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Stochastic resonance induced by a multiplicative periodic signal in a logistic growth model with correlated noises

Research Article

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Abstract: The stochastic resonance (SR) phenomenon induced by a multiplicative periodic signal in a logistic growth model with correlated noises is studied by using the theory of signal-to-noise ratio (SNR) in the adiabatic limit. The expressions of the SNR are obtained. The effects of multiplicative noise intensity α and additive noise intensity D, and correlated intensity λ on the SNR are discussed respectively. It is found that the existence of a maximum in the SNR is the identifying characteristic of the SR phenomena. In comparison with the SR induced by additive periodic signal, some new features are found: (1) When SNR as a function of λ for fixed ratio of α and D, the varying of α can induce a stochastic multi-resonance, and can induce a re-entrant transition of the peaks in SNR vs λ; (2) There exhibits a doubly critical phenomenon for SNR vs D and λ, *i.e.*, the increasing of D (or λ) can induce the critical phenomenon for SNR with respect to λ (or D); (3) The doubly stochastic resonance effect appears when α and D are simultaneously varying in SNR, *i.e.*, the increment of one noise intensity can help the SR on another noise intensity come forth.

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Keywords: tumor cell growth model • stochastic resonance • multiplicative periodic signal

1. Introduction

Stochastic resonance (SR) was extensively investigated both theoretically and experimentally due to its potential applications [1–24]. The concept of stochastic resonance was originally put forward in the seminal papers by Benzi and collaborators [1, 2] where they address the problem of the periodically recurrent ice ages. Nicolis and Nicolis independently raised this suggestion that stochastic resonance might rule the periodicity of the recurrent ice ages [3, 4]. The SR phenomenon was subsequently observed in a Schmitt trigger circuit [6] and in a He-Ne bidirectional ring laser [7]. The SR is a typical example for the construction role of noise. The adiabatic elimination theory [8], linear response theory [9, 10], and perturbation theory [11] were employed to characterize the SR phenomenon. The typical manifestation of SR is the existence of a maximum of the output signal and of the the signal-to-noise ratio as a function of noise intensity.

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The residence-time distribution offers another possibility to characterize SR [12–14]. Existing literature concerning SR included three indispensable basic ingredients: an energetic activation barrier or a form of threshold, a weak coherent input (such as a periodic signal) and a source of noise that is inherent in the system [15]. Currently, many new types of SR are known such as stochastic multi-resonance [17], stochastic giant-resonance [18], linear stochastic resonance [18–21], and doubly stochastic resonance [22], and so on. It is noteworthy that in most cases the signal is taken in an additive fashion, however, in some situations, the signal should be multiplicative in some systems, such as in a membrane-protein system investigated by Fuliński and Góra [21]. Collins *et al.* introduced a multiplicative aperiodic signal into a symmetric bistable potential with the presence of additive noise, and found aperiodic stochastic resonance [23]. Recently, Nicolis and Nicolis studied the stochastic resonance in the presence of slowly varying control parameters and a multiplicative periodic signal in a bistable system [24]. Their results may give rise to a new method for the control of the transition rates [25].

Logistic growth is one of the most popular equations not only in the mathematical ecology but also in other applications. Firstly it was introduced by Verhulst for saturated proliferation at a single-species [26], and has been extended to include spatial dynamics by Fisher [27] and by Kolmogoroff *et al.* [28]. It is now one of the classical examples of self-organization in many natural and artificial systems [29]. Effects of noise on periodic orbits of the logistic map also has been investigated [30]. This equation was proposed to describe the growth of the Ehrlich ascities tumour (EAT) in a mouse [31]. It appears that even such a simple ordinary differential equation can be used to model such a complicated process. So far, much attention has been paid to the statistical properties of the logistic growth model with correlated noises [32–37]. It was found that the correlations between noises affect the stationary and transient properties of the model fundamentally, including the stochastic resonance induced by an additive periodic signal [37]. Since the periodic signal has external origin here in most cases of the logistic growth model, the consideration of a multiplicative periodic signal is more realistic than an additive periodic signal. However, the stochastic resonance induced by a multiplicative signal has not been discussed.

In section 2, an approximate analytical expression of output signal-to-noise ratio (SNR) is derived for the case of a multiplicative periodic signal, the effects of noise intensity and correlated intensity on the SNR of the system are presented in detail. In section 3, a discussion of the results concludes the paper.

2. The signal-to-noise ratio of the logistic growth model with multiplicative signal and correlated noises

The equation of the logistic growth model reads

$$\frac{dx}{dt} = ax - bx^2. \tag{1}$$

Here x is the population density, so is confined to positive real numbers, a is the growth rate and b is the decay rate. If the environmental fluctuation due to some external factors (such as temperature, radiotherapy, geological events) is considered, the parameter a should be modified as $a + \xi(t) + A\cos(\omega t)$ [38], and some factors such as internal fluctuation, restrain the number of the population, giving rise to a negative additive noise [32]. As a result, the dimensionless form of the Langevin equation for this model reads

$$\frac{dx}{dt} = ax(t) - bx^2(t) + x(t)A\cos(\omega t) + x(t)\xi(t) - \eta(t), \tag{2}$$

where $\xi(t)$ and $\eta(t)$ are Gaussian white noises with

$$\langle \xi(t) \rangle = \langle \eta(t) \rangle = 0, \tag{3}$$

$$\langle \xi(t)\xi(t') \rangle = 2\alpha\delta(t-t'), \tag{4}$$

$$\langle \eta(t)\eta(t') \rangle = 2D\delta(t-t'), \tag{5}$$

and

$$\langle \xi(t)\eta(t') \rangle = \langle \eta(t)\xi(t') \rangle = 2\lambda\sqrt{D\alpha}\,\delta(t-t'), \quad |\lambda| < 1. \tag{6}$$

In Eq. (2), A is the amplitude of the periodic signal and ω is the frequency. In Eqs. (2)–(5) α and D are the intensities of the noise, λ is the cross-correlated intensity between the noises. The deterministic potential

$$V(x) = -\frac{a}{2}x^2 + \frac{b}{3}x^3 \tag{7}$$

corresponds to Eq. (2) has two steady states $x_1 = \frac{a}{b}$ (stable) and $x_2 = 0$ (unstable).

The Fokker-Planck equation corresponding to the Langevin equation (2) with Eqs. (2)–(5) can be written as

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[ax - bx^2 + xA\cos(\omega t) + \alpha x - \lambda\sqrt{D\alpha}\right]P(x,t) + \frac{\partial^2}{\partial x^2}\left(\alpha x^2 - 2\lambda\sqrt{D\alpha}x + D\right)P(x,t). \tag{8}$$

In the presence of the multiplicative periodic signal $A\cos(\omega t)$, the potential of the system is modulated by the periodic signal. However, here the signal amplitude is assumed small enough (*i.e.*, $A \ll 1$) that, in the absence of any noise, it is insufficient to force a particle to move from one well to the other, and it can be considered that x_1 and x_2 are still the stable state and unstable state of the system respectively. On the other hand, we also assume that the variation of the periodic signal is slow enough (*i.e.*, $\omega \ll 1$ or the adiabatic limit) that there is enough time to make the system reach local equilibrium in the period of $1/\omega$. Thus, the associated quasi-steady-state distribution function can be derived from Eq. (8) in the adiabatic limit [8]

$$P_{qst}(x,t) = N\left(\alpha x^2 - 2\lambda\sqrt{D\alpha}x + D\right)^{-1}\exp\left[-\frac{\Phi(x,t)}{\alpha}\right], \tag{9}$$

where N is a normalization constant, and

$$\begin{aligned}\Phi(x,t) = {} & bx + \left(2\lambda\sqrt{\frac{D}{\alpha}}b - \frac{\alpha}{2} - \frac{a}{2}\right)\times\ln\left|\alpha x^2 - 2\lambda\sqrt{D\alpha}x + D\right| - \left[bD + \left(a - 2\lambda\sqrt{\frac{D}{\alpha}}b\right)\lambda\sqrt{D\alpha}\right] \\ & \times\arctan\left(\frac{\sqrt{\alpha/D}x - \lambda}{\sqrt{1-\lambda^2}}\right) - \left[\frac{1}{2}\ln\left|\alpha x^2 - 2\lambda\sqrt{D\alpha}x + D\right| + \sqrt{\frac{\alpha}{D(1-\lambda^2)}}\times\arctan\left(\frac{\sqrt{\alpha/D}x - \lambda}{\sqrt{1-\lambda^2}}\right)\right]A\cos(\omega t),\end{aligned} \tag{10}$$

In order to calculate the transition rates W out of the x_1 state we consider the mean-first-passage ($MFPT$) T of the system from $x_1 = \frac{a}{b}$ to $x_2 = 0$. When α and D are small in comparison with the height of energy barrier, *i.e.*,

$$D, \alpha < |\Phi(x_2) - \Phi(x_1)|, \tag{11}$$

by use of the steepest-descent approximation [39–42], the modified $MFPT$ can be obtained

$$T(x_1 \to x_2) = \frac{2\pi}{\left[|V''(x_1)V''(x_2)|\right]^{1/2}}\exp\left[\frac{\Phi(x_2) - \Phi(x_1)}{\alpha}\right], \tag{12}$$

the prime of $V(x)$ denotes the differentiation with respect to the variable x, and the transition rate is [8, 43]

$$\begin{aligned}W = {} & \frac{1}{T(x_1 \to x_2)} = \frac{a}{2\pi}\exp\left[\frac{\Phi(x_1) - \Phi(x_2)}{\alpha}\right] = \frac{a}{2\pi}\exp\left\{-\frac{1}{\alpha}\left[-a - \left(2\lambda\sqrt{\frac{D}{\alpha}}b - \frac{\alpha}{2} - \frac{a}{2}\right)\right.\right. \\ & \times\ln\left|\frac{a^2\alpha}{b^2D} - 2\lambda\frac{a}{b}\sqrt{\frac{\alpha}{D}} + 1\right| + \left[bD + (a - 2\lambda\sqrt{\frac{D}{\alpha}}b)\lambda\sqrt{D\alpha}\right]\times\left(\arctan\frac{-\lambda + \frac{a}{b}\sqrt{\frac{\alpha}{D}}}{\sqrt{1-\lambda^2}} + \arctan\frac{\lambda}{\sqrt{1-\lambda^2}}\right) \\ & \left.\left. + \left[\sqrt{\frac{\alpha}{D(1-\lambda^2)}}\left(\arctan\frac{\lambda}{\sqrt{1-\lambda^2}} + \arctan\frac{-\lambda + \frac{a}{b}\sqrt{\frac{\alpha}{D}}}{\sqrt{1-\lambda^2}}\right) + \frac{1}{2}\ln\left|\frac{a^2\alpha}{b^2D} - 2\lambda\frac{a}{b}\sqrt{\frac{\alpha}{D}} + 1\right|\right]A\cos(\omega t)\right]\right\}.\end{aligned} \tag{13}$$

Within the framework of the theory of SR presented by McNamara and Wiesenfeld [8], the expression of the signal-to-noise ratio (SNR) of the logistic growth system in the adiabatic limit can be obtained from the two-state approach and can be given by

$$SNR = \frac{\pi W_1^2 A^2}{4W_0}\left[1 - \frac{W_1^2 A^2}{2\left(W_0^2 + \omega^2\right)}\right]^{-1}, \tag{14}$$

where

$$\begin{aligned}W_0 = 2W\Big|_{A\cos(\omega t)=0} = {} & \frac{a}{\pi}\exp\left\{-\frac{1}{\alpha}\left[-a - \left(2\lambda\sqrt{\frac{D}{\alpha}}b - \frac{\alpha}{2} - \frac{a}{2}\right)\times\ln\left|\frac{a^2\alpha}{b^2D} - 2\lambda\frac{a}{b}\sqrt{\frac{\alpha}{D}} + 1\right|\right.\right. \\ & \left.\left. + \left[bD + \left(a - 2\lambda\sqrt{\frac{D}{\alpha}}b\right)\lambda\sqrt{D\alpha}\right]\times\left(\arctan\frac{-\lambda + \frac{a}{b}\sqrt{\frac{\alpha}{D}}}{\sqrt{1-\lambda^2}} + \arctan\frac{\lambda}{\sqrt{1-\lambda^2}}\right)\right]\right\},\end{aligned} \tag{15}$$

$$W_1 = 2\frac{dW}{dA\cos(\omega t)}\bigg|_{A\cos(\omega t)=0} = -\left\{\sqrt{\frac{1}{D\alpha(1-\lambda^2)}}\left(\arctan\frac{\lambda}{\sqrt{1-\lambda^2}} - \arctan\frac{-\lambda+\frac{a}{b}\sqrt{\frac{\alpha}{D}}}{\sqrt{1-\lambda^2}}\right) + \frac{1}{2\alpha}\ln\left|\frac{a^2\alpha}{b^2 D} - 2\lambda\frac{a}{b}\sqrt{\frac{\alpha}{D}} + 1\right|\right\} W_0. \quad (16)$$

By virtue of the expression Eq. (14) of SNR, the effects of multiplicative and additive noises, and correlated intensity on the SNR can be investigated by numerical calculation.

The SNR as a function of α and λ is presented in Fig. 1, and the ratio of α and D is fixed. A maximum of SNR vs λ for small α denotes the SR in a broad sense [20]. With α increasing, a second optimal value of λ for SR appears, and two peaks appear in the SNR simultaneously. This means that stochastic multi-resonance [17] occurs. As the α further increases, the second maximum disappears, the stochastic multi-resonance becomes a single resonance again, *i.e.*, the re-entrant transition between one peak and two peaks and then to one peak again happens in SNR.

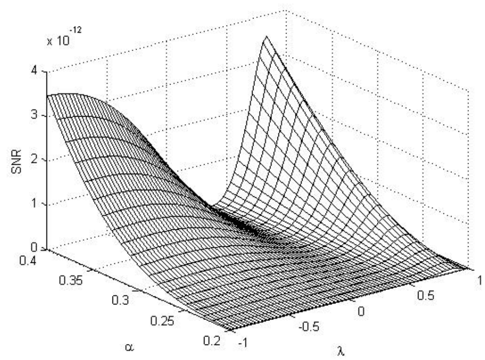

Figure 1. The signal-to-noise ratio SNR as a function of α and λ, with $a = 1$, $b = 0.1$, $A = 0.0001$, $\omega = 0.0001$, and $D = 0.7\alpha$.

Fig. 2 shows that the SNR as a function of λ, α, and D, is fixed. For very small α, a peak in SNR vs λ denotes the SR in a broad sense [20]. This peak structure in SNR is vary sensitive to the varying of α, and a slight increment of α induces this peak to disappear. A peak structure appears again in SNR vs λ with the further increasing of α. The small and large α display opposite effects on the SR when SNR is a function of λ.

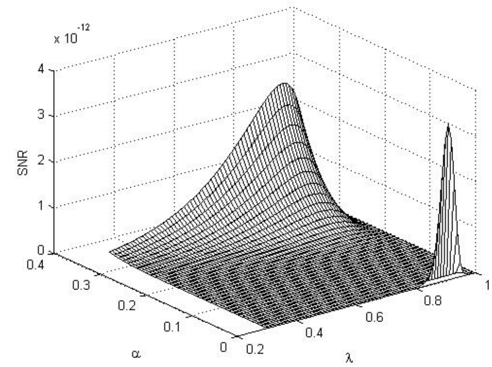

Figure 2. The signal-to-noise ratio SNR as a function of α and λ with $a = 1$, $b = 0.1$, $A = 0.0001$, $\omega = 0.0001$ and $D = 0.1$.

The effects of D and λ on the SNR are shown in Fig. 3. A coronary surface is the characteristic of this figure. The increasing of D (or λ) will enhance the SR firstly then suppress it for the SNR as a function of λ (or D), *i.e.*, both D and λ have critical values in the SNR as a function of one of them. This manifests as the doubly critical phenomenon in the SR induced by D and λ.

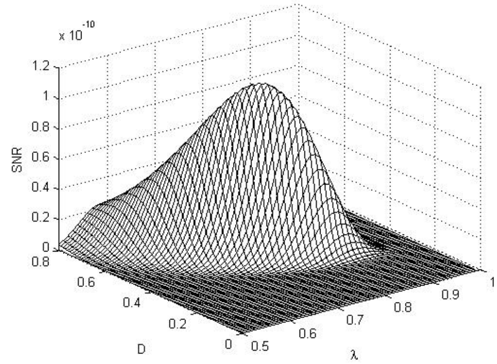

Figure 3. The signal-to-noise ratio SNR as a function of D and λ with $a = 1$, $b = 0.1$, $A = 0.0001$, $\omega = 0.0001$, and $\alpha = 0.01$.

Finally the SNR vs D and α is given in Fig. 4. It can be found that if the D or α is small, no resonance exists. The increasing of D (or α) can induce a maximum in SNR vs α (or D). The two noise intensities enhance the SR on each other. This denotes the effect of doubly stochastic resonance [22].

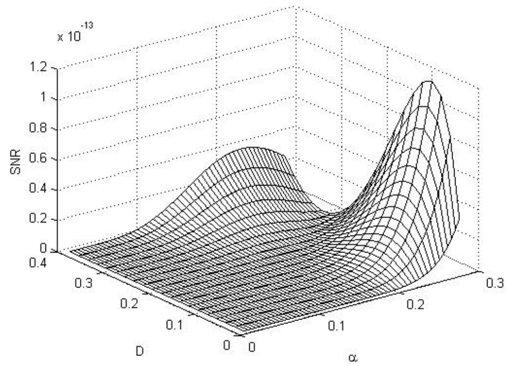

Figure 4. The signal-to-noise ratio SNR as a function of D, α with $a = 1$, $b = 0.1$, $A = 0.0001$, $\omega = 0.0001$, and $\lambda = 0.5$.

3. Discussion and conclusion

The SR phenomenon induced by multiplicative periodic forcing in a logistic growth model with correlated noises is studied by using the theory of SNR proposed by McNamara and Wiesenfeld [8]. The expression of the SNR is obtained. By virtue of the expression of SNR and through the numerical computation, it is found that the existence of a maximum in the SNR is the identifying characteristic of the SR phenomenon. In comparison with the SR induced by an additive periodic signal, three important aspects are found: (1) stochastic multi-resonance, this phenomenon is induced by the varying of α in SNR vs λ; (2) doubly critical phenomenon, this feature occurs in SNR as a function of D and λ; (3) doubly stochastic resonance, which describes α (or D) as inducing the SR in SNR versus D (or α). These results indicate that the SR induced by the multiplicative periodic signal can display many interesting and resounding behaviors.

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