Intelligent Systems for Stabilizing Mode-Locked Lasers and Frequency Combs: Machine Learning and Equation-Free Control Paradigms for Self-Tuning Optics

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Abstract: We demonstrate that a software architecture using innovations in machine learning and adaptive control provides an ideal integration platform for self-tuning optics. For mode-locked lasers, commercially available optical telecom components can be integrated with servo-controllers to enact a training and execution software module capable of self-tuning the laser cavity even in the presence of mechanical and/or environmental perturbations, thus potentially stabilizing a frequency comb. The algorithm training stage uses an exhaustive search of parameter space to discover best regions of performance for one or more objective functions of interest. The execution stage first uses a sparse sensing procedure to recognize the parameter space before quickly moving to the near optimal solution and maintaining it using the extremum seeking control protocol. The method is robust and equation-free, thus requiring no detailed or quantitatively accurate model of the physics. It can also be executed on a broad range of problems provided only that suitable objective functions can be found and experimentally measured.

Keywords: mode-locked lasers, frequency combs, machine learning, extremum-seeking control

1 Introduction

Modern-day engineering systems are of ever-growing complexity, often combining a myriad of networked, interacting components to produce a near-optimal output. And although modeling and simulation of such systems are still of critical importance for characterizing the underlying dynamics and range of potential behaviors, system nonlinearities and sensitivity to a multi-dimensional parameter space often make it difficult to provide quantitatively accurate predictions of system performance. Such an assessment is true for mode-locked lasers [1–3], where for nearly three decades the dominant physical effects have been known, but for which a quantitatively predictive theory of performance has remained elusive. Such lack of quantitative agreement underscores the fundamental, highly sensitive nature of man stochastically varying fluctuations in physical parameters that interact with the networked, nonlinear system. In mode-locked fiber lasers, for instance, it is well known that the fiber birefringence varies randomly along the fiber itself. Environmental changes in temperature and physical manipulation of the fiber (bending), even to a small degree, can greatly change the birefringence and the resulting mode-locking performance. Thus commercial fiber lasers are glued in place and shielded from environmental fluctuations in an attempt to stabilize performance. An alternative solution to environmental shielding would be to apply a robust control algorithm on the laser in order to recognize parameter changes and adjust the parameter space adaptively to produce the near-optimal performance. In what follows, we highlight how emerging methods from machine learning and adaptive control can provide a new paradigm for equation-free architectures in self-tuned, intelligent systems.

The term equation-free should first be explained. The term does not necessarily imply that we do not know the governing physics, and their underlying mathematical structure, of the problem at hand. In fact, for mode-locked lasers, it has been known from the earliest days what critical physical effects interact to produce mode-locked pulses. Rather, one can view the term in at least two distinct ways. First, it can suggest that due to the nonlinearity...
and/or sensitivity of the system, there may be unmodeled, even perturbatively small, physical effects that no longer allows the system to be quantitatively accurate. Thus the governing equations under consideration are an accurate reflection of the physics, but they lack the capability for accurate prediction. A second interpretation is that there are critical interactions that happen in the complex, networked architecture that are currently beyond our ability to frame mathematically for analysis and simulation, thus underlining any ability to provide accurate modeling. By acknowledging these limitations in the theoretical infrastructure, the equation-free strategy developed here uses data collected directly from the system to build libraries of learned dynamical states that are consistent with the data. Thus no underlying physics are prescribed and stochastic fluctuations are handled in an adaptive, unbiased manner. For the purposes of control, only short time future state predictions are necessary, greatly confining the scope and requirements of the equation-free architecture. The recently developed extremum-seeking control (ESC) [4–6] protocol can provide an ideal management system for mode-locked lasers as it (i) takes advantage of the equation-free framework, (ii) is robust and adaptive, and (iii) can be partnered with machine learning for significant performance enhancement. More broadly, ESC provides and ideal framework for many complex, networked systems as constructing quantitatively accurate models for such systems is extremely difficult due to their potential sensitivity and dependence upon a multi-dimensional parameter space.

In this manuscript, we combine these emerging, data-driven methods towards building an algorithm capable of achieving self-tuning and near-optimal performance in a mode-locked laser. The algorithmic infrastructure has both a learning module and an execution module. The learning module performs a principled exploration of parameter space, mapping out regions of optimal performance and building a library of these parameter regimes. The execution module capitalizes on the learned behavior by first identifying its current parameter regime and quickly moving towards the most optimal mode-locking regime. The ESC then maintains the mode-locking in its near-optimal state. We demonstrate the success of the algorithm on a numerical model of fiber laser mode-locked by a set of waveplates and polarizers.

2 Mode-Locked Lasers: Theoretical Considerations

To demonstrate the use of machine learning and equation-free control, we will consider a specific mode-locked laser model. The model is ideal for demonstrating the key concepts advocated here, especially as modeling the birefringence has remained, even after several decades, a challenging proposition. The birefringence further highlights the sensitivity to a specific parameter that is stochastic in nature, ultimately making quantitative prediction through first-principles modeling untenable.

Although a specific model is proposed for study, the methods are independent of this model and can be applied generically to any mode-locking configuration, i.e. the method is equation-free and does not require knowledge of the specific physics of the laser under consideration. Indeed, one of the compelling advantages of the proposed strategy is that the learning module of the algorithm explores parameter space and determines the best parameter sets available for the objective functions constructed.

2.1 Governing Equations and Pulse Propagation

The intra-cavity dynamics of the mode-locked laser of interest must account for, among other things, the nonlinear polarization dynamics and energy equilibration responsible for initiating the mode-locking process. Figure 1 demonstrates a typical experimental cavity configuration for achieving stable and robust mode-locking [1, 3]. Although we consider the passive polarizer and waveplates as discrete elements in the laser cavity, the remaining physical terms are lumped together into an averaged propagation equation that includes the chromatic dispersion, Kerr nonlinearity, attenuation and bandwidth-limited, saturating gain [7, 8]:

\[
i \frac{\partial u}{\partial z} + \frac{D}{2} \frac{\partial^2 u}{\partial t^2} - Ku + (|u|^2 + A|v|^2)u + Bv^2 u^* = ig(z) \left( 1 + \Omega \frac{\partial^2}{\partial t^2} \right) u - i\Gamma u
\]

\[
i \frac{\partial v}{\partial z} + \frac{D}{2} \frac{\partial^2 v}{\partial t^2} + Kv + (A|u|^2 + |v|^2)v + Bu^2 v^* = ig(z) \left( 1 + \Omega \frac{\partial^2}{\partial t^2} \right) v - i\Gamma v
\]

The left-hand side of this equation is the coupled nonlinear Schrödinger equations (CNLS). This system models the averaged propagation of two orthogonally polarized elec-
300 femtoseconds is the full width at half-maximum of the dispersion in the laser cavity. Specifically, in the normalized normal, the fiber cavity. The parameter and $A$ represent the pulse at 10 cm and linear attenuation (cavity losses). Here, $s$ and $p$ denote the gain (pumping) strength and cavity saturation length defined by $T_0^2 / |\beta_2|$. The birefringence strength parameter $K$ determines the effective relative phase velocity difference between the $u$ and $v$ fields. The material properties of the optical fiber determine the values of nonlinear coupling parameters $A$ and $B$, which satisfy $A + B = 1$ by axisymmetry, specifically $A = 2/3$ and $B = 1/3$ [9, 10].

With establishment of the intra-cavity propagation dynamics, it only remains to apply the discrete effects of the waveplates and passive polarizer in the laser cavity to induce mode-locking. Jones matrices are used to model the effects of waveplates and polarizer [11]. When the principle axes of these devices are aligned with the fast axis of the fiber, the Jones matrices of the quarter-waveplate, half-waveplate and polarizer are given respectively by:

$$W_1 = \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}, \quad (4)$$

$$W_2 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad (5)$$

$$W_p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \quad (6)$$

For arbitrary orientation $\alpha_j$ (see Fig. 1) with respect to the fast axis of the fiber, the above matrices are modified by:

$$J_j = R(\alpha_j)WR(-\alpha_j) \quad (7)$$

![Figure 1: Experimental configuration of a ring laser cavity that includes quarter-waveplates (QWP), passive polarizer, half-waveplate (HWP), ytterbium-doped (or Erbium-doped) amplification and output coupler. The Yb-doped section of fiber is fused with standard single-mode fiber and treated in a distributed fashion. The angles of the waveplates and polarizer are denoted by $\alpha_j$ with $j = 1, 2, 3, p$. Each of these is on a servo-driven motor denoted by $S_j$. The right panel shows the physical configuration for tuning the laser. The fiber birefringence is modified through mechanical disturbances (bending) and/or environmental fluctuations in temperature, thus greatly compromising mode-locking performance and frequency comb stability.](image-url)
where $W$ is one of the matrices in Eq. (6) and

$$R(\alpha_j) = \begin{pmatrix} \cos(\alpha_j) & -\sin(\alpha_j) \\ \sin(\alpha_j) & \cos(\alpha_j) \end{pmatrix}. \quad (8)$$

This arbitrary rotation was considered in previous work [12, 13] under the restricted condition $\alpha_0 = 0$ and $\cos(\alpha_1 + \alpha_2) = 0$. General values of $\alpha_j$ have a significant impact on the coefficients of the Hauss master modelocking model, for instance [16].

To help make clear the model of the laser cavity dynamics subject to Eqs. (1)–(8), consider a single round trip passage through the cavity. The propagation of the field starts right after the polarizer with orientation $\alpha_j$ for which the pulse is linearly polarized. The quarter-waveplate (with angle $\alpha_3$) to the left of the polarizer converts the polarization state from linear to elliptical, thus creating a polarization ellipse. Upon passing through the laser cavity, the polarization ellipse is subjected to a intensity-dependent rotation as well as amplification as governed by (1). At the end of fiber, the half-waveplate (with angle $\alpha_2$) further rotates the polarization ellipse through a certain angle. The quarter-waveplate (with angle $\alpha_3$) converts the polarization state from elliptical back to linear, and the polarizer finally aligns the field with its own principal axis.

The CNLS (1), together with Jones matrices (7), gives a full description of pulse propagation in the laser system. The numerical procedure employed for solving the partial differential equation uses a fourth-order Runge-Kutta method in $z$ and spectral (Fourier) transforms in the time variable $t$. The principle of operation involves iterations of solving the CNLS over one round trip and applying the Jones matrices of the waveplates and polarizer sequentially. The discrete application of Jones matrices after each cavity round trip acts like a filter that can be tuned to control the mode-locking behavior. Depending on their orientations, the waveplates and the polarizer can either destabilize the field propagating in the cavity or provide an effective intensity discriminating mechanism to lock it into a robust pulse.

### 2.2 Tuning the Mode-Locking: Selecting an Objective Function

Given the governing equations, extensive numerical simulations can be performed in order to identify parameter regimes where mode-locking occurs. Each of these regimes can in turn be evaluated for their ability to produce high-energy, high-peak-power mode-locked states. In addition to being a costly exercise, such studies also rarely match the real cavity dynamics since, for instance, parameters like the fiber birefringence $K$ are unknown. This motivates our use of machine learning, optimization and adaptive control strategies for characterizing the laser cavity. Interestingly, the integration of all three methods can be achieved without a detailed theoretical knowledge of the cavity equations, i.e. they are equation-free methods in the sense that learning the laser characteristics and applying adaptive control only relies on experimental measurements of the underlying system.

For any such data-driven strategy to be effective, we require an objective function, with local maxima that correspond to our targeted mode-locking output. Often high-energy mode-locked solutions are sought. However, for frequency-comb applications, phase stabilization of pulses is the ultimate objective, not the power, short-pulse durations, or bandwidth-limited output. For the moment, we will seek high-energy solutions, there are many chaotic waveforms that have significantly higher energy than mode-locked solutions. Therefore, energy alone is not a good objective function. Instead, we divide the energy function $E$ by the fourth moment (kurtosis) $M_4$ of the Fourier spectrum of the waveform

$$O = \frac{E}{M_4}$$

which is large for undesirable chaotic solutions. This objective function, which has been shown to be successful for applying adaptive control, is large when we have a large amount of energy in a tightly confined temporal wave packet [15, 16].

Figure 2 shows the new objective function (solid black), energy (red dash), and the fourth moment of the spectrum (blue dots) for various slices of the rotation angles $\alpha_j$. In each panel, one of the angles is rotated from $-90^\circ$ to $90^\circ$ while the other angles are held fixed at values that locally maximize the objective function. These values are obtained by applying the extremum-seeking controller developed in the next sections. In each panel, the maximal energy occurs away from the regions of single-pulse mode locking, and tracking energy alone would lead to chaotic solutions. In contrast, the new objective function has local maxima in the single-pulse mode-locked regions, and there is a buffer region between the peak in objective function and the gray region where mode-locking fails. The fourth-moment of the spectrum (blue dots) is relatively small in the single-pulse mode-locking regions (white) and is much larger in the gray regions because of multi-pulse or chaotic solutions. Thus, dividing energy by the fourth moment of the spectrum penalizes non-mode-locking solutions.
2.3 Difficult to Model Physics

The component by component physics-based model advocated in this section provides a qualitatively accurate description of the mode-locking process. And despite extensive experimental insight, the first-principles approach has remained qualitative in nature for more than two decades. The underlying and primary reason is fiber birefringence [9, 10, 17, 18] (See also the recent review article by Gordon and Kogelnik [19]). Fiber birefringence has been intensely studied due to its impact on fiber optic communications. It is known that the fiber birefringence is stochastic in nature, varying randomly along the length of the fiber laser cavity and highly susceptible (and sensitive) to environmental factors such as bend, twist, anisotropic stress and ambient conditions such as temperature. Unlike optical communications, where ultra-long distances are propagated and statistical averages can be used to understand the quantitative impact of birefringence, a fiber cavity is only 1–10 meters in length and essentially represents a single, short realization of a stochastically varying parameter, thus statistical averages are ineffective for quantitative assessment.

As in optical communications, a ring fiber laser cavity design propagates pulses over ultra-long distances in fractions of a second. Signal distortions due to the chromatic dispersion and nonlinearity accumulate after many round trips of the laser cavity, as does the signal distortion due to the fiber birefringence [9, 10, 17–19]. Successful pulsed laser operation is achieved when the linear and nonlinear cavity effects from dissipation and dispersion balance each other resulting in stable mode-locked pulses [2, 3]. Although single-mode fibers are typically used for such laser cavities, the so-called single-mode fibers, in fact, support two modes simultaneously, which are orthogonally polarized. In an idealized circular-core fiber, these two modes will propagate with the same phase velocity. However, practical fibers are not perfectly circularly symmetric. As a result, the two modes propagate with slightly different phase and group velocities due to small differences in the effective index of refraction experienced by each. While this birefringence is small in absolute terms in standard optical fibers, approximately $10^{-7}$ index of refraction difference in the two modes, the corresponding beat length $L_B$ is about 10 meters with variations occurring on lengths of 100 meters, which is often on the same order as the dispersive and/or nonlinear length scales. As a result, the birefringence can have a significant impact on mode-locking dynamics. Such system sensitivity has prevented it from major performance advances, limiting power and pulsewidths. Moreover, failure to accurately model the stochastic and sensitive birefringence fluctuations in the cavity have deprived the community for more than two decades of a quantitatively accurate model of this highly successful laser system.

Our objective in this manuscript is to make use of modern data-analysis methods, i.e. machine learning techniques, to help discover a proxy measure for the effective cavity birefringence. Unlike optical communication lines where over long distances a statistical average might be experienced by a pulse, here a single realization of a stochastic variation of the birefringence is what drives the laser cavity dynamics (See Fig. 3). If the cavity is perturbed by bend, twist, anisotropic stress and/or ambient temperature, then a new realization results. For optimizing performance, it is critical to characterize, or recognize, the fiber birefringence correctly in order to determine the waveplate and polarizer settings, for instance, required to give...
the best energy performance. Ultimately the goal of the learning module in our algorithm is to not reconstruct, or model, the birefringence, but rather to characterize its impact on performance and discover the best regimes of operation given the specific birefringence.

3 Machine Learning and Pattern Recognition

Data-driven discovery methods are having a profound impact across the physical, engineering and biological sciences. The goal of these emerging methods is to mine big data content for low-dimensional patterns of activity and/or clustering of behaviors. Such pattern recognition algorithms, typically falling under the aegis of machine learning methods, are then capitalized on for control, optimization or characterization of the system under consideration. The success of these methods, such as self-driving automobiles, have brought society to the brink of a technology revolution. As we show, optical systems are far less complex than self-driving algorithms, thus suggesting that self-tuning methods should be easily implementable in the much more limited parameter space afforded by, for example, mode-locked fiber lasers.

Data mining of optical systems begins with measurements generated from a broad exploration of parameter space. For mode-locked lasers, we advocate a toroidal search of parameter space. The search algorithm can be mined for characterizing the optimal mode-locking performance as a function of birefringence. In practice, this algorithm could take a few to tens of minutes to execute given the fact that mode-locking itself occurs in microseconds. Thus the only limitation to how fast the algorithm can be executed is the speed of the servos [20, 21]. The actuation speed of the servos also determines how quickly the laser can be self-tuned once the search algorithm is concluded. In particular, it would take a matter of seconds to move the servos to the optimal mode-locking state. Thus the control occurs on a much slower scale than the cavity round trip time.

The first step in the toroidal search is to identify all the input parameters capable of actuating the system and changing performance. In the experimental setting of an NPR-base mode-locked fiber laser, the actuation units are easily identified, consisting of the polarizer and three waveplates that can be rotated from 0 to 2π, thereby creating a parameter space that is a 4-torus. One can also modify the cavity to include polarization ears that are capable of changing the cavity birefringence. But for the moment, we will consider the average birefringence to be fixed at some unknown value. Interestingly, the method advocated here may be the only reasonable way of training a multiple NPR laser cavity, let’s say N of them, to achieve optimal, high-power performance [22–24].

In order to sample or data-mine the resulting parameter space (4N-torus), a toroidal search algorithm is developed. If we want to sample this 4N-dimensional torus,
Figure 4: Visualization of the toroidal search algorithm with a waveplate and the polarizer. (a, c, e) 2-torus of $\alpha_3$ and $\alpha_p$ with sample points shown (dots) for different sample rates (1.25Hz-black, 5Hz-magenta, 20Hz-blue, the global optimum is marked in red). (b, d, f) The time-series of the corresponding objective function with the global optimum again marked in red. (h) Zoomed in objective function (red) plot near the global optimum, pulse energy (black) and kurtosis (blue) are also shown (all normalized to the same scale for comparison). (g) Experimental confirmation of the approach as given in Refs. [7, 8]. The red patches are experimentally determined mode-locked states while the blue regions are not mode-locked.

then $4N$ time series are constructed from:

$$\theta_j(t) = \omega_j t + \theta_{j0}$$

for $j \in [1, \ldots, 4N]$, where $\theta_{j0}$ are initial parameter values, $\omega_j$ are angular frequencies which are incommensurate so that $m\omega_j + n\omega_k = 0$ doesn’t have an integer solution. In other words, $\omega_j/\omega_k$ is irrational for any $j, k \in [1, \ldots, 4N]$, given $j \neq k$. It is easy to prove that under such conditions, $[\theta_1(t) \ldots \theta_{4N}(t)]$ is dense on the torus [25]. Thus using this method, it is guaranteed that one can sample any torus sufficiently well if sampling for a long enough time or using a high enough sample rate. In what follows, we consider a 2-torus for a single NPR laser as a simple example case. However, the methods mentioned in this paper can be applied to toroidal parameter spaces of any dimensionality. Note that any parameter input can be made periodic for the toroidal search by first progressing from its minimum value to its maximum value and then back to its minimum value, thus constituting a periodic modulation.

Figure 4 shows how the toroidal sampling works on a 2-torus comprised of parameters $\alpha_3$ and $\alpha_p$. Specifically, the resulting time series of the objective function $O$ is demonstrated as the 2-torus is sampled. In Fig. 4(a), the torus in under-sampled and aliasing of the objective function occurs. However, as the sampling rate is increased, as shown in Fig. 4(e), the objective function is fully constructed and an evaluation of best performance can be ascertained. Indeed, the red dot in Fig. 4(f) shows the optimal global maxima of the laser cavity. The narrow shaded region around this peak performance is highlighted in Fig. 4(h) where an additional evaluation is made of whether the solution is mode-locked or not.

The best solutions found are stored in a library of known behavior, thus preserving for future reference the exact waveplate and polarizer settings for a given birefringence. Interestingly, to date only one experimental result mapping out the toroidal search characteristics is available [7, 8]. These results are reproduced here in Fig. 4(g),
where the red regions are the mode-locked regimes and the blue regions are the non-mode-locked regimes as a function of the three waveplate settings. Such red regions would be stored in a library of learned, optimal behavior, i.e. the data mining discovered advantageous patterns of activity to be exploited.

4 Sparse Sampling for Classification of Dynamics

The sensitivity of mode-locking to the difficult-to-model birefringence has already been mentioned several times. The toroidal search can be performed for the laser cavity a number of times with the cavity itself subject to mechanical disturbances and/or environmental changes. The simplest way to execute such a sensitivity study is to place a polarization ear component in the cavity and modulate it in a principled way to sweep through a large number of birefringence values. From an execution viewpoint, the first step in applying self-tuning is to recognize the current birefringence.

In practice, for a given birefringence, toroidal sampling is used to produce a time series of the objective function as shown in the middle panels of Fig. 4. Once produced, the time series are used to compute spectrograms in our library building process, i.e. the characterization of birefringence is encoded in the spectrogram produced. Each spectrogram in the exploration process is used to construct optimal parameter settings (libraries) that are kept for future use. In order to develop a robust algorithm that matches the current objective function time series with the library entries, we want to utilize both the temporal and spectral (frequency) signatures of the time series. As a result, we introduce the Gábor transform and construct a spectrogram [26] of the optimal solution.

The time series collected from toroidal sampling are comprised of various frequency components that are exhibited at different times. Although the Fourier transform of the signal contains all frequency information, there is no indication of when each frequency component occurs in time. Indeed, by definition, the Fourier transform eliminates all time-domain information since we integrate over all time. Gábor proposed a formal method for keeping information in both time and frequency by making a simple modification to the definition of the Fourier transform kernel: $g_{t,\omega}(\tau) = e^{i\omega\tau}g(\tau-t)$, where the filter $g(\tau-t)$ was introduced with the aim of localizing both time and frequency. The Gábor transform, also known as the short-time Fourier transform, is then defined as:

$$\tilde{f}_g(t, \omega) = \int_{-\infty}^{\infty} f(\tau)g(\tau-t)e^{-i\omega\tau} \, d\tau,$$

where the bar denotes the complex conjugate of the function. Thus the function $g(\tau-t)$ acts as a time filter for localizing the signal and its frequency content over a specific window of time, allowing for the construction of a spectrogram. A spectrogram represents a time series (signal) in both the time and spectral domain. A key observation is that these spectrograms are unique for varying cavity birefringence [27]. Thus the spectrogram serves as a proxy measure for classifying the underlying cavity birefringence. By definition, the spectrograms are symmetric in frequency, for storage and computation efficiency concerns, we only use the positive frequency part of the spectrograms for classification purposes. These spectrograms serve as the basis of a pattern recognition/classification scheme for determining the value of cavity birefringence.

The library of spectrograms $S_k (k = 1, 2, \ldots, n)$, constructed from a large number ($n$) of possible birefringence values, is then dimensionally reduced through its dominant correlation structures via a singular value decomposition (SVD) [26] $S_k = U_k \Sigma_k V_k^*$ and $u_k = [u_{k1}, u_{k2}, \ldots, u_{kn}]$ where $u_{kj}$ are column vectors, or principal components associated with $S_k$. A rank $m$ approximation to the library is made by keeping the first $m$ ($m < n$) modes (low-rank approximation) of $U_k$. These are stored in the birefringence library $U_L$ such that

$$U_k = [\tilde{U}_1, \tilde{U}_2, \ldots, \tilde{U}_M]$$

where the $k$-th sub-library $\tilde{U}_k$ contains the first $m$ modes of $U_k$.

Once we have constructed our dimensionally reduced library modes, we can perform a short toroidal search of the laser system (the objective function) and compute its spectrogram. The sampling time does not have to be of the same length as the time series collected when the library was built. Indeed, we can sample much shorter time-series (short toroidal searches) and still achieve the desired results. Specifically, we perform an SVD reduction on the measured spectrogram and keep the first $m$ modes as before. With the most important (dominant) modes from the measurement in hand, we can do an $L_1$-norm library search, thus promoting sparsity in our solution [26]. In the $L_1$-norm search, our objective is to find a vector

$$a = \arg \min_a ||a||_1 \quad \text{subject to} \quad U_L a = u_{m_1}.$$

Here we require the number of library modes to be greater than the dimensionality of the frequency domain. Given
Figure 5: Example of the extremum-seeking control paradigm and associated control diagram. The input parameters $\alpha_j$ are modulated ($\alpha_1$ in (a)) by servo controllers on (b) all waveplates and the polarizer. The input waveplate angle modulation (in blue) generates a response (in red) of the objective function, giving a method for choosing an updated setting for the polarizer angle. This generates a gradient search towards the extremum as shown in (c). A schematic for single-input, single-output (SISO) ESC is given in (d).

this condition, this becomes an underdetermined linear system of equations. The $L_1$-norm minimization produces a sparse vector $a$ whose non-zero elements act as a classifier (indicator function) for identifying which sub-library the birefringence falls into. Thus if the largest element falls into the $i$-th sub-library, the recognized birefringence value is equal to $K_i$. This sparsity promoting optimization, when used in conjunction with the unique spectrograms, gives a rapid and accurate classification scheme for the fiber birefringence. Thus birefringence recognition can be easily accomplished.

This learning and execution algorithm can be used on the NPR mode-locking model advocated here. In testing, the birefringence $K$ is varied following a gaussian random walk. For each trial, the spectrogram corresponding to the current birefringence is computed and the $L_1$-norm sparse search is executed. The recognition algorithm is tested in two scenarios: (i) well-aligned data given the assumption that the servo motors that control the waveplates and polarizers work without error, and (ii) the mis-aligned data that considers the error in the initial angle of the servo motors. A birefringence recognition rate of 98% is achieved with aligned data while 88% recognition is achieved in the mis-aligned (time series) scenario. It should also be noted that even when our recognition algorithm fails to find the correct birefringence value, the error between the true birefringence and the recognized value is very small. Thus, even if we use the mis-classified birefringence, is is likely that the predicted optimal parameters will be near the true optimal parameters where the ESC controller can be used [15, 16].

5 Equation-Free Modeling and Control

Extremum-seeking control (ESC) is an adaptive control law that finds and tracks local maxima of an objective function by sinusoidally varying a set of input parame-
ters and measuring the consequent variation of the objective function \([4–6]\). The resulting controller does not rely on a model of the dynamics that relate the input parameters to the objective function, making it especially useful for complex, nonlinear systems with disturbances that are difficult to model. From this perspective, it can then be considered an equation-free control strategy. Application of the technique compares the varying input signal to the resulting variation in the objective function, thus producing a gradient direction, or estimate, capable of dynamically improving the estimate of the optimal input parameters. Designed correctly, extremum-seeking is guaranteed to stably converge to a neighborhood of the control input \(u^*\) that yields a local maximum of the objective function.

Figure 5 shows an extremum-seeking controller for the laser system with a single-input and a single-output (SISO). The input variable modulated is the angle \(\alpha_1\), and the output is the objective function \(O = E/M_4\) which is denoted by \(f(u)\). The algorithm works by adding a perturbation signal \(a \sin(\omega t + \beta)\) to the best guess of the input \(\hat{u}\) that maximizes the quantity of interest, namely, the objective function. The perturbation passes through the system and results in a perturbation in the output. The high-pass filter of this output is a signal \(\rho\) that oscillates about zero mean. Multiplying the high-pass filtered output by the input perturbation yields a demodulated signal \(\xi\) that is positive when \(\hat{u} < u^*\) and negative when \(\hat{u} > u^*\). Finally, integrating \(\xi\) into our estimate \(\hat{u}\) brings the estimate \(\hat{u}\) closer to the optimal value \(u^*\) corresponding to a local maximum.

Overall, this ESC provides a highly intuitive approach to control: modulate an input parameter and see if the objective function responds with an in-phase or out-of-phase response. This dictates how to update the input parameter setting to be closer to optimal. The ESC for SISO can

Figure 6: Flow chart showing the (a) training and (b) execution modules associated with the mode-locked laser cavity. The flow chart is annotated with key mathematical concepts and equations from the paper. Such a training and execution scheme can be applied a wide variety of photonic and optical systems.
be modified to handle multiple-input and a single-output (MISO). The algorithm is generalized by using a number of separate extremum-seeking loops, each with their own perturbation signal and magnitude, high-pass filter, and integrator. It is generally more involved to develop a well-tuned extremum-seeking controller for a MISO system, although guidelines for stable controllers do exist [6]. As illustrated in Fig. 5, the input space generating the output objective function is four dimensional.

6 Stabilizing and Optimizing Mode-Locking

The data methods outlined in previous sections can be combined to provide a robust self-tuning algorithm for mode-locked lasers. Specifically, the toroidal search is used to search the parameter space and discover candidate regimes for optimal mode-locking. The search can be performed and evaluated using multiple objective functions, thus as requirements for the cavity performance change, objective functions can be modified to meet the current needs. The best solutions are stored, or encoded, in a library structure using its spectrogram representation. The execution module takes advantage of the learned behavior by first doing a short toroidal search to determine the current birefringence. Once determined, the optimal waveplate and polarizer settings can be recalled from the library of optimal behaviors. The ESC is then used to make adaptive adjustments to the mode-locked laser in order to keep it optimally tuned.

Figure 6 gives a detailed breakdown of the self-tuning algorithm. The flow chart also references the key mathematical, or data analytic, concepts from the previous sections. The flow chart provides a fairly simple procedure for the self-tuning architecture. Moreover, the technology exists today to implement this in a realistic system, be it related to optics or not. It is anticipated that the training module could in practice be executed on the matter of hours or less. In contrast, the execution module is quite rapid. The short toroidal search takes on the order of a matter of minutes. Once the birefringence is identified, the ESC keeps the mode-locked state optimized in real-time. If a mechanical perturbation is applied, or a rapid environmental change occurs, then the short toroidal search is repeated.

A demonstration of the execution module is given in Fig. 7. In this numerical experiment, the average cavity birefringence is changed slowly (and randomly) over the course of 60 minutes. The execution module is easily able to adapt to these changes and drive the waveplates and polarizers to new, optimal settings in order to maintain peak performance. Without the controller being active, the mode-locked laser would behave in an erratic fashion, falling out of a mode-locked state repeatedly as the birefringence is fluctuating. This figure highlights the power and efficiency of the self-tuning algorithm.

7 Conclusions and Outlook

Data-driven strategies for analyzing complex systems are of growing importance across all fields in the physical, engineering and biological sciences. These methods are not just capable of giving us insight into patterns of activity, but rather they can be used to provide control protocols to systems with high-dimensional actuation (input) spaces. Hand tuning such systems can be untenable, unwieldy or extremely time consuming. With the advent of machine learning, such systems can be optimized in an automated fashion with methods exemplified in this text. Indeed, we have demonstrated a clear path forward towards self-tuning mode-locked lasers, and more broadly, optical systems. Such strategies are needed now in the nanophotonics arena where self-tuning is critical for integration of such components in larger networked component structures, i.e. the broader application depends upon reliable performance from its constitutive parts.

An attractive feature of the algorithm developed here is that it is equation-free, and thus does not rely on the user having detailed, and quantitatively accurate, information about the system under consideration. Rather, data is accumulated from the system, in this case optical spectrum and power measurements, to construct a suitable objective function. Interestingly, the toroidal search learning algorithm which explores parameter space can be evaluated using as many objective functions as desired. For mode-locked lasers and frequency comb applications, this may mean looking for the shortest pulses, most energetic pulses, or most stable phase-locking between round trips. If a suitable objective function can be constructed, then self-tuning can be achieved. All the hardware exists today to implement the strategy, thus our work highlights the need to partner the hardware with software and algorithms to greatly enhance the potential technological achievements.

More broadly, such data-driven strategies and learning algorithms are just beginning to see wide spread use throughout the sciences. Two recent papers of note use similar strategies for the control of turbulent flows [28] and
the beam steering in metamaterial antennas [29]. In both cases, the methods have not only been theoretically proposed, but reduction to practice has been demonstrated. These innovations and successful implementations help advocate for such techniques in the nanophotonics community.

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