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Non-extensive equilibration in relativistic matter

Research Article

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Abstract: We present a view of the non-extensive thermodynamics based on general composition rules. A formal logarithm maps these rules to the addition, which can be used to generate stationary distributions by standard techniques. We review the most commonly used rules and as an application we discuss the Tsallis-Pareto distribution of transverse momenta of energetic hadrons, which emerge from relativistic heavy-ion collisions.

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1. Introduction

It is an old suggestion to apply the machinery of statistical physics to the description of high energy particle production which dates back to Rolf Hagedorn. The study of event - averaged single particle spectra, however, shows deviations from the one expected in a thermal model, namely form the Boltzmann-Gibbs statistics. This fact may be interpreted in different ways, starting with the finite time available for a possible equilibration, continuing with finite size (microcanonical) effects causing fluctuations in the intensive parameters of a thermodynamical model, and stretching to the particular proposition of using concepts of a generalized, non-extensive thermodynamics. The latter provides a treatment in which, – after some transformations and re-interpretations of the usual thermal quantities, like the temperature, – the usual procedures produce non-exponential particle spectra and other deviations from the predictions of the naive thermal models. While in studying spectra stemming from small systems, like e^+e^- or pp and $p\overline{p}$ collisions the most characteristic features are related to a restriction in the longitudinal phase space showing in properties of the rapidity distribution dN/dy and in the occurence of the negative binomial multiplicity distribution [1–5], the transverse momentum distributions rather show an enlarged and possibly more fractally filled phase space [6, 7]. There are several speculations on possible causes for such deformations, measured in a summerized way by a parameter q used in Tsallis' non-extensive [8–12] entropy formula, or Rényi's extensive one [13]. Fluctuations of temperature or energy equipartition [14–16], anomalous diffusion [17] or multiplicative noise [18, 19], or alternatively an altering in the two-body energy composition rule [20–24] were investigated in theoretical approaches. In order to solidify the motivation for such an approach we sketch some arguments about possible occurences of non-extensivity in thermodynamically treateble, large systems (the size is

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measured not by a volume, but by the number of particles produced and analyzed).

In non-extensive systems those thermodynamic variables, which usualy scale with system size (with volume V and particle number N), violate somehow this scaling. Generally the entropy and energy of an N-particle system, S_N and E_N, are composed from individual quantities, S_1 and E_1, by counting for interaction and correlation corrections. Considering pair interactions, mediated by the pair-potential $v(r)$ (which depends on the relative coordinates $r = r_1 - r_2$), and a particular form of the two-particle phase space occupation, $\rho_{12} = f(p_1)f(p_2)g(r)$ with the pair correlation function $g(r)$, approaching one for uncorrelated pairs, the specific ratios can easily be derived, $S_N/N = S_1 - \overline{n} \int g \ln g \, d^dr$, $E_N/N = E_1 + \overline{n} \int gv \, d^dr$. The volume and the particle number are related via the mean density: $V = N/\overline{n} = \int d^dr$. In these expressions $S_1 = 2\overline{s} \int g(r) \, d^dr$ with $\overline{s} = -\int f(p) \ln f(p) \, d^dp$ and $E_1 = 2\overline{e} \int g(r) \, d^dr$ with $\overline{e} = \int f(p)K(p) \, d^dp$, where $K(p)$ denotes the kinetic energy of a single particle. In principle the pair distribution function, $g(r)$, can be obtained from the interaction $v(r)$ in a stationary state (if exists), but this is a very complicated and difficult calculation for most of the known physical systems. Non-extensivity occurs, whenever the specific ratios, like E_N/N and S_N/N diverge in the $N \to \infty$ limit. One considers this limit at constant mean number density, $\overline{n}$.

It is easy to construct examples for non-extensive energy at extensive entropy: in the crude approximation, when $g(r)$ is either zero (up to a characteristic short range length) or one (towards infinity), the entropy correction is zero and hence the large-N entropy is extensive. However, with a pair potential of the form $v(r) \sim r^{-\alpha}$, whenever $\alpha \leq d$ with d being the spatial dimension ($d = 3$ for isotropic systems), the correction to the energy may diverge in the large-N limit[1] [25]: $E_N/N \approx E_1 + \text{const.} \, N^{1-\alpha/d}$. In systems with long range correlations (with scale independence, like in some random networks) $g(r)$ may differ from one even at large distances. In such cases the entropy also may pick up a non-extensive contribution. Another example may be given by confinement: the correlation $g(r)$ approaches zero in this case for confined pairs at large relative distances, but the integral of $g \ln g$ may have a non-vanishing contribution which – if of power-law type – may diverge with the total volume logarithmically.

The composition of small sytems into a big one (extensivity) and the composition of two large systems (additivity) are related problems. Non-extensive systems are always non-additive by using the original definitions for energy and entropy. In a quite broad class of cases, however, another additive quantity may be constructed. The existence of such a quantity relies on special properties (specifically on the associativity) of the composition rule. The mapping of a non-additive quantity to an additive one, the formal logarithm, usually contains parameters, which describe the degree of non-extensivity. This is the basis of the construction of non-additive entropy (and energy) formulas.

This way a basic problem occurs for any non-extensive thermodynamics: how do large subsystems equilibrate, whose thermal state is described not only by a temperature, T, but also by a non-extensivity parameter, say q. In particular, for the Aczel-Daroczy-Chrvat-Tsallis entropy formula, how does equilibration occur between different (q, T) systems? Given two preheated systems, does a common stationary distribution occur, will it be a Tsallis-Pareto distribution, and is the temperature, defined by this equilibration process, universal (absolute)? We attacked such questions in the framework of a particular parton cascade model, using non-extensive energy composition rules in a Boltzmann equation type simulation [26] recently.

We note that the often cited entropy formula [8–12],

$$S_T = \frac{1}{1-q} \sum_i \left(w_i^q - w_i \right), \tag{1}$$

using normalized probabilities $\sum_i w_i = 1$, follows the special composition rule $S_{12} = S_1 + S_2 + (1-q)S_1S_2$. For factorizing probabilities, $w_{ij}^{(12)} = w_i^{(1)} w_j^{(2)}$, it can be mapped to an additive rule for $S_R = Y(S_T)$, given as

$$S_R = \frac{1}{a} \ln \left(1 + aS_T \right). \tag{2}$$

Here we used the parameter $a = (1-q)$. The result is the well-known Rényi entropy [13],

$$S_R = \frac{1}{1-q} \ln \left(\sum_i w_i^q \right), \tag{3}$$

which is additive but still contains the extra parameter q. For this additive, and hence extensive, entropy formula the question towards the two-parameter equilibration also holds [27–29].

2. Abstract composition rules generalize non-extensivity

The infinite repetition of an arbitrary pairwise, iterable composition rule is an *associative* rule [20]. It is a mathematical property that associative rules always possess

[1] *C. Tsallis, private communication*

a strict monotonic function, called here the formal logarithm, in terms of which they can be expressed [30]. We denote an abstract pairwise composition rule by the mapping $(x, y) \to h(x, y)$. The associativity of such a rule is expressed by

$$h(h(x, y), z) \;=\; h(x, h(y, z)), \tag{4}$$

for x, y and z being real quantities. The general solution of the associativity equation (4) is given by

$$h(x, y) = X^{-1}\left(X(x) + X(y)\right), \tag{5}$$

with $X(x)$ being a strict monotonic function, the formal logarithm. It maps the arbitrary composition rule $h(x, y)$ to the addition by taking the X-function of Eq. (5):

$$X(h(x, y)) = X(x) + X(y). \tag{6}$$

Due to this construction there are generalized analogs to classical extensive (and additive) quantities; they are formal logarithms. As a consequence stationary distributions, in particular by solving generalized Boltzmann equations [31], are proportional to the Gibbs exponentials of the formal logarithm,

$$f(x) = \frac{1}{Z} e^{-\beta X(x)}. \tag{7}$$

A general non-additive entropy formula can be derived based on the inverse of the formal logarithm (inverting the $f \sim \exp \circ X$ function):

$$S = \int f X^{-1}(-\ln f). \tag{8}$$

The rule leading to the q-exponential distribution is given by $h(x, y) = x+y+axy$ with the parameter a proportional to $q-1$. In this case one obtains the formal logarithm as being $X(x) = \frac{1}{a}\ln(1+ax)$. This formal logarithm leads to a stationary distribution with power-law tail as the function composition $exp \circ X$ on the power $-\beta$:

$$f(E) = \frac{1}{Z} e^{-\frac{\beta}{a}\ln(1+aE)} = \frac{1}{Z}\left(1 + aE\right)^{-\beta/a}. \tag{9}$$

The corresponding non-additive entropy formula is constructed as the expectation value of the inverse of this function, $L^{-1} \circ \ln$ of $1/f$:

$$S = \int f\, \frac{e^{-a\ln(f)} - 1}{a} \;=\; \frac{1}{a}\int \left(f^{1-a} - f\right). \tag{10}$$

Its formal logarithm is The Rényi entropy.

Further examples for non-additive rules can be easily given. The power-rule, $h(x, y) = (x^b + y^b)^{1/b}$, leads to a streched exponential, $f(x) \propto \exp(-\beta x^b)$, in the stationary state. Kaniadakis [32] suggested a composition rule, $h(x, y) = x\sqrt{1 + \kappa^2 y^2} + y\sqrt{1 + \kappa^2 x^2}$ with the corresponding formal logaritm being the inverse sine hyperbolic function, $X(x) = \frac{1}{\kappa}\mathrm{Arsh}(\kappa x)$. The stationary distribution, $f(x) = \frac{1}{Z}\left(\kappa x + \sqrt{1 + \kappa^2 x^2}\right)^{-\beta/\kappa}$, develops a power-law tail for large $|x|$. The corresponding entropy formula is the average of $X^{-1} \circ \ln$ of $1/f$ over the allowed phase space:

$$S_K = -\int \frac{f}{\kappa}\sinh(\kappa \ln f) = \int \frac{f^{1-\kappa} - f^{1+\kappa}}{2\kappa}. \tag{11}$$

Regarding $\kappa x = p/mc$, the formal logarithm is proportional to the rapidity. This would imply a stationary distribution like $exp(-|\eta|)$ with η being the rapidity. Finally we note that the Tsallis rule $h(x, y) = x + y + axy$ is particular, being the most general symmetric second order formula satisfying $h(x, 0) = x$.

3. Equilibration of large subsystems

Seeking for a canonical equilibrium state we have to maximize a total entropy given by a general composition rule, $S(E_1, E_2)$, at the same time satisfying a constraint which is in the general case also non-additive: $h(E_1, E_2)$ is constant. For the moment we neglect the dependence on further thermodynamical variables; usually the particle number N and the volume V is regarded to be proportional and extensive. In the traditional case both the entropy and the energy are combined additively: $S(E_1, E_2) = S(E_1) + S(E_2)$ and $h(E_1, E_2) = E_1 + E_2$. In the general case by using corresponding formal logarithms the quantities $Y(S)$ and $X(E)$ have to be considered as additive. Since for associative rules the formal logarithm is srict monotonic, the maximum of the total entropy is achieved where $Y(S)$ has its extremum. The general canonical principle is therefore given by $Y(S) - \beta X(E) = \max$. The parameter β at this point is a Lagrange multiplier. Applying this for the equilibration of two large subsystems, and assuming that the entropy of each system depends only on its own energy, one arrives at the equilibrium condition

$$\frac{Y'(S(E_1))}{X'(E_1)}\, S'(E_1) = \frac{Y'(S(E_2))}{X'(E_2)}\, S'(E_2) = \frac{1}{T}. \tag{12}$$

Comparing this with the general canonical maximization form mentioned above we obtain $\beta = 1/T$, so here T

is an absolute temperature in the clasiscal thermodynamical sense. Its relation to the entropy, however, has been generalized. In particular for an additive entropy, but non-additive energy composition rule, one arrives at $1/T = S'(E)/X'(E)$. The relation of this quantity to the logarithmic spectral slope, $1/T_{\rm slope} = -d \ln f/dE$ leads to a practical tool for the analysis of particle spectra in experiments. For the Pareto-Tsallis disitribution it is given by $T_{\rm slope} = T/X'(E) = T(1+aE) = T + (q-1)E$. The naive effort to extract a temperature from energy spectra of particles, as it is a widespread praxis in relativistic heavy ion studies, only works if $q = 1$, *i.e.* for spectra exponential in the particle energy. Otherwise an energy dependent slope, and a curved spectrum in the logarithmic plot has to be interpreted.

4. Spectral temperatures in relativistic heavy ion collisions

It is helpful to describe shortly, how a temperature can be conjectured from observations on particle spectra produced in relativistic heavy ion collisions. The detected particles have relativistic velocities and different masses. One intriguing way is to look at the transverse momentum, p_T-, spectra around mid-rapidity (in the center of mass system for equal colliding heavy ions). The different identified hadrons, mostly pions, kaons, protons and antiprotons, have to show that their abundance in the momentum space depends on their energy; this phenomenon at zero rapidity is the so-called m_T-scaling. The transverse mass is given as $m_T = \sqrt{m^2 + p_T^2}$, at strictly zero rapidity this is the total relativistic energy.

The analysis is made a little more involved by the fact that the source emitting the detected hadrons is not at rest. The most prominent feature is a transverse flow, with relativistic velocity, v_T (and a corresponding Lorentz factor $\gamma_T = 1/\sqrt{1-v_T^2}$ in units where $c = 1$). The relativistic energy of a particle in the frame of the emitting source cell is given by the Jüttner variable:

$$E = u_\mu p^\mu = \gamma_T m_T \cosh(y-\eta) - \gamma_T v_T p_T \cos(\varphi - \Phi). \quad (13)$$

Here the four-velocity of the source, $u_\mu = (\gamma_T \cosh\eta, \gamma_T \sinh\eta, \gamma_T v_T \cos\Phi, \gamma_T v_T \sin\Phi)$, and the actual four-momentum of the particle, $p_\mu = (m_T \cosh y, m_T \sinh y, p_T \cos\varphi, p_T \sin\varphi)$, are parametrized by rapidity and angle variables y, η and Φ, φ, respectively. We consider a thermal model for the particle spectra; then the yield is supposed to depend on the Jüttner variable E given by Eq. (13). Assuming a general distribution $f(E) \sim \exp(-X(E)/T)$, which is

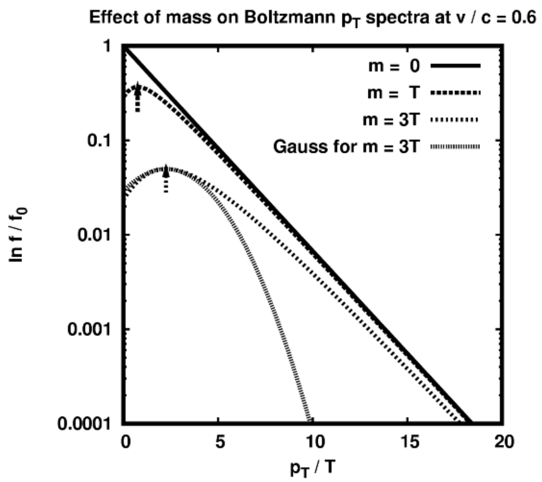

Figure 1. General shape of p_T spectra for massive particles in the presence of a bulk transverse flow with radial velocity component $v_T = 0.6$. The Gaussian approximation to the maximum reveals both a Lorentz-enhanced mass, $m^* = m\gamma_T$, and a Lorentz-enhanced temperature parameter, $T\gamma_T$.

monotonic decreasing, one finds its maximum at the minimum of E. This variable is minimal at the rapidity $y_{\rm min} = \eta$, and angle $\varphi_{\rm min} = \Phi$, giving

$$E_{\rm min} = \gamma_T m_T - \gamma_T v_T p_T. \quad (14)$$

This Lorentz-boosted transverse energy reaches its minimum at the transverse momentum value $p_{T,\rm min} = m\gamma_T v_T$, leading to $m_{T,\rm min} = m\gamma_T$ and $E_{\rm min} = m$. The expansion around this minimum in the p_T-distribution is an effective Gaussian:

$$e^{-(E-m)/T} \approx \exp\left(-\frac{(p_T - m\gamma_T v_T)^2}{2m\gamma_T\, T\gamma_T}\right). \quad (15)$$

Such spectra are plotted in Fig. 1 for a typical transverse flow of $v_T = 0.6$ for massless and massive particles with masses of $m = T$ and $m = 3T$. The curves show typical shapes for light meson and heavy baryon spectra occuring in relativistic heavy ion collisions.

In fact, according to experimental findings at RHIC the observed particle spectra have to be corrected for a transverse flow in order to reach m_T-scaling. On the other hand the formula near the maximum, Eq. (15), may shed some light to the classical Einstein-Ott-Planck discussion about the temperature of relativistically moving bodies from an unexpected corner of modern experimental observations. Namely by using the Gaussian approximation both the particle mass and the effective spectral temperature gain a Lorentz factor, γ_T.

5. Non-extensivity in quark matter and in hadron matter

We conjecture that the power-law tails observed in hadronic spectra may stem from the non-extensivity of the preformed quark matter, which hadronizes rapidly. We make a connection between quark and hadron spectra by the quark coalescence model. A coalescence of two quarks (actually a quark and an antiquark) into a meson produces a yield proportional to the following quantity:

$$F(\vec{p}) = \int f\left(E\left(\vec{P}/2+\vec{q}\right)\right) f\left(E\left(\vec{P}/2-\vec{q}\right)\right) C(\vec{q})\, d^3q. \tag{16}$$

Here we integrate over the relative momentum of the quarks with a coalescence factor, $C(\vec{q})$, for which a simple model has been utilized [33, 34]. For common momenta much larger than the relative one $|\vec{P}| \gg |\vec{q}|$ (otherwise the quarks do not coalesce!) on gets

$$F(\vec{P}) \approx f^2\left(E\left(\vec{P}/2\right)\right) \int C(\vec{q})\, d^3q. \tag{17}$$

In particular light hadrons made from massless quarks follow the quark-scaling rule:

$$f_{\text{hadron}}(E) \propto f^n(E/n). \tag{18}$$

As a consequence particular properties of the non-extensive thermal model between quark and hadron matter also scale: $T_{\text{mesons}} = T_{\text{baryons}} = T_{\text{quarks}}$ for the temperature, while $q_{\text{mesons}} - 1 = (q_{\text{quarks}} - 1)/2$ for mesons and $q_{\text{baryons}} - 1 = (q_{\text{quarks}} - 1)/3$ for baryons. Experimentally these relations are still to be checked. These predictions of the non-extensive phenomenology meet the curves from pQCD calculations smoothly, with the following surmised properties of quark matter at RHIC: $T = 140 \ldots 180$ MeV, $q = 1.22$, $v_T = 0.6$ [35].

Summarizing we have shown that non-extensive behavior can be mapped to additive properties of a formal logarithm of the original quantity in the general case of associative composition rules. That such rules necessarily arise in the thermodynamical limit is demonstrated in Ref. [20]. Using formal logarithms all the classical concepts and techniques can be applied to describe thermal equilibrium or to generate distributions accordingly. The thermal equilibration of large subsystems are subject to straightforward generalizations of the familiar rules. As an exaple we discussed certain particle spectra arising in relativistic heavy ion collisions from a thermal and non-extensive quark matter. These results qualitatively agree with experimental findings.

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