X, F^k = \{g_n\}_{n=0}^{\infty})$ of $(X, F = \{f_n\}_{n=0}^{\infty})$. Suppose that $(S(X), G^k)$ is the functional envelope of $(X, F^k = \{g_n\}_{n=0}^{\infty})$, it can to see that if orbit closure of $\varphi \in S(X)$ in $(S(X), G)$ is compact, then for every $k \in \mathbb{N}$, orbit closure of $\varphi \in S(X)$ in $(S(X), G^k)$ is compact, for the converse, in Example 3.11 we show that there is a non-autonomous discrete system $(X, F = \{f_n\}_{n=0}^{\infty})$ such that for every $k \in \mathbb{N}$, $\overline{\mathcal{O}_{G^k}(\varphi)}$, is compact, but $\overline{\mathcal{O}_G(\varphi)}$ is not compact. Theorem 3.12 implies that if $F = \{f_n\}_{n=0}^{\infty}$ converge to a map f and for some $k > 1$, $\overline{\mathcal{O}_{G^k}(\varphi)}$ is compact, then $\overline{\mathcal{O}_G(\varphi)}$ is compact.

It is known that in any non-autonomous discrete system $(X, F = \{f_n\}_{n=0}^{\infty})$ on a compact metric space X, Limit set of every point is non-empty. Example 4.2 shows that there is a non- autonomous discrete system such that ω-limit set of every point $\varphi \in S(X)$ in functional envelope $(S(X), G)$ is empty set. In Theorem 4.1, we give conditions to imply that ω-limit set of some point $\varphi \in S(X)$ in functional envelope $(S(X), G)$ is non-empty set. It is known that every autonomous discrete system on compact metric space has a recurrent point, in Example 4.3 we show that there is a non-autonomous discrete system with empty recurrent set. Example 4.2 shows that it may be happen that a non-autonomous discrete system has a recurrent point but its functional envelope has no recurrent point. In Theorem 4.5, we give conditions to imply that the functional envelope of a non-autonomous discrete system has recurrent point.

2 Transitive points, Recurrent points and non-wandering points

Given a compact metric space (X, d) and the set (semigroup) $S(X)$ of all continuous maps from $\varphi : X \to X$, we consider the following metric on $S(X)$: for $\varphi, \psi \in S(X)$

$$d_U(\varphi, \psi) = \sup_{x \in X} d(\varphi(x), \psi(x)).$$

The corresponding space will be denoted by $S(X)$.

Definition 2.1. If $F = \{f_n\}_{n=0}^{\infty}$ is a non- autonomous discrete system on compact metric space (X, d), then the family of maps $G_n : S(X) \to S(X)$ defined by $G_n(\varphi) = F_n \circ \varphi$, is called the functional envelope of non-autonomous discrete system $(X, F = \{f_n\}_{n=0}^{\infty})$ and it denoted by $(S(X), G = \{G_n\}_{n=0}^{\infty})$.

Since (X, d) is a compact metric space, uniform continuity of F_n on X implies that $G_n : S(X) \to S(X)$ is uniform continuous. Also G_n is injective on $S(X)$ if and only if F_n is injective, but the following example shows that the surjective property of F_n does not imply that G_n is surjective:

Example 2.2. ([1]) Let $X = [0, 1]$ with the usual metric d defined by $d(x, y) = |x - y|$ and

$$f_k(x) = \begin{cases} 2x & x \in [0, \frac{1}{4}], \\ \frac{1}{2} & x \in [\frac{1}{4}, \frac{3}{4}] \\ 2x - 1 & x \in [\frac{3}{4}, 1]. \end{cases}$$

It is easy to see that F_n is surjective continuous map but there is not $\varphi \in S(X)$, with $F_n \circ \varphi = id_X$, thus G_n is not surjective continuous.

Definition 2.3. Let $(S(X), G = \{G_n\}_{n=0}^{\infty})$ be the functional envelope of non-autonomous discrete system $(X, F = \{f_n\}_{n=0}^{\infty})$. The orbit of $\varphi \in S(X)$ in the system $(S(X), G = \{G_n\})$ has the form

$$\mathcal{O}_G(\varphi) = \{\varphi, F_1 \circ \varphi, F_2 \circ \varphi, \dots\}.$$

The following example shows that dynamics of functional envelope $(S(X), G = \{G_n\}_{n=0}^{\infty})$ of non-autonomous discrete system $(X, F = \{f_n\}_{n=0}^{\infty})$, may be different from dynamics of the non-autonomous discrete system $(X, F = \{f_n\}_{n=0}^{\infty})$.

Example 2.4. Let (X, d) be a compact metric space, there is countable set $\{x_n\}_{n=0}^{\infty} \subseteq X$ with $\overline{\{x_n\}_{n=0}^{\infty}} = X$. Put $f_n = x_n$ for every $n = 0, 1, 2 \dots$. It is easy to see that non-autonomous discrete system $(X, F = \{f_n\}_{n=0}^{\infty})$ is minimal i.e. for every $x \in X$, $\overline{\mathcal{O}_F(x)} = X$, but for every $\varphi \in S(X)$, $\overline{\mathcal{O}_G(\varphi)} \neq S(X)$.

In the following theorem, the set of all $\varphi \in S(X)$ with $\overline{\mathcal{O}_G(\varphi)} = S(X)$ is denoted by $Trans(G)$ and the set of $x \in X$ with $\overline{\mathcal{O}_F(x)} = X$ is denoted by $Trans(F)$.

Theorem 2.5. *Let $(S(X), G = \{G_n\}_{n=0}^{\infty})$ be the functional envelope of non-autonomous discrete system $(X, F = \{f_n\}_{n=0}^{\infty})$.*

- *For $\varphi \in Trans(G)$ we have:*
  1. *every $\psi \in \mathcal{O}_G(\varphi)$ is injective,*
  2. *$\varphi \neq Const_a$, $\forall a \in X$ where $Const_a : X \to X$ is defined by $Const_a(x) = a$.*
  3. *$\varphi(X) \subseteq Trans(F)$ i.e. $\forall x \in X$, $\overline{\mathcal{O}_F(x)} = X$,*
- *If X is a compact manifold, then $Trans(G) = \emptyset$*

Proof. 1. Let $\psi \in \mathcal{O}_G(\varphi)$. Assume that $\psi(x) = \psi(y)$. Since $\psi \in \mathcal{O}_G(\varphi)$, it is easy to see that, there is $m \in \mathbb{N}$ such that for every $n > m$, $F_n \circ \varphi(x) = F_n \circ \varphi(y)$. The condition $\overline{\mathcal{O}_G(\varphi)} = S(X)$ implies that there is a subsequence $\{n_k\}_{k=0}^{\infty}$ of $\mathbb{N}$ such that $F_{n_k}(\varphi) \to id_X$ as $n_k \to \infty$. This shows that ψ is injective.

2. If $Const_a \in Trans(G)$, for $id_X \in S(X)$, there is a subsequence $\{n_k\}_{k=0}^{\infty}$ of $\mathbb{N}$ such that $F_{n_k}(\varphi) \to id_X$ as $n_k \to \infty$ but $F_{n_k} \circ Const_a = Const_{F_{n_k}(a)}$. This means that $Const_{F_{n_k}(a)} \to id_X$ that is a contradiction,
3. Let $\varphi \in Trans(G)$ and $B_\epsilon(a)$ be an open set in X, since $\overline{\mathcal{O}_G(\varphi)} = S(X)$, for $Const_a \in S(X)$ and $\epsilon > 0$, there is $n_k \in \mathbb{N}$ such that $d_U(F_{n_k}(\varphi), Const_a) < \epsilon$. If $y = \varphi(x)$, we have $d(F_{n_k}(y), a) < \epsilon$. This means that if $y \in \varphi(X)$, then $\overline{\mathcal{O}_F(x)} = X$.

- By contradiction suppose $\varphi \in Trans(G)$. Since X is a compact manifold, there is an open set $U \subseteq S(X)$ such that every $\psi \in U$ is non-injective. $\varphi \in Trans(G)$ implies that there is an injective map $F_n \circ \varphi \in U$, that is a contradiction.

□

In autonomous discrete system (X, f), A point $x \in X$ is called a recurrent point (non-wandering point) of f, if for every open neighborhood U of x, there exists $n > 0$ such that $f^n(x) \in U$ ($f^n(U) \cap U \neq \emptyset$). Also $x \in X$ is called a minimal point, $x \in M(f)$, if $\overline{\mathcal{O}_f(x)}$ is a minimal set. The set of all recurrent points (resp. non-wandering points) of f is denoted by $\mathcal{R}(f)$ (resp. $\Omega(f)$). In non-autonomous discrete system $(X, F = \{f_n\}_{n=0}^{\infty})$, for $x \in X$ and $A \subseteq X$, put

$$N_F(x, A) = \{n \in \mathbb{N} | F_n(x) \in A\}.$$

A point $x \in X$ is called a recurrent point of $F = \{f_n\}_{n=0}^{\infty}$, $x \in \mathcal{R}(F)$, if for every open set U of x and every $n \in \mathbb{N}$, there is $k > n$ with $k \in N_F(x, U)$, also $x \in X$ is called a minimal point of $F = \{f_n\}_{n=0}^{\infty}$, if for every open set U of x, $N_F(x, U)$ is syndetic i.e. there is $k \in \mathbb{N}$, such that for all $n \in \mathbb{N}$, $[n, n+k] \cap N_F(x, U) \neq \emptyset$, the set of all minimal points of $F = \{f_n\}_{n=0}^{\infty}$, is denoted by $M(F)$. A point $x \in X$, is called a non-wandering point of $F = \{f_n\}_{n=0}^{\infty}$, if for every open set U of x and every $m \in \mathbb{N}$, there is $n > m$ such that $F_n(U) \cap U \neq \emptyset$. The set of all non-wandering points is denoted by $\Omega(F)$. It is clear that $M(F) \subseteq \mathcal{R}(F) \subseteq \Omega(F)$. In general in $F = \{f_n\}_{n=0}^{\infty}$, there are no relations between minimal points, recurrent points and non-wandering points of f_i with minimal points, recurrent points and non-wandering points of non-autonomous discrete system $(X, F = \{f_n\}_{n=0}^{\infty})$. For example let $X = [0, 1]$ and $f_{2n-1}(x) = x^2$ and $f_{2n}(x) = \sqrt{x}$, it can to see that $M(f) = \mathcal{R}(f) = \Omega(f) = \{0, 1\}$ but $f_{2n} \circ f_{2n-1} = id_X$ implies that $M(F) = \mathcal{R}(F) = \Omega(F) = [0, 1]$. It is easy to see that if $F = \{f_n\}_{n=0}^{\infty}$ is a k-periodic sequence, that is for all $n \in \mathbb{N}$, $f_{n+k} = f_n$, then minimal point, recurrent point and non-wandering point of autonomous discrete system $(X, f_k \circ f_{k-1} \circ \ldots f_1 \circ f_0)$ is a minimal point, recurrent point and non-wandering point of non-autonomous discrete system $(X, F = \{f_n\}_{n=0}^{\infty})$, respectively.

Definition 2.6. The ω-limit set of $\varphi \in S(X)$ under functional envelope $(S(X), G = \{G_n\}_{n=0}^{\infty})$ is the set $\omega_G(\varphi)$ of all limit points of $\mathcal{O}_G(\varphi)$.

For open set U of $\varphi \in S(X)$ put

$$N_G(\varphi, U) = \{n \in \mathbb{N} | F_n \circ \varphi \in U\}.$$

$\varphi \in S(X)$ is called a recurrent point of $(S(X), G = \{G_n\}_{n=0}^{\infty})$, $\varphi \in \mathcal{R}(G)$, if for every open set $U \subseteq S(X)$ of φ and every $m \in \mathbb{N}$ there is $n > m$ with $n \in N_G(\varphi, U) \neq \emptyset$. $\varphi \in S(X)$ is called a minimal point of $(S(X), G = \{G_n\}_{n=0}^{\infty})$, $\varphi \in M(G)$, if for every open set U of $\varphi \in S(X)$, $N_G(\varphi, U)$ is syndetic. A point $\varphi \in S(X)$ is called a non-wandering point, $\varphi \in \Omega(G)$, if for every open set U of $\varphi \in S(X)$ and every $m \in \mathbb{N}$, there are $n > m$ and $\psi \in U$ with $F_n \circ \psi \in U$.

Theorem 2.7. *Let $(S(X), G = \{G_n\}_{n=0}^{\infty})$ be the functional envelop of non-autonomous discrete system $(X, F = \{f_n\}_{n=0}^{\infty})$. Then we have*

1. *If $\varphi \in \mathcal{R}(G)$ and $g \in S(X)$, then $\varphi \circ g \in \mathcal{R}(G)$.*
2. *If $\varphi \in M(G)$ and $g \in S(X)$, then $\varphi \circ g \in M(G)$.*
3. *If $\varphi \in \Omega(G)$ and $g \in S(X)$, then $\varphi \circ g \in \Omega(G)$.*

Proof. For (1), (2), it is enough to show that for every $\epsilon > 0$,

$$N_G(\varphi, B_\epsilon(\varphi)) \subseteq N_G(\varphi \circ g, B_\epsilon(\varphi \circ g)).$$

Let $n \in N_G(\varphi, B_\epsilon(\varphi))$. Since

$$d_U(F_n(\varphi \circ g), \varphi \circ g) \leq d_U(F_n(\varphi), \varphi),$$

we have $n \in N_G(\varphi \circ g, B_\epsilon(\varphi \circ g))$.
(3) For $\epsilon > 0$, $\varphi \in \Omega(G)$ and $g \in S(X)$, we show that there is $n \in \mathbb{N}$ such that

$$F_n(B_\epsilon(\varphi \circ g)) \bigcap B_\epsilon(\varphi \circ g) \neq \emptyset.$$

Since $\varphi \in \Omega(G)$, for $\epsilon > 0$, there is $\psi \in S(X)$ with $d_U(F_n(\psi), \varphi) < \epsilon$ and $d_U(\psi, \varphi) < \epsilon$. Also for every $g \in S(X)$, since

$$d_U(\psi \circ g, \varphi \circ g) < d_U(\psi, \varphi) < \epsilon$$

and

$$d_U(F_n(\psi \circ g), \varphi \circ g) < d_U(F_n(\psi), \varphi) < \epsilon$$

thus

$$F_n(B_\epsilon(\varphi \circ g)) \cap B_\epsilon(\varphi \circ g) \neq \emptyset.$$

□

Lemma 2.8. *Let* $(S(X), G = \{G_n\}_{n=0}^{\infty})$ *be the functional envelope of a non autonomous discrete system* $(X, \{f_n\}_{n=0}^{\infty})$. *Then*

1. $\forall \varphi \in S(X)$ *and* $\forall \epsilon > 0$, $N_G(\varphi, B_\epsilon(\varphi)) \subseteq N_F(y, B_\epsilon(y))\ \ \forall y \in \varphi(X)$
2. $\forall a \in X$, $N_G(Const_a, B_\epsilon(Const_a)) = N_F(a, B_\epsilon(a))$

Pro1of. Let $n \in N_G(\varphi, B_\epsilon(\varphi))$, for $y \in \varphi(X)$, there is $x \in X$, with $y = \varphi(x)$, we have $d(F_n \circ \varphi(x), \varphi(x)) < d_U(F_n \circ \varphi, \varphi) < \epsilon$, thus $d(F_n(y), y) < \epsilon$.

2. The equality $d_U(F_n \circ Const_a, Const_a) = d(F_n(a), a)$ and (1), imply that:

$$\forall a \in X,\ \ N_G(Const_a, B_\epsilon(Const_a)) = N_F(a, B_\epsilon(a)).$$

□

In the following theorem, the set of all recurrent points, minimal points and non-wandering points of $(S(X), G = \{G_n\}_{n=0}^{\infty})$ is denoted by $\mathcal{R}(G)$, $M(G)$, $\Omega(G)$, respectively, also the set of recurent points, minimal points and non-wandering points of non-autonomous discrete system $(X, F = \{f_n\}_{n=0}^{\infty})$ is denoted by $\mathcal{R}(F)$, $M(F)$, $\Omega(F)$, respectively.

Theorem 2.9. *Let* $(S(X), G = \{G_n\}_{n=0}^{\infty})$ *be the functional envelope of a non-autonomous discrete system* $(X, F = \{f_n\}_{n=0}^{\infty})$.

1. *If* $\varphi \in \mathcal{R}(G)$ *then* $\varphi(X) \subset \mathcal{R}(F)$, *also* $a \in \mathcal{R}(F)$ *if and only if* $Const_a \in \mathcal{R}(G)$,
2. *If* $\varphi \in M(G)$, *then* $\varphi(X) \subseteq M(F)$, *also* $a \in M(F)$ *if and only if* $Const_a \in M(G)$
3. *If* $\varphi \in \Omega(G)$, *then* $\varphi(X) \subseteq \Omega(F)$, *Also* $a \in \Omega(F)$ *if and only if* $Const_a \in \Omega(G)$

Proof. (1), (2) are clear by Lemma 2.8.
(3). Let $\varphi \in \Omega(G)$, $y \in \varphi(X)$ and $\epsilon > 0$. There are $\psi \in S(X)$ and $n \in \mathbb{N}$ such that $d_U(\varphi, \psi) < \epsilon$ and $d_U(F_n(\psi), \varphi) < \epsilon$. This means that if $y = \varphi(x)$, then for $\psi(x) \in X$, we have $d(\psi(x), \varphi(x)) < \epsilon$ and $d(F_n(\psi(x)), \varphi(x)) < \epsilon$, therefore $\psi(x) \in f^n(B_\epsilon(\varphi(x))) \cap B_\epsilon(\varphi)$.

□

3 Equicontinuity and compactness of orbit closure of a map

Given a metric space (X, d) and a family of continuous maps $g : X \to X$, $\mathcal{G}$, is equicontinuous on a set $A \subseteq X$, if for every $\epsilon > 0$, there is $\delta > 0$ such that for every $x, y \in A$ with $d(x, y) < \delta$ and for every $g \in \mathcal{G}$ we have $d(g(x), g(y)) < \epsilon$. If $A = X$, we simply say that $\mathcal{G}$ is equicontinuous. If $B \subseteq A$ and $\mathcal{G}$ is equicontinuous on A, then it is equicontinuous on B. Note that if A is compact, then for the equicontinuity of $\mathcal{G}$ on A, it is sufficient and necessary that $\mathcal{G}$ be equicontinuous at every point of A. This means that given $\epsilon > 0$, there is $\delta > 0$ such that if $d(x, a) < \delta$, then $d(f(x), f(a)) < \epsilon$, $\forall f \in \mathcal{G}$. Also it can to see that $\mathcal{G} \subseteq S(X)$ is equicontinuous if and only if $\overline{\mathcal{G}} \subseteq S(X)$ is equicontinuous.

Definition 3.1. Non- autonomous discrete dynamical system $F = \{f_n\}_{n=0}^{\infty}$ is called equicontinuous, whenever the family of maps $\{F_n\}_{n=0}^{\infty}$, $F_n = f_n \circ f_{n-1} \circ \ldots \circ f_1 \circ f_0$, is equicontinuous on X.

The following example shows that in general there are no relations between equicontinuity of the maps f_i and equicontinuity of the non-autonomous discrete system $F = \{f_n\}_{n=0}^{\infty}$.

Example 3.2. 1. Let $X = I \times S^1$ be the unit cylinder. Suppose $f_0, f_1 : I \times S^1 \to I \times S^1$ are defined by $f_0((r, \theta)) = (r, r + \theta)$ and $f_1(r, \theta) = (r, \theta - r)$ and $(X, F = \{f_n\}_{n=0}^{\infty})$ be the corresponding non-autonomous discrete system. It can to see that $f_i, i = 0, 1$ is not equicontinuous [7], but $f_1 \circ f_0 = id_X$ and the system $(X, F = \{f_n\}_{n=0}^{\infty})$ is equicontinuous.

2. Let $X = [0, 1]$ and $f_i : X \to X$, $i = 0, 1$, are defined by

$$f_0(x) = \begin{cases} 2x & x \in [0, \frac{1}{2}], \\ \frac{3}{2} - x & x \in [\frac{1}{2}, 1]. \end{cases}$$

and

$$f_1(x) = \begin{cases} \frac{1}{2} - x & x \in [0, \frac{1}{2}], \\ 2x - 1 & x \in [\frac{1}{2}, 1]. \end{cases}$$

It can to see that f_i, $i = 0, 1$ is equicontinuous map but $F = \{f_0, f_1\}$ is not equicontinuous.

Lemma 3.3. *([1]) Let X, Y be metric space, X a compact metric spaces, and $\varphi : X \to Y$ be a surjective continuous map. Let $a \in Y$ and let $\delta > 0$. Then there is an $\eta > 0$ such that*

$$B_\eta(a) \subseteq \bigcup_{x \in \varphi^{-1}(a)} \varphi(B_\delta(x)).$$

Since $S(X)$ is not compact in general, the orbit closure $\overline{\mathcal{O}_G(\varphi)}$ may be non-compact. In the following theorem, we give conditions that the orbit of φ is compact.

Theorem 3.4. *Let $\varphi \in S(X)$. Then the following conditions are equivalent:*
1. *The orbit closure $\overline{\mathcal{O}_G(\varphi)}$ is compact.*
2. *The family of maps $\{\varphi, F_1 \circ \varphi, F_2 \circ \varphi, \dots\}$ is equicontinuous on X.*
3. *The non-autonomous discrete system $F = \{f_n\}_{n=0}^{\infty}$ is equicontinuous on $range(\varphi)$.*

Proof. (1) $\Leftrightarrow$ (2). Since X is compact, φ is bounded on X. It implies that $\{\varphi, F_1 \circ \varphi, F_2 \circ \varphi, \dots\}$ is a bounded set. Thus by Arzela- Ascoli theorem, $\overline{\mathcal{O}_G(\varphi)}$ is compact if and only if $\{\varphi, F_1 \circ \varphi, F_2 \circ \varphi, \dots\}$ is equicontinuous, therefor (1), (2) are equivalence.
2 $\Rightarrow$ 3. We show that non-autonomous discrete system $F = \{f_n\}_{n=0}^{\infty}$ is equicontinuous at every $a = \varphi(x) \in range(\varphi)$. Since the family of maps $\{\varphi, F_1 \circ \varphi, F_2 \circ \varphi, \dots\}$ is equicontinuous on X, for $\epsilon > 0$ there is $\delta > 0$ such that if $d(x_1, x_2) < \delta$ then $d(F_k(\varphi(x_1)), F_k(\varphi(x_2))) < \epsilon$, for $k = 0, 1, \dots$. For $\delta > 0$, there is $\eta > 0$ satisfies in Lemma 3.3. Suppose $y = \varphi(x_1) \in \varphi(X)$ with $d(y, a) < \eta$, by Lemma 3.3, $d(x, x_1) < \delta$, thus $d(F_k(\varphi(x_1)), F_k(\varphi(x))) < \epsilon$, that is, $d(F_k(y), F_k(a)) < \epsilon$ for all $k = 0, 1, \dots$.
3 $\Rightarrow$ 2. We show that for every $\epsilon > 0$, there is $\delta > 0$ such that if $d(x, y) < \delta$, then $d(F_k(\varphi(x)), F_k(\varphi(y))) < \epsilon$. Since non-autonomous discrete system $F = \{f_n\}_{n=0}^{\infty}$ is equicontinuous on $range(\varphi)$, for $\epsilon > 0$, there is $\eta > 0$ such that if $d(\varphi(x), \varphi(y)) < \eta$, then $d(F_k(\varphi(x)), F_k(\varphi(y))) < \epsilon$, by uniform continuity of $\varphi : X \to X$, for $\eta > 0$ there is $\delta > 0$, such that if $d(x, y) < \delta$, then $d(\varphi(x), \varphi(y)) < \eta$,thus for $\epsilon > 0$ there is $\delta > 0$ such that if $d(x, y) < \delta$, then $d(F_k(\varphi(x)), F_k(\varphi(y))) < \epsilon$, for all $k \in \mathbb{N}$. □

Corollary 3.5. *The following statement are equivalent:*
1. *Non-autonomous discrete system $(X, F = \{f_n\}_{n=0}^{\infty})$ is equicontinuous,*
2. *the family $\{G_n : S(X) \to S(X) | G_n(\varphi) = F_n \circ \varphi\}$, is equicontinuous on $S(X)$.*
3. *orbit closure of all maps are compact*
4. *orbit closure of id_X is compact,*

Proof. 1 $\Rightarrow$ 2. For $\epsilon > 0$, there is $\delta > 0$ such that if $d(x, y) < \delta$, then $d(F_k(x), F_k(y)) < \epsilon$. If $d_U(\varphi, \psi) < \delta$, then for every $x \in X$, $d(\varphi(x), \psi(x)) < \delta$, thus for every $x \in X$, $d(F_k(\varphi(x)), F_k(\varphi(y))) < \epsilon$, this means that $d_U(G_n(\varphi), G_n(\psi)) < \epsilon$
2 $\Rightarrow$ 3. Since the family $\{G_n : S(X) \to S(X) | G_n(\varphi) = F_n \circ \varphi\}$, is equicontinuous at every $\varphi \in S(X)$, thus by Theorem 3.4, the orbit closure $\overline{\mathcal{O}_F(\varphi)}$ is compact.
(3) $\Rightarrow$ (4) is clear,
(4) $\Rightarrow$ (1). Since orbit closure of id_X is compact, if $\varphi = id_X$, by Theorem 3.4(2), $\{F_1, F_2, \dots\}$ is equicontinuous.
□

Definition 3.6. A linear non-autonomous discrete system is a triple $(X, d, \{f_n\}_{n=0}^{\infty})$ consisting of a separable Frechet space X, a translation invariant metric d on X and operators $f_n : X \to X$.

In the following theorem, we state a result about the relationship between dense orbit and equicontinuity in linear non-autonomous discrete system.

Theorem 3.7. *Let $(X, \{f_n\}_{n=0}^{\infty})$ be a linear non-autonomous discrete system. If $(X, \{f_n\}_{n=0}^{\infty})$ has a dense orbit, then it is not equicontinuous system.*

Proof. Since (X, d) is separable Frechet space, an important feature of this metric is that it is translation-invariant, that is, for every $x, y, z \in X$, $d(x + z, y + z) = d(x, y)$. Let $z \in X$ ba such that $(X, \{f_n\}_{n=0}^{\infty})$ is equicontinuous in z. Then for $\epsilon > 0$, there is $\delta > 0$ such that if $d(w, z) < \delta$, then for every $n \in \mathbb{N}$, $d(F_n(w), F_n(z)) < \epsilon$. For every $\epsilon > 0$ and $\delta > 0$, put $U = \{z | d(z, 0) < \delta\}$ and $V = \{z | d(z, 0) > \epsilon\}$, by Theorem 6 in [6], the set of transitive points (a point that its orbit is dense), is a dense G_δ set, it implies that there are $m > 0$ and transitive point $x \in U$ with $F_m(x) \in V$. If $z \in X$ and $w = x+z$, then $d(w, z) = d(x, 0) < \delta$ and $d(F_m(w), F_m(z)) = d(F_m(x), 0) > \epsilon$, that is contradiction by equicontinuity in z. □

In the following corollary, we give a property of the functional envelope of a linear non-autonomous discrete system:

Corollary 3.8. *Let $(S(X), \{f_n\}_{n=0}^{\infty})$ be the functional envelope of a linear non-autonomous discrete system $(X, \{f_n\}_{n=0}^{\infty})$. For $\varphi \in S(X)$, every one of the following condition implies that the orbit closure of φ, $\overline{\mathcal{O}_G(\varphi)}$, is not compact.*

1. *$F = \{f_n\}_{n=0}^{\infty}$ has a dense orbit,*
2. *constant maps belong to the closure of the orbit $\mathcal{O}_G(\varphi) = \{\varphi, F_1(\varphi), F_2(\varphi), \ldots\}$.*

Proof. 1. Theorem 3.7, implies that $F = \{f_n\}_{n=0}^{\infty}$ is not equicontinuous on $range(\varphi)$, thus by Theorem 3.4, the orbit closure of φ, is not compact.

2. Let $a \in X$ and $\epsilon > 0$ be given. If $Const_a : X \to X$ is defined by $Const_a(x) = a$, then since $Const_a \in \overline{\mathcal{O}_G(\varphi)}$, there is $n \in \mathbb{N}$ with $d_U(Const_a, F_n(\varphi)) < \epsilon$. Thus for $y \in \varphi(X)$, we have $F_n(y) \in B_\epsilon(a)$. This means that every point of $\varphi(X)$ has dense orbit in $F = \{f_n\}_{n=0}^{\infty}$, thus the orbit closure of φ, is not compact.

□

Definition 3.9. ([8]) Let $(X, F = \{f_n\}_{n=0}^{\infty})$, $F = \{f_n\}_{n=0}^{\infty}$, be a non-autonomous discrete system, for every positive integer k, denote

$$(f_1^k, f_{k+1}^k, \ldots, f_{km+1}^k, \ldots) = F^k, \quad f_i^k = f_{k-1+i} \circ f_{n-2+i} \circ \ldots \circ f_{i+1} \circ f_i$$

we know (X, F^k) also is a non-autonomous discrete system, and we call this k- th iteration system of $(X, F = \{f_n\}_{n=0}^{\infty})$.

Let $(S(X), G^k)$ be the functional envelope of a non- autonomous discrete system (X, F^k), it is clear that for $\varphi \in S(X)$, $\mathcal{O}_{G^k}(\varphi) \subseteq \mathcal{O}_G(\varphi)$. Thus if $\overline{\mathcal{O}_G(\varphi)}$ is a compact set, then $\overline{\mathcal{O}_{G^k}(\varphi)}$ is a compact set too, this and Corollary 3.5, imply that:

Theorem 3.10.
1. *If a non-autonomous discrete system $(X, F = \{f_n\}_{n=0}^{\infty})$ is equicontinuous, then (X, F^k) is equicontinuous, $\forall k \in \mathbb{N}$.*
2. *If the functional envelope of the non-autonomous discrete system $(X, F = \{f_n\}_{n=0}^{\infty})$ is equicontinuous, then functional envelope of k- th iteration system of $(X, F = \{f_n\}_{n=0}^{\infty})$, (X, F^k), is equicontinuous, $\forall k \in \mathbb{N}$.*

In the following example, we show that the converse of Theorem 3.10, is not true in general:

Example 3.11. Let (X, d) be a compact metric space and $h : X \to X$ is c- expansive homeomorphism, that is if $x \neq y$ then there is $n \in \mathbb{N}$ with $d(h^n(x), h^n(y)) > c$. It is known that h is expansive if and only if h^k is expansive,

$\forall k \in \mathbb{N}$. Put

$$f_{i,2}(x) = \begin{cases} h^{i+1}(x) & i \in \{2n-1 | n \in \mathbb{N}\}, \\ h^{-i}(x) & i \in \{2n | n \in \mathbb{N}\} \end{cases}$$

Since $h^2 : X \to X$ is expansive, for $x \neq y$, there is $l \in \mathbb{N}$ such that $d(h^{2l}(x), h^{2l}(y)) > c$. If $i = 2l - 1$ it is easy to see that $f_{i,2} \circ f_{i-1,2} \circ \dots f_{1,2} = h^{2l}$, thus $(X, F = \{f_{i,2}\})$ is not equicontinuous but $f_{2n,2} \circ f_{2n-1,2} = id_X$, implies that (X, F^2) is equicontinuous. Also put

$$f_{i,k}(x) = \begin{cases} h^{i+(k-1)}(x) & i \in \{kn+1 | n \geq 0\}, \\ h^{-(i+k-2)}(x) & i \in \{kn+2 | n \geq 0\}, \\ \text{x} & \text{o.w} \end{cases}$$

Since h^k is expansive, for $x \neq y$ there is $l \in \mathbb{N}$ such that $d(h^{lk}(x), h^{lk}(y)) > c$, if $i = kl + 1$ it is easy to see that $f_{i,k} \circ f_{i-1,k} \circ \dots \circ f_{1,k} = h^{kl}$, thus the non-autonomous $(X, F = \{f_{i,k}\}_{i=0}^{\infty})$ is not equicontinuous, but $f_{(n+1)k,k} \circ \dots f_{nk+1,k} = id_X$ implies that (X, F^k) is equicontinuous.

This example shows that it may be for $\varphi \in S(X)$, for every $k > 1$, $\overline{\mathcal{O}_{G^k}(\varphi)}$ is compact but $\overline{\mathcal{O}_G(\varphi)}$ is not compact. The following theorem shows that if a non-autonomous discrete system $(X, \{f_n\}_{n=0}^{\infty})$ converges to a map f and for some $k > 1$, $\overline{\mathcal{O}_{G^k}(\varphi)}$ is compact, then $\overline{\mathcal{O}_G(\varphi)}$ is compact.

Theorem 3.12. *Let a non-autonomous discrete system $(X, \{f_n\}_{n=0}^{\infty})$ converges uniformly to a map f. If for some $k > 1$, (X, F^k) is equicontinuous at x, then $(X, F = \{f_n\}_{n=0}^{\infty})$ is equicontinuous at x.*

Proof. By contradiction, suppose that $(X, F = \{f_n\}_{n=0}^{\infty})$ is not equicontinuous at x. There is $\epsilon > 0$ such that for every $\delta > 0$ there exist $y(x, \delta) \in B_\delta(x)$ and $n(x, \delta) \in \mathbb{N}$ such that $d(F_{n(x,\delta)}(x), F_{n(x,\delta)}(y(x, \delta))) > \epsilon$. Since $\{f_n\}_0^{\infty}$ converges uniformly to a map f, it can to see that $\forall i \in \mathbb{N}$, $\{F_{[n,n+i]}\}_{n=0}^{\infty}$ converges to f^i, see Lemma 2.1 in [8], also by uniform continuity of $f^i : X \to X$, for $\epsilon > 0$ there is $\delta > 0$ such that

$$d(y, x) < \delta \Rightarrow d(f^i(x), f^i(y)) < \epsilon, \quad i = 1, \dots, k,$$

thus the inequality

$$d(F_{[n,n+i]}(y), F_{[n,n+i]}(x)) < d(F_{[n,n+i]}(y), f^i(y)) \quad + d(f^i(y), f^i(x)) + d(f^i(x), F_{[n,n+i]}(x))$$

implies that for $\epsilon > 0$ there exist $\delta > 0$ and $N > 2k$ such that for $n > N$

$$d(y, x) < \delta \Rightarrow d(F_{[n,n+i]}(y), F_{[n,n+i]}(x)) < \epsilon, \quad i = 1, \dots, k.$$

Uniform continuity $F_i : X \to X$ implies that for $\epsilon > 0$ there is $\eta > 0$ such that

$$d(x, y) < \eta \Rightarrow d(F_i(x), F_i(y)) < \epsilon \;\; i = 1, \dots, 3N.$$

Also $d(F_{n(x,\eta)}(x), F_{n(x,\eta)}(y(x, \eta))) > \epsilon$ implies that $n(x, \eta) > 3N > 6k$. Suppose $n(x, \eta) = lk + i$, $i \in \{1, \dots, k - 1\}$.

$$d(F_{n(x,\eta)}(x), F_{n(x,\eta)}(y(x, \eta))) = d(F_{[lk+1,n(x,\eta)]}(F_{lk}(x)), F_{[lk+1,n(x,\eta)]}(F_{lk}(y(x, \eta)))$$

implies that for all $\eta > 0$, there exist $y(x, \eta) \in B_\eta(x)$ and $l \in \mathbb{N}$ such that $d(F_{lk}(x), F_{lk}(y(x, \eta))) > \delta$, this is a contradiction, because (X, F^k) is equicontinuous at x. □

4 Equicontinuity and ω- limit set

Let $(X, \{f_n\}_{n=0}^{\infty})$ be a non-autonomous discrete system on compact metric space (X, d). For every $x \in X$, $\omega_F(x) \neq \emptyset$ because $\mathcal{O}_F(x)$ is a infinite set in compact metric space X. But for functional envelope of a non-autonomous discrete systems we have:

Theorem 4.1. *Let $\varphi \in S(X)$, then the following are equivalent:*
1. $\omega_G(\varphi) \neq \emptyset$
2. *there is a sequence $n_i \to \infty$, such that the family $\{F_{n_1} \circ \varphi, F_{n_2} \circ \varphi, \dots\}$is equicontinuous on X,*
3. *there is a sequence $n_i \to \infty$, such that the family $\{F_{n_i}\}$ is equicontinuous on $range(\varphi)$.*

Proof. (1) $\Rightarrow$ (2). Suppose $\psi \in \omega_G(\varphi)$, there is sequence $n_i \to \infty$, such that $d_U(F_{n_i}(\varphi), \psi) \to 0$. Since $\psi : X \to X$ is uniform continuous, for $\epsilon > 0$, there is $\delta > 0$ such that if $d(x, y) < \delta$, then $d(\psi(x), \psi(y)) < \frac{\epsilon}{3}$, also for $\epsilon > 0$, there is $N \in \mathbb{N}$ such that if $n_i > N$, then $d(F_{n_i}(\varphi(x)), \psi(x)) < \frac{\epsilon}{3}$ and $d(F_{n_i}(\varphi(y)), \psi(y)) < \frac{\epsilon}{3}$, this means that if $d(x, y) < \delta$, then $d(F_{n_i}(\varphi(x)), F_{n_i}(\varphi(y))) < \epsilon$, for every $n_i > N$. Thus there is a sequence $n_i \to \infty$, such that the family $\{F_{n_1} \circ \varphi, F_{n_2} \circ \varphi, \dots\}$is equicontinuous on X.

(2) $\Rightarrow$ (3) Similarly as in the proof of 2 $\Rightarrow$ 3 of the theorem3.4.
(3) $\Rightarrow$ (2) is clear
(2) $\Rightarrow$ (1) Since X is compact, for every $f, g \in \{F_{n_1} \circ \varphi, F_{n_2} \circ \varphi, \dots\}$, $d_U(f, g) \leq diam(X)$, thus by Arzela-Askoli theorem equicontinuity of the family $\{F_{n_1} \circ \varphi, F_{n_2} \circ \varphi, \dots\}$ implies that closure of $\{F_{n_1} \circ \varphi, F_{n_2} \circ \varphi, \dots\}$ is compact, this means that $\omega_G(\varphi) \neq \emptyset$. □

We know that in any non-autonomous discrete system $(X, F = \{f_n\}_{n=0}^{\infty})$, $\forall x \in X$, $\omega_F(x) \neq \emptyset$, but in the following example, we give a non-autonomous discrete system , $(X, F = \{f_n\}_{n=0}^{\infty})$, such that $\forall \varphi \in S(X)$, $\omega_G(\varphi) = \emptyset$.

Example 4.2. Let $\{c_n\}_{n=0}^{\infty} = \mathbb{Q} \cap [0, 1]$ and $f_n : [0, 1] \to [0, 1]$ is defined by $f_n(x) = c_n x$. It is easy to see that for $x \in [0, 1]$, there is not a sequence $n_i \to \infty$ such that the family $\{F_{n_i}\}$ is equicontinuous at x, thus by Theorem 4.1, for every $\varphi \in S(X)$, $\omega_G(\varphi) = \emptyset$.

It is known that every autonomous discrete dynamical system (X, f) on compact metric space has a recurrent point, in the following example we show that there is a non-autonomous discrete system with empty recurrent set.

Example 4.3. For every $n \in \mathbb{N}$, $f_n : S^1 \to S^1$ is defined by $f_n(x) = x + \frac{\pi}{2^n}$, since $F_k(x) = x + \Sigma_{n=1}^{k} \frac{\pi}{2^n}$ therefor $\frac{\pi}{2} \leq |F_k(x) - x|$, implies that for all $k \in \mathbb{N}$, $F_k(x) \notin B_{\frac{1}{2}}(x)$, i.e. $\mathcal{R}(F) = \emptyset$.

If $F = \{f_n\}_{n=0}^{\infty}$ is a k- periodic sequence i.e. for all $n \in \mathbb{N}$, $f_{n+k} = f_n$, then it is easy to see that $\mathcal{R}(f_n \circ f_{n-1} \circ \dots \circ f_1) \subseteq \mathcal{R}(F)$, this implies that:

Corollary 4.4. *If (X, d) is a compact metric space and $F = \{f_n\}_{n=0}^{\infty}$ be k- periodic sequence i.e. for all $n \in \mathbb{N}$, $f_{n+k} = f_n$. then non-autonomous discrete system $(X, F = \{f_n\}_{n=0}^{\infty})$ has recurrent point i.e. $\mathcal{R}(F) \neq \emptyset$.*

Example 4.2 shows there is non-autonomous discrete system $(X, F = \{f_n\}_{n=1}^{\infty})$ on compact metric space (X, d) such that $\mathcal{R}(F) \neq \emptyset$ but recurrent set of its functional envelope is empty set i.e. $\mathcal{R}(G) = \emptyset$.

Theorem 4.5. *Let $F = \{f_n\}_{n=0}^{\infty}$ be k- periodic sequence such that $f_i \circ f_j = f_j \circ f_i$. Also suppose that $f_i \circ \varphi = \varphi \circ f_i$, for some $\varphi \in S(X)$. If $\theta \in \omega_G(\varphi)$, then θ is recurrent and $range(\theta) = range(\varphi)$.*

Proof. Since $\theta \in \omega_G(\varphi)$, there are $i \in \{0, 1, \dots, k\}$ and $n_l \to \infty$, such that $d_U(F_i \circ F_k^{n_l} \circ \varphi, \theta) \to 0$, It is easy to see that $F_k^l \circ F_i \circ F_k^m = F_i \circ F_k^{m+l}$, thus

$$(d_U(F_i \circ F_k^{n_l} \circ \varphi, \theta) \to 0) \Rightarrow F_k^l \circ \theta = \theta \circ F_k^l.$$

Also for $\epsilon > 0$, there are $m, n \in \mathbb{N}$ such that $d(F_i \circ F_k^{m-n} \circ \varphi, \theta) < \frac{\epsilon}{2}$ and $d_U(F_i \circ F_k^m \varphi, \theta) < \frac{\epsilon}{2}$, since

$$F_k^n \circ \theta = \theta \circ F_k^n \;\; , \; F_k^n \circ \varphi = \varphi \circ F_k^n$$

and

$$d_U(F_k^n(F_i \circ F_k^{m-n}(\varphi)), F_k^n(\theta)) < d_U(F_i \circ F_k^{m-n}(\varphi), \theta) < \frac{\epsilon}{2},$$

we have

$$d_U(F_i \circ F_k^m(\varphi), F_k^n(\theta)) < \frac{\epsilon}{2}, \quad d_u(F_i \circ F_k^m(\varphi), \theta) < \frac{\epsilon}{2}.$$

This means that $d_U(F_k^n(\theta), \theta) < \epsilon$. Thus θ is recurrent.

Let $\theta \in \omega_G(\varphi)$ and $z \in range(\theta)$, there are $x \in X$ with $\theta(x) = z$, There is $n_l \to \infty$ such that $d(F_{n_l} \circ \varphi(x), z) \to 0$. Since X is compact, $\{F_{n_l}(x)\}_{l=0}^{\infty}$ has a limit point y. $F_{n_l} \circ \varphi = \varphi \circ F_{n_l}$ implies that $\varphi(y) = z$. This means that $range(\theta) \subseteq range(\varphi)$. For proof of $range(\varphi) \subseteq range(\theta)$, suppose $y \in \varphi(X)$, since F_i is surjective, there is $x_l \in X$ with $F_i \circ F_k^{n_l}(x_l) = y$, without loss of generally we may $x_l \to x$, then $\varphi(F_i \circ F_k^{n_l}(x_l)) \to \varphi(y)$, thus $\varphi(y) = \theta(x)$. Hence $range(\varphi) \subseteq range(\theta)$

□

Definition 4.6. ([1]) Let $(S(X), G = \{G_n\})$ be functional envelop of non-autonomous discrete system $(X, F = \{f_n\}_{n=0}^{\infty})$. P is said to be a range down property if whenever φ satisfies P and range$\theta \subseteq range\ \varphi$, then θ satisfies P.

In the following theorem, we show that compactness of an orbit closure, having a non-empty omega- limit set and recurrent are range property:

Theorem 4.7. *Let $(S(X), \{G_n\})$ be functional envelop of $(X, F = \{f_n\}_{n=0}^{\infty})$. Then every one of the following property, are range down property:*

1. *the compactness of an orbit closure ,*
2. *having a nonempty ω- limit set,*
3. *recurrence.*

Pro1of. If $range(\theta) \subseteq range(\varphi)$, and $\overline{\mathcal{O}_G(\varphi)}$ is compact then by Theorem 3.3 non-autonomous discrete system $F = \{f_n\}_{n=0}^{\infty}$ is equicontinuous on $\overline{range(\varphi)}$, thus non-autonomous discrete system $F = \{f_n\}_{n=0}^{\infty}$ is equicontinuous on $range(\theta)$, it implies that $\overline{\mathcal{O}_G(\theta)}$ is compact.

2. Let $range(\theta) \subseteq range(\varphi)$, and $\omega_G(\varphi) \neq \emptyset$. By Theorem 4.1, there is a sequence $n_i \to \infty$, such that the family $\{F_{n_i}\}$ is equicontinuous on $range(\varphi)$, since $range(\theta) \subseteq range(\varphi)$, $\{F_{n_i}\}$ is equicontinuous on $range(\theta)$, therefor $\omega_G(\theta) \neq \emptyset$.
3. let $\varphi \in \omega_G(\varphi)$ and $range(\theta) \subseteq range(\varphi)$, there is a sequence $n_k \to \infty$, with $F_n(\varphi) \to \varphi$. For every $x \in X$, there is $y \in X$ with $\theta(x) = \varphi(y)$ thus for every $n_k \in \mathbb{N}$,

$$d(\theta(x), F_{n_k}(\theta(x))) = d(\varphi(y), F_{n_k}(\varphi(y))) < d_U(\varphi, F_{n_k}(\varphi))$$

It implies that $d_U(\theta, F_{n_k}(\theta) \to 0$ i.e. $\theta \in \omega_G(\theta)$.

□

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