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Oscillation of Solutions of Nonlinear Difference Equation With a Super-linear Neutral Term

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Abstract: This paper deals with the oscillation of solutions of certain class of neutral difference equation

$$\Delta(a_n\Delta(x_n + p_n x^{\alpha}_{n-k})) + q_n x^{\beta}_{n+1-l} = 0,$$

where α and β are ratio of odd positive integers. New sufficient conditions are obtained for the oscillation of studied equation and examples illustrating the main results are provided.

Keywords: Oscillation, difference equation, nonlinear, super-linear neutral term

MSC: 39A11

1 Introduction

In this paper, we are concerned with the nonlinear neutral difference equation of the form

$$\Delta(a_n\Delta(x_n + p_n x^{\alpha}_{n-k})) + q_n x^{\beta}_{n+1-l} = 0, \ \ n \geq n_0, \tag{1.1}$$

where $n_0 \in \mathbb{N} = \{0, 1, 2, \dots\}$, subject to the following conditions:

(C_1) $\{a_n\}$ is a positive real sequence for all $n \geq n_0$;
(C_2) $\{p_n\}$ and $\{q_n\}$ are nonnegative real sequences for all $n \geq n_0$;
(C_3) $0 \leq p_n < 1$ for all $n \geq n_0$;
(C_4) k is a positive integer, and l is a nonnegative integer;
(C_5) $\alpha \geq 1$, and β are ratio of odd positive integers.

Let $\theta = \max\{k, l\}$. By a solution of equation (1.1), we mean a real sequence $\{x_n\}$ defined for all $n \geq n_0 - \theta$, and satisfies equation (1.1) for all $n \geq n_0$. A nontrivial solution of equation (1.1) is said to be oscillatory if the terms of the sequence are neither eventually positive nor eventually negative, and nonoscillatory otherwise.

In recent years there is a great interest in studying the oscillatory and asymptotic behavior of solutions of various classes of difference equations, see, for example [1–4, 6–10] and the references cited therein. In particular in [7], the authors considered the equation of the form (1.1) and obtained criteria for the oscillation of equation (1.1) under the conditions either

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n} = \infty \tag{1.2}$$

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or

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n} < \infty, \tag{1.3}$$

for the case $0 < \alpha \le 1$. In [9], the authors investigated the oscillatory behavior of equation (1.1) under the assumption (1.2), and $\alpha > 1$. In order to solve the problem completely we examine the other case (1.3), which appears to be more difficult than the former. To accomplish this is the main purpose of the present paper. Thus the results obtained in this paper are new, and complement to that of in [3, 6–10].

2 Oscillation Results

In this section, we obtain sufficient conditions for the oscillation of all solutions of equation (1.1) under the assumption (1.3). Note that from the hypotheses, it is enough to state and prove the results for the case $\{x_n\}$ is eventually positive since the proof for the opposite case is similar.

In the following, for convenience we denote

$$\begin{aligned} z_n &= x_n + p_n x_{n-k}^{\alpha}, \\ A_n &= \sum_{s=n}^{\infty} \frac{1}{a_s}, \text{ and } R_n = \sum_{s=N}^{n-1} \frac{1}{a_s} \end{aligned}$$

where $n \ge N \ge n_0$.

Lemma 2.1. *Let* $\{x_n\}$ *be an eventually positive solution of equation* (1.1). *Then the corresponding function* $\{z_n\}$ *satisfies one of the following two cases for all sufficiently large* n*:*

1. $z_n > 0,\ \Delta z_n > 0,\ \Delta(a_n \Delta z_n) \le 0;$
2. $z_n > 0,\ \Delta z_n < 0,\ \Delta(a_n \Delta z_n) \le 0.$

Proof. The proof is similar to that of Lemma 2.1 of [7], and hence the details are omitted. □

Lemma 2.2. *Let* $\{x_n\}$ *be an eventually positive solution of equation* (1.1) *and the corresponding function* $\{z_n\}$ *satisfies Case(1) of Lemma 2.1. Then there exists an integer* $N (N \ge n_0)$ *such that for every constant* $M > 0$

$$x_n \ge (1 - M^{\alpha-1} p_n R_{n-k}^{\alpha-1}) z_n \tag{2.1}$$

for all $n \ge N$.

Proof. From Case (1) of Lemma 2.1, we have $a_n \Delta z_n$ is decreasing and

$$z_n = z_N + \sum_{s=N}^{n-1} \frac{a_s \Delta z_s}{a_s} \ge R_n a_n \Delta z_n, \tag{2.2}$$

which implies

$$\Delta\left(\frac{z_n}{R_n}\right) \le 0 \text{ for all } n \ge N. \tag{2.3}$$

On the other hand, from the definition z_n, we have $z_n \ge x_n$, and

$$\begin{aligned} x_n &\ge z_n - p_n z_{n-k}^{\alpha} \ge (1 - p_n z_{n-k}^{\alpha-1}) z_n \\ &\ge (1 - M^{\alpha-1} p_n R_{n-k}^{\alpha-1}) z_n \end{aligned}$$

where we have used $\frac{z_n}{R_n}$ is decreasing and $\frac{z_n}{R_n} \le M$ for $M > 0$. This completes the proof. □

Lemma 2.3. *Let $\{x_n\}$ be an eventually positive solution of equation (1.1) and the corresponding function $\{z_n\}$ satisfies Case(2) of Lemma 2.1. Then there exists an integer $N(N \geq n_0)$ such that for every constant $K > 0$*

$$x_n \geq \left(1 - \frac{K^{\alpha-1} p_n A_{n-k}^{\alpha}}{A_n^{\alpha}}\right) z_n \tag{2.4}$$

for all $n \geq N$.

Proof. Since $a_n \Delta z_n$ is decreasing, we have

$$\Delta z_s \leq \frac{a_n \Delta z_n}{a_s} \quad \text{for } s \geq n.$$

Summing the last inequality, we obtain

$$0 \leq z_n + A_n a_n \Delta z_n, \; n \geq N \tag{2.5}$$

which implies

$$\Delta\left(\frac{z_n}{A_n}\right) \geq 0, \; n \geq N. \tag{2.6}$$

On the other hand, from the definition z_n, we have $z_n \geq x_n$, and

$$x_n \geq z_n - p_n \frac{z_{n-k}^{\alpha}}{A_{n-k}^{\alpha}} A_{n-k}^{\alpha} \geq z_n - p_n \frac{z_n^{\alpha}}{A_n^{\alpha}} A_{n-k}^{\alpha} \geq \left(1 - p_n \frac{K^{\alpha-1}}{A_n^{\alpha}} A_{n-k}^{\alpha}\right) z_n$$

where we have used (2.6) and $z_n \leq K$ for all $n \geq N$. This completes the proof □

Theorem 2.4. *Let $\beta > 1$ and condition (1.3) be hold. If there exists a positive, nondecreasing real sequence $\{\rho_n\}$ such that*

$$\lim_{n\to\infty} \sup \sum_{s=N}^{n-1} \left[\rho_s q_s \left(1 - M^{\alpha-1} p_{s+1-l} R_{s+1-k-l}\right)^{\beta} - \frac{a_{s-l}(\Delta\rho_s)^2}{4\beta K_1^{\alpha-1} \rho_s}\right] = \infty, \tag{2.7}$$

and

$$\lim_{n\to\infty} \sup \sum_{s=N}^{n-1} \left[A_{s+1}^{\beta} q_s \left(1 - \frac{K^{\alpha-1} p_{s+1-l} A_{s+1-k-l}^{\alpha}}{A_{s+1-l}^{\alpha}}\right)^{\beta} - \frac{\beta A_s^{\beta-1}}{4 M_1^{\beta-1} a_s A_{s+1}^{\beta}}\right] = \infty \tag{2.8}$$

hold for all constants $M > 0$, $M_1 > 0$, $K > 0$ and $K_1 > 0$, then every solution of equation (1.1) is oscillatory.

Proof. Assume the contrary that equation (1.1) has an eventually positive solution $\{x_n\}$, that is, there is an integer $n_1 \geq n_0$ such that $x_n > 0$, $x_{n-k} > 0$, and $x_{n-l} > 0$ for all $n \geq n_1$. From the definition of z_n, we have $z_n > 0$ for all $n \geq N \geq n_1$, where N is chosen so that two cases of Lemma 2.1 hold for all $n \geq N$. We shall prove that in each case we are led to a contradiction.

Case(1). From equation (1.1) and (2.1), we have

$$\Delta(a_n \Delta z_n) + q_n (1 - M^{\alpha-1} p_{n+1-l} R_{n+1-l-k}^{\alpha-1})^{\beta} z_{n+1-l}^{\beta} \leq 0, \; n \geq N. \tag{2.9}$$

Define

$$w_n = \rho_n \frac{a_n \Delta z_n}{z_{n-l}^{\beta}}, \; n \geq N.$$

Then $w_n > 0$ for $n \geq N$, and

$$\begin{aligned} \Delta w_n &= \rho_n \frac{\Delta(a_n \Delta z_n)}{z_{n+1-l}^{\beta}} + \Delta\rho_n \frac{a_{n+1} \Delta z_{n+1}}{z_{n+1-l}^{\beta}} - \rho_n \frac{a_n \Delta z_n}{z_{n+1-l}^{\beta} z_{n-l}^{\beta}} \Delta z_{n-l}^{\beta} \\ &\leq -\rho_n q_n (1 - M^{\alpha-1} p_{n+1-l} R_{n+1-l-k}^{\alpha-1})^{\beta} + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \frac{\rho_n}{\rho_{n+1}} w_{n+1} \frac{\Delta z_{n-l}^{\beta}}{z_{n-l}^{\beta}}. \end{aligned} \tag{2.10}$$

By Mean value theorem, we have

$$\Delta z_{n-l}^{\beta} \geq \beta z_{n-l}^{\beta-1} \Delta z_{n-l},$$

and using this in (2.10) we obtain

$$\Delta w_n \leq -\rho_n q_n (1 - M^{\alpha-1} p_{n+1-l} R_{n+1-l-k}^{\alpha-1})^{\beta} + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \beta \frac{\rho_n}{\rho_{n+1}^2} \frac{w_{n+1}^2}{a_{n-l}} K_1^{\alpha-1} \tag{2.11}$$

where we have used $a_n \Delta z_n \leq a_{n-l} \Delta z_{n-l}$ and $z_n \geq K_1 > 0$ for all $n \geq N$. Now, using the method of completing the square in (2.11), we obtain

$$\Delta w_n \leq -\rho_n q_n (1 - M^{\alpha-1} p_{n+1-l} R_{n+1-l-k}^{\alpha-1})^{\beta} + \frac{a_{n-l}(\Delta \rho_n)^2}{4\beta K_1^{\alpha-1} \rho_n}, \ n \geq N.$$

Summing the last inequality from N to n, one obtains

$$\sum_{s=N}^{n} \left[\rho_s q_s (1 - M^{\alpha-1} p_{s+1-l} R_{s+1-l-k}^{\alpha-1})^{\beta} - \frac{a_{s-l}(\Delta \rho_s)^2}{4\beta K_1^{\alpha-1} \rho_s} \right] \leq w_N$$

which contradicts (2.7) as $n \to \infty$.
Case(2). From equation (1.1) and (2.4), we have

$$\Delta(a_n \Delta z_n) + q_n \left(1 - \frac{K^{\alpha-1} p_{n+1-l} A_{n+1-l-k}^{\alpha}}{A_{n+1-l}^{\alpha}} \right)^{\beta} z_{n+1}^{\beta} \leq 0, \ n \geq N \tag{2.12}$$

where we have used z_n is decreasing. Define

$$u_n = \frac{a_n \Delta z_n}{z_n}, \ n \geq N. \tag{2.13}$$

Then $u_n \leq 0$ for $n \geq N$. From (2.5) we have

$$\frac{a_n \Delta z_n}{z_n} A_n \geq -1, \ n \geq N. \tag{2.14}$$

Thus

$$\frac{-a_n \Delta z_n (-a_n \Delta z_n)^{\beta-1} A_n^{\beta}}{z_n^{\beta}} \leq 1$$

for $n \geq N$. So, by $(-a_n \Delta z_n) > 0$ and (2.13), we have

$$-\frac{1}{L^{\beta-1}} \leq u_n A_n^{\beta} \leq 0 \tag{2.15}$$

where $L = -a_N \Delta z_N$. From (2.13), we obtain

$$\Delta u_n = \frac{\Delta(a_n \Delta z_n)}{z_{n+1}^{\beta}} - \frac{a_n \Delta z_n}{z_n^{\beta} z_{n+1}^{\beta}} \Delta z_n^{\beta}, \ n \geq N. \tag{2.16}$$

By Mean value theorem, we have

$$\Delta z_n^{\beta} \leq \beta z_{n+1}^{\beta-1} \Delta z_n,$$

and using this and (2.12) in (2.16) we obtain

$$\Delta u_n \leq -q_n \left(1 - \frac{K^{\alpha-1} p_{n+1-l} A_{n+1-l-k}^{\alpha}}{A_{n+1-l}^{\alpha}} \right)^{\beta} - \beta \frac{u_n^2}{a_n} z_n^{\beta-1}, \ n \geq N. \tag{2.17}$$

Since $\frac{z_n}{A_n}$ is increasing there is a constant $M_1 > 0$ such that $\frac{z_n}{A_n} \geq M_1$ for $n \geq N$, and hence (2.17) yields

$$\Delta u_n \leq -q_n \left(1 - \frac{K^{\alpha-1} p_{n+1-l} A_{n+1-l-k}^{\alpha}}{A_{n+1-l}^{\alpha}} \right)^{\beta} - \beta \frac{M_1^{\beta-1} A_n^{\beta-1}}{a_n} u_n^2, \ n \geq N. \tag{2.18}$$

Multiplying (2.18) by A_{n+1}^{β} and then summing the resulting inequality from N to $n-1$, one obtains

$$\sum_{s=N}^{n-1} A_{s+1}^{\beta}\Delta u_s + \sum_{s=N}^{n-1} A_{s+1}^{\beta} q_s \left(1-\frac{K^{\alpha-1}p_{s+1-l}A_{s+1-l-k}^{\alpha}}{A_{s+1-l}^{\alpha}}\right)^{\beta} + \sum_{s=N}^{n-1}\frac{\beta M_1^{\beta-1}A_{s+1}^{\beta}A_s^{\beta-1}}{a_s}u_s^2 \le 0$$

or

$$A_n^{\beta}u_n - A_N^{\beta}u_N + \sum_{s=N}^{n-1}\frac{\beta A_s^{\beta-1}}{a_s}u_s + \sum_{s=N}^{n-1}\frac{\beta M_1^{\beta-1}A_{s+1}^{\beta}A_s^{\beta-1}}{a_s}u_s^2 + \sum_{s=N}^{n-1} A_{s+1}^{\beta} q_s \left(1-\frac{K^{\alpha-1}p_{s+1-l}A_{s+1-l-k}^{\alpha}}{A_{s+1-l}^{\alpha}}\right)^{\beta} \le 0$$

which yields

$$\sum_{s=N}^{n-1}\left[A_{s+1}^{\beta}q_s\left(1-\frac{K^{\alpha-1}p_{s+1-l}A_{s+1-k-l}^{\alpha}}{A_{s+1-l}^{\alpha}}\right)^{\beta} - \frac{\beta A_s^{\beta-1}}{4M_1^{\beta-1}a_sA_{s+1}^{\beta}}\right] \le \frac{1}{L^{\beta-1}} + A_N^{\beta}u_N$$

when using (2.15). This contradicts (2.8) as $n\to\infty$, and the proof is now completed. □

Theorem 2.5. *Let* $0<\beta<1$, *and condition* (1.3) *be hold. If*

$$\sum_{n=N}^{\infty} q_n\left(1-M^{\alpha-1}p_{n+1-l}R_{n+1-k-l}^{\alpha-1}\right)^{\beta}R_{n+1-l}^{\beta}=\infty, \tag{2.19}$$

and

$$\limsup_{n\to\infty}\sum_{s=N}^{n-1}\left[K^{\beta-1}A_{s+1}q_s\left(1-\frac{K^{\alpha-1}p_{s+1-l}A_{s+1-k-l}^{\alpha}}{A_{s+1-l}^{\alpha}}\right)^{\beta}-\frac{1}{4a_sA_{s+1}}\right]=\infty \tag{2.20}$$

hold for all constants $M>0$, *and* $K>0$, *then every solution of equation* (1.1) *is oscillatory.*

Proof. Proceeding as in the proof of Theorem 2.4, we see that two cases of Lemma 2.1 hold for all $n\ge N$.
Case(1). From equation (1.1) and (2.1), we have (2.9). Further using (2.2) in (2.9), we obtain

$$\Delta(a_n\Delta z_n)+q_n(1-M^{\alpha-1}p_{n+1-l}R_{n+1-l-k}^{\alpha-1})^{\beta}R_{n+1-l}^{\beta}(a_{n+1-l}\Delta z_{n+1-l})^{\beta}\le 0 \tag{2.21}$$

for $n\ge N$. Let $w_n=a_n\Delta z_n>0$, and from the inequality (2.21) we see that $\{w_n\}$ is a positive solution of the inequality

$$\Delta w_n+q_n(1-M^{\alpha-1}p_{n+1-l}R_{n+1-l-k}^{\alpha-1})^{\beta}R_{n+1-l}^{\beta}w_{n+1-l}^{\beta}\le 0. \tag{2.22}$$

Since $0<\beta<1$, we see that by condition (2.19) and a Theorem 1 of [5], the inequality (2.22) has no positive solution, a contradiction.
Case(2). Proceeding as in the proof of Case(2) of Theorem 2.4, we obtain (2.12). Define

$$u_n=\frac{a_n\Delta z_n}{z_n},\ \ n\ge N. \tag{2.23}$$

Then $u_n<0$ for all $n\ge N$, and from (2.23), we obtain

$$\begin{aligned}\Delta u_n &= \frac{\Delta(a_n\Delta z_n)}{z_{n+1}}-\frac{a_n\Delta z_n}{z_nz_{n+1}}\Delta z_n\\ &\le -q_n\left(1-\frac{K^{\alpha-1}p_{n+1-l}A_{n+1-k-l}^{\alpha}}{A_{n+1-l}^{\alpha}}\right)^{\beta}\frac{z_{n+1}^{\beta}}{z_{n+1}}-\frac{u_n^2}{a_n}.\end{aligned} \tag{2.24}$$

Since z_n is positive and decreasing there is a constant $K > 0$ such that $z_n \le K$ for all $n \ge N$. Using the last inequality in (2.24), we have

$$\Delta u_n \le -\frac{q_n}{K^{1-\beta}}\left(1-\frac{K^{\alpha-1}p_{n+1-l}A^{\alpha}_{n+1-l-k}}{A^{\alpha}_{n+1-l}}\right)^{\beta}-\frac{u_n^2}{a_n},\ n\ge N.$$

Multiplying the last inequality by A_{n+1} and then summing the resulting inequality from N to $n-1$, we obtain

$$\begin{aligned}A_nu_n - A_Nu_N &+ \sum_{s=N}^{n-1}\frac{u_s}{a_s}+\sum_{s=N}^{n-1}\frac{u_s^2}{a_s}A_{s+1}\\ &+ \sum_{s=N}^{n-1}\frac{A_{s+1}q_s}{K^{1-\beta}}\left(1-\frac{K^{\alpha-1}p_{s+1-l}A^{\alpha}_{s+1-l-k}}{A^{\alpha}_{s+1-l}}\right)^{\beta}\le 0\end{aligned}$$

which yields

$$\sum_{s=N}^{n-1}\left[K^{\beta-1}A_{s+1}q_s\left(1-\frac{K^{\alpha-1}p_{s+1-l}A^{\alpha}_{s+1-k-l}}{A^{\alpha}_{s+1-l}}\right)^{\beta}-\frac{1}{4a_sA_{s+1}}\right]\le 1+A_Nu_N$$

when using (2.14). This contradicts (2.20) as $n\to\infty$, and the proof is now completed. □

3 Examples

In this section, we present two examples to illustrate the main results.

Example 3.1. *Consider a second order neutral difference equation*

$$\Delta\left(n(n+1)\Delta\left(x_n+\frac{1}{3n^2}x^3_{n-2}\right)\right)+n^3x^3_{n-1}=0,\ n\ge 1. \tag{3.1}$$

Here $a_n = n(n+1)$, $p_n = \frac{1}{3n^2}$, $q_n = n^3$, $k = 2$, $l = 2$, $\alpha = \beta = 3$. A simple calculation shows that $A_n = \frac{1}{n}$, and $R_n = \frac{n-1}{n}$. By taking $\rho_n \equiv 1$, one can see that the conditions (2.7) and (2.8) are satisfied, and hence by Theorem 2.4, every solution of equation (3.1) is oscillatory.

Example 3.2. *Consider a second order neutral difference equation*

$$\Delta\left(2^n\Delta\left(x_n+\frac{1}{4^n}x^3_{n-1}\right)\right)+4^nx^{1/3}_{n-1}=0,\ n\ge 1. \tag{3.2}$$

Here $a_n = 2^n$, $p_n = \frac{1}{4^n}$, $q_n = 4^n$, $k = 1$, $l = 2$, $\alpha = 3$ and $\beta = 1/3$. Since $A_n = \frac{1}{2^{n-1}}$, $R_n = \frac{2^n-2}{2^n}$, and it is easy to verify that all conditions of Theorem 2.5 are satisfied, and hence every solution of equation (3.2) is oscillatory.

Remark 3.3. *The results obtained in this paper extend some of the results in [6, 7, 9, 10] for $\alpha > 1$. Also the results of [9] cannot be applied to equations* (3.1) *and* (3.2) *since $\sum_{n=1}^{\infty}\frac{1}{a_n} < \infty$. Hence our results generalize and extend some of the results reported in the literature.*

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