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Phenomenology of the nucleon-nucleon potential

Research Article

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Abstract: A local nucleon-nucleon potential expansion is developed in terms of orthogonal projectors. Considering the nucleon-nucleon (NN) potential as a completely phenomenological structure, the expansion provides an opportunity to obtain the NN scattering phase shifts that can be described by applying a restricted set of operators, dependent on angular and spin-isospin degrees of freedom of the interacting nucleons. The results obtained with an approximation for eight basic operators (central, spin-orbit and tensorial) are consistent with experience in the field, and provide directions for further modifications of realistic NN potentials.

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1. Introduction

The success achieved in atomic physics with the use of the Coulomb potential to describe interactions between pairs of charged particles, created expectations of a similar success for nuclear physics. A popular belief was that solving the Schrödinger equation by applying some analogous local nucleon-nucleon (NN) potential to describe the interaction between nucleon pairs would provide a basic understanding of the properties of nuclei. Hence, the problem of deriving NN potentials attracted, and still attracts, the attention of nuclear theorists. The most accurate realistic potentials are created by carefully fitting NN scattering data and deuteron properties [1–3]. These potentials describe a rich set of phase shifts, for orbital momentum ranging from zero to values where the centrifugal barrier completely dominates over any strong short-ranged potential. However, considerable problems with deuteron and some phase shifts descriptions still exist; the numerous efforts to refine these potentials have, at least for the deuteron description, not yet been successful. It is well-known that an S-phase description ensures the central part of the potential, the tensor interaction is necessary for a deuteron description, and a P-phase description requires the spin-orbit potential. For fitting some partial waves these parts of potential cannot produce required precision. Potentials having quadratic momentum and quadratic spin-orbit dependence are necessary. For higher partial waves description exotic, momentum-dependent and quadratic momentum-dependent terms of NN potential, sometimes without clear physical motivation, recently have been under consideration [4].

However, the problems with the two nucleon data description are not the most important. Every indication of NN potential failure occurs in light nuclei calculations with realistic NN potentials. These potentials underbind all

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nuclei, including those composed of a few nucleons. Obviously, the structure of nucleons modifies their characteristics and interactions when they are present in atomic nuclei, *i.e.* are surrounded by other nucleons. The known solution to this problem is to use three-nucleon forces. However, phenomenological three-nucleon potentials depend on the functional form of the NN potentials. Moreover, the most impressive light nuclei calculations applying the Green's Function Monte Carlo (GFMC) method [5–9] are performed taking only part of the NN potential, consisting of eight basic operators; a description of the NN scattering phases cannot be obtained by using only this part of the NN potential. Including more potential terms in GFMC calculations results in large statistical uncertainties. Most probably, these uncertainties are caused by correlations of different NN channel potentials, and hence, the definition of potential.

As mentioned, this activity is based on the belief that some very restricted (due to the strong force invariance with respect to different symmetries) choice of operators can reproduce a complete set of phase-shifts and the deuteron properties or, in other words, that a realistic NN potential is able to ensure the fundamental description of the strong interaction. However, experience in the field teaches that the real picture is not so optimistic. Perhaps the NN potential definition as a fundamental structure needs some revision. Our approach to this problem is based on an original and unusual assumption that the realistic local NN potential is a completely phenomenological construct. Given this assumption, there are two questions: (1) how many phase shifts can one describe with a restricted set of operators? and (2) what set of additional operators is necessary to describe the required array of phase shifts?

2. Structure of realistic NN potential

From its first definition [10, 11] to recent modifications [1–3], the realistic nucleon-nucleon (NN) potential is understood to be the sum of functions of radial variable multiplied by operators, dependent on the angular and spin-isospin degrees of freedom of the two interacting nucleons. After integration, this sum can be present as a set of functions $V^{j^\pi t}(r)$, defined in different two-nucleon channels $j^\pi t$, where j is the total momentum, t is the isospin and π is the parity of the channel given by $\pi = \text{sign}\,(-1)^l \equiv \text{sign}\,(-1)^{s+t-1}$. Namely, the set of quantum numbers $j^\pi t$ identifies the states of atomic nuclei. Deuteron is not an exception; hence, this set, consisting of exact quantum numbers, is also the best for the description of two nucleon channels. The widely accepted spectroscopic notation for two nucleon states (partial waves) is ${}^{2s+1}l_j$. However, the orbital momentum quantum number l is not a good quantum number for this system; hence, in some cases the channel contains two partial waves. As a consequence, in coupled channels satisfying the condition $(-1)^{s+t} = (-1)^j$ ($j \neq 0$)) the NN potential is given by three functions, defined as a symmetrical second order matrix. The two most important channels of this kind are 1^+0, whose standard spectroscopic notation is ${}^3S_1 - {}^3D_1$, and 2^-1, whose standard spectroscopic notation is ${}^3P_2 - {}^3F_2$. We apply a slightly different set of quantum numbers to describe the NN partial waves. The set, consisting of quantum numbers $jlst$ (quantum numbers $\{jls\}$ must compose a triangle) is more acceptable for us, because it contains quantum numbers st, describing the so called supermultiplet [12]. We also apply the set of quantum numbers jst for two-nucleon channel identification; the set of quantum numbers st uniquely defines the channel identifiers πt and vice versa, due to the relation $\pi = \text{sign}\,(-1)^{s+t-1}$. The list of NN channels and states with the lowest values of angular momentum ($l \leq 5$) of different supermultiplets, as well as the standard spectroscopic notations of these states, are presented in Tab. 1.

The parameterization of the NN potential as a set of channel potentials $V^{j^\pi t}(r)$ or $V^{jst}(r)$ is best for fitting phase shifts in different channels. Moreover, this is the best parametrization of the potential for nuclear calculations, especially when the basis, applied for the exact wave-function expansion, ensures the exact quantum numbers for any two-nucleon subsystem.

Generally, the NN potential is a complex operator dependent on two sets, each consisting of seven variables $r\Omega$, where r is the radial coordinate, and $\Omega \equiv \left(\theta\varphi\sigma_i\sigma_j\tau_i\tau_j\right)$ marks the angular and spin-isospin degrees of freedom of the interacting nucleons. We consider local NN potentials $V(r\Omega)$, defined by the expression

$$V\left(r\Omega, r'\Omega'\right) = V(r\Omega)\frac{1}{r^2}\delta\left(r - r'\right)\delta\left(\cos\theta - \cos\theta'\right)\delta\left(\varphi - \varphi'\right)\delta_{\sigma_i,\sigma_i'}\delta_{\sigma_j,\sigma_j'}\delta_{\tau_i,\tau_i'}\delta_{\tau_j,\tau_j'}. \quad (1)$$

The local NN potential can be written as a sum of the above-mentioned operators

$$V(r\Omega) = \sum_p v_p(r)\, O_p(\Omega). \quad (2)$$

The sum over $p = 1, 2, \ldots, p_0$ is very restricted due to the condition that $O_p(\Omega)$ must satisfy translational, rotational, parity and time-reversal invariance (see [12, 13] and [14]). The optimal choice of this set is given in [2], and consists of 18 operators: 14 charge-independent (CI) among them, and 8 basic CI operators among the last ones:

$$O_{p=1,8}(\Omega) = \left[\mathbf{1}, \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j, \mathbf{L}\cdot\mathbf{S}, \mathbf{S}_{ij}\right] \otimes \left[\mathbf{1}, \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j\right]. \quad (3)$$

Here, $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ mark the spin and isospin operators of interacting ($i-th$ and $j-th$) nucleons, $\mathbf{L}\cdot\mathbf{S}$ is a spin-orbit operator, and

$$\mathbf{S}_{ij} = \sqrt{24\pi}\left\{Y_2(\theta\varphi)\otimes\{\mathbf{S}\otimes\mathbf{S}\}_2\right\}_{00} \equiv \frac{1}{2}\left[3\frac{(\boldsymbol{\sigma}_i\cdot\mathbf{r})(\boldsymbol{\sigma}_j\cdot\mathbf{r})}{r^2} - (\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)\right] \quad (4)$$

is the tensor interaction operator. Any potential linear in momentum can be expressed by these operators.

It is well-known that these eight basic CI operators are sufficient to fit the CI averages of the S- and P-waves and to ensure acceptable deuteron properties. These two nucleon channels are $j^\pi t = 0^+1$ (jst=001, $jlst$=0001; the usual spectroscopic notation of this channel, consisting of one partial wave, is ${}^1S_0$), $j^\pi t = 0^-1$ ($jst = 011, jlst = 0111$;${}^3P_0$), $j^\pi t = 1^+0$ ($jst = 110$, this channel consists of two states $jlst = 1010$ and 1210; ${}^3S_1 - {}^3D_1$), $j^\pi t = 1^-0$ ($jst = 100$, $jlst$=1100;${}^1P_1$), $j^\pi t = 1^-1$ (jst=111,${}^3P_1$) and P wave of channel $j^\pi t = 2^-1$ (jst=211, $jlst$=2111; ${}^3P_2$). Hence, for this set one has complete balance of the potential and mentioned phase shifts: eight potentials $v_p(r)$ define the same number of phase shifts and corresponding potentials $V^{j^\pi t}(r)$, because the channel 1^+0 requires three such functions.

The six additional CI operators allowed by the mentioned invariances are

$$O_{p=9,14}(\Omega) = \left[\mathbf{L}^2, (\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)\mathbf{L}^2, (\mathbf{L}\cdot\mathbf{S})^2\right]\otimes\left[\mathbf{1}, \boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j\right]. \quad (5)$$

These operators are quadratic in angular momentum which is necessary for higher phases and coupled channel 2^-1 $\left({}^3P_2 - {}^3F_2\right)$, hence the NN potentials in the corresponding channels description. The charge-dependent operators $O_{p=15,18}(\Omega)$ are necessary for a description of higher order effects, so we omit them, as well as the Coulomb potential, from further consideration.

Continuing with the eight basic CI operators, the potential is given by

$$V(r\Omega) = v_c(r)\mathbf{1} + v_{cs}(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j) + v_{ct}(r)(\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j) + v_{cst}(r)(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)(\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j) + v_{ls}(r)(\mathbf{L}\cdot\mathbf{S}) + v_{lst}(r)(\mathbf{L}\cdot\mathbf{S})(\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j) + v_t(r)\mathbf{S}_{ij} + v_{tt}(r)\mathbf{S}_{ij}(\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j). \quad (6)$$

Three kinds of members of this expression correspond to the known parts of the NN potential: the central ($v_{c\ldots}(r)$), spin-orbit ($v_{ls\ldots}(r)$), and tensorial ($v_{t\ldots}(r)$) operators, enriched by all possible products of operators, ($\boldsymbol{\sigma_i}\cdot\boldsymbol{\sigma_j}$) and ($\boldsymbol{\tau_i}\cdot\boldsymbol{\tau_j}$). The first four members of the expression make up the central part of the potential and contain operators, present in the spin-singlet and spin-triplet ($s = 0$ and $s = 1$) channels. The spin-orbit and tensorial operators have non-zero entries only in spin-triplet states.

This part of the potential has eight operator terms and serves as a basis for Green's Function Monte Carlo (GFMC) light nuclei calculations; application of the full interaction gives very large statistical errors, as mentioned above.

The purpose of this paper is to study the possibilities of describing phase shifts and NN potential while applying this very limited set of operators $O_{p=1,8}(\Omega)$ and considering NN potential as completely phenomenological structure.

Table 1. The channels ($j^\pi t$, and jst) and partial waves (${}^{2s+1}l_j$ and $jlst$) of different supermultiplets (st):

st	$j^\pi t$	jst	${}^{2s+1}l_j$	$jlst$
00	1^-0	100	${}^1P_1$	1100
	3^-0	300	${}^1F_3$	3300
	5^-0	500	${}^1H_5$	5500
	...	...	...	...
01	0^+1	001	${}^1S_0$	0001
	2^+1	201	${}^1D_2$	2201
	4^+1	401	${}^1G_4$	4401
	...	...	...	...
10	1^+0	110	${}^3S_1$	1010
			${}^3D_1$	1210
	2^+0	210	${}^3D_2$	2210
	3^+0	310	${}^3D_3$	3210
			${}^3G_3$	3410
	4^+0	410	${}^3G_4$	4410
	5^+0	510	${}^3G_5$	5410
	...	...	...	...
11	0^-1	011	${}^3P_0$	0111
	1^-1	111	${}^3P_1$	1111
	2^-1	211	${}^3P_2$	2111
			${}^3F_2$	2311
	3^-1	311	${}^3F_3$	3311
	4^-1	411	${}^3F_4$	4311
			${}^3H_4$	4511
	5^-1	511	${}^3H_5$	5511
	...	...	...	...

3. Techniques of orthogonal projectors to supermultiplets

Expression of the NN potential in terms of orthogonal projectors is well-known for the central part of the po-

tential [12]. This part of the potential is independent of angular variables and the problem simplifies because the dimension of subspace of spin-isospin functions for two nucleons equals sixteen. The basic coupled momenta functions spanning this space are

$$\alpha_{sm_s tm_t}\left(\sigma_i\sigma_j\tau_i\tau_j\right) = \alpha_{sm_s}\left(\sigma_i\sigma_j\right)\alpha_{tm_t}\left(\tau_i\tau_j\right) \equiv \left\{\alpha_{1/2}\left(\sigma_i\right)\otimes\alpha_{1/2}\left(\sigma_j\right)\right\}_{sm_s}\left\{\alpha_{1/2}\left(\tau_i\right)\otimes\alpha_{1/2}\left(\tau_j\right)\right\}_{tm_t}, \tag{7}$$

where $\alpha_{1/2\mu_\sigma}\left(\sigma_i\right)$ and $\alpha_{1/2\mu_\tau}\left(\tau_i\right)$ are the spin and isospin eigenfunctions of $i-th$ nucleon. Moreover, due to the scalarity of the operators present in the potential expression, the eigenvalues are degenerate and are independent of the projection quantum numbers m_s, m_t. Thus, the dimension of this space reduces to four, corresponding to four possible sets of eigenvalues $st = 00, 01, 10, 11$, which define the supermultiplets. The matrices of operators, dependent on spin-isospin variables of two nucleons, present in the central part of NN potential, are

$$\mathbf{1}_{st} = \mathbf{1}_s\otimes\mathbf{1}_t = \begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \tag{8}$$

$$\left(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j\right)_{st} = \left(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j\right)_s\otimes\mathbf{1}_t = \begin{pmatrix}-3&0\\0&1\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix} = \begin{pmatrix}-3&0&0&0\\0&-3&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \tag{9}$$

$$\left(\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j\right)_{st} = \mathbf{1}_s\otimes\left(\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j\right)_t = \begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes\begin{pmatrix}-3&0\\0&1\end{pmatrix} = \begin{pmatrix}-3&0&0&0\\0&1&0&0\\0&0&-3&0\\0&0&0&1\end{pmatrix}, \tag{10}$$

and

$$\left[\left(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j\right)\left(\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j\right)\right]_{st} = \left(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j\right)_s\otimes\left(\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j\right)_t = \begin{pmatrix}-3&0\\0&1\end{pmatrix}\otimes\begin{pmatrix}-3&0\\0&1\end{pmatrix} = \begin{pmatrix}9&0&0&0\\0&-3&0&0\\0&0&-3&0\\0&0&0&1\end{pmatrix}. \tag{11}$$

Here $\otimes$ denotes the direct product of corresponding matrices. These matrices are linearly independent and form the complete basis for diagonal fourth order matrices. As a consequence, they can be present as linear sums of projection operators $\mathbf{P}_{st}$ to different supermultiplets st. The projection operators are

$$\mathbf{P}_{00} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{P}_{01} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (12)$$

$$\mathbf{P}_{10} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{P}_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (13)$$

The expressions for operators in terms of projectors are obvious,

$$\mathbf{1}_{st} = \mathbf{P}_{00} + \mathbf{P}_{01} + \mathbf{P}_{10} + \mathbf{P}_{11}, \quad (14)$$

$$\left(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j\right)_{st} = -3\mathbf{P}_{00} - 3\mathbf{P}_{01} + \mathbf{P}_{10} + \mathbf{P}_{11}, \quad (15)$$

$$\left(\boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j\right)_{st} = -3\mathbf{P}_{00} + \mathbf{P}_{01} - 3\mathbf{P}_{10} + \mathbf{P}_{11}, \quad (16)$$

$$\left[\left(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j\right)\left(\boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j\right)\right]_{st} = 9\mathbf{P}_{00} - 3\mathbf{P}_{01} - 3\mathbf{P}_{10} + \mathbf{P}_{11}. \quad (17)$$

The central part of the NN potential in terms of these projection operators can be written

$$V_c^{00}(r)\,\mathbf{P}_{00} + V_c^{01}(r)\,\mathbf{P}_{01} + V_c^{10}(r)\,\mathbf{P}_{10} + V_c^{11}(r)\,\mathbf{P}_{11}, \quad (18)$$

where

$$\begin{pmatrix} V_c^{00}(r) \\ V_c^{01}(r) \\ V_c^{10}(r) \\ V_c^{11}(r) \end{pmatrix} = \begin{pmatrix} 1 & -3 & -3 & 9 \\ 1 & -3 & 1 & -3 \\ 1 & 1 & -3 & -3 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} v_c(r) \\ v_{cs}(r) \\ v_{ct}(r) \\ v_{cst}(r) \end{pmatrix}. \quad (19)$$

The analogous procedure with slightly modified the same matrix can be applied for the spin-orbit

$$\begin{pmatrix} V_{ls}^{00}(r) \\ V_{ls}^{01}(r) \\ V_{ls}^{10}(r) \\ V_{ls}^{11}(r) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & -3 & -3 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} v_{ls}(r) \\ 0 \\ v_{lst}(r) \\ 0 \end{pmatrix} \quad (20)$$

and tensorial

$$\begin{pmatrix} V_t^{00}(r) \\ V_t^{01}(r) \\ V_t^{10}(r) \\ V_t^{11}(r) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & -3 & -3 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} v_t(r) \\ 0 \\ v_{tt}(r) \\ 0 \end{pmatrix} \quad (21)$$

parts of the potential. The expression for the potential, present in Eq. (6) is

$$V(r\Omega) = \sum_{st} V^{st}(r)\,\mathbf{P}_{st}\left(\sigma_i\sigma_j\tau_i\tau_j\right), \quad (22)$$

where

$$V^{st}(r) = \sum_{q=c,ls,t} V_q^{st}(r). \quad (23)$$

Due to orthogonality of the projection operators

$$\mathbf{P}_{st}\mathbf{P}_{s't'} = \delta_{st,s't'}\mathbf{P}_{st}, \quad (24)$$

the supermultiplet potentials $V^{st}(r)$ are independent; hence, this presentation is the best for further consideration and generalizations.

However, this is not the definition of channel potentials necessary for fitting and calculations. To derive an analogous expansion for the entire NN potential more refined techniques are necessary, because it is a complex operator, dependent on all two-nucleon system variables, including the angular ones.

4. Orthogonal projectors to NN channels

We now present the method and recipes to obtain projectors to the defined channels, *i.e.* the orthogonal projectors present in the expansion

$$V(r\Omega) = \sum_{jst} V^{jst}(r)\,\mathbf{P}_{jst}(\Omega). \quad (25)$$

Again, these projectors must be orthogonal, *i.e.* satisfy the condition

$$\mathbf{P}_{jst} \cdot \mathbf{P}_{j's't'} = \mathbf{P}_{jst}\delta_{jst,j's't'} \quad (26)$$

to avoid dependence and correlations between channel potentials.

To simplify the presentation we introduce matrix representations for the generalized projection operators. These representations can be realized by calculating matrices of those operators in the basis of eigenfunctions of operators, whose quantum numbers are present as subscripts of the projector. For projectors $\mathbf{P}_{st}$ these operators are $\mathbf{S}^2$ and $\mathbf{T}^2$.

Let us start the construction of corresponding matrices for the operators $\mathbf{P}_{jst}(\Omega)$. These projectors can be present as a products of the introduced projector and a simpler coprojector, projecting to the total momentum value j for a given supermultiplet st

$$\mathbf{P}_{jst}(\Omega) = \mathbf{P}_{j[st]}(\theta\varphi) \otimes \mathbf{P}_{st}\left(\sigma_i\sigma_j\tau_i\tau_j\right). \quad (27)$$

The allowed values of j can be numbered as follows:

If $(-1)^{j+s+t} = -1$, the projector is one-dimensional, because only one value of orbital momentum $l = j$ is allowed. Obviously, the dimension of the submatrix corresponding to this projector equals one.
If $(-1)^{j+s+t} = 1$, $s = 1$, the projector is of second order, *i.e.* two values of $l = j \pm 1 > 0$ are allowed. The submatrix of the projector to the coupled channel is the second order idempotent matrix of the first rank, *i.e.*

$$\mathbf{p}_{j[st],1} = \begin{pmatrix} a & -\sqrt{a(1-a)} \\ -\sqrt{a(1-a)} & 1-a \end{pmatrix}, \qquad (28)$$

where $0 < a < 1$ is an arbitrary real number. The second order projectors appear in pairs of orthogonal idempotent matrices. The second matrix for this projector is

$$\mathbf{p}_{j[st],2} = \begin{pmatrix} 1-a & \sqrt{a(1-a)} \\ \sqrt{a(1-a)} & a \end{pmatrix}. \qquad (29)$$

The entire matrix, corresponding to the projector $\mathbf{P}_{j[st]}$ of the coupled channel is a set of two orthogonal idempotent matrices with submatrices of the second order present in the arbitrary place, corresponding to the given total momentum quantum number j.
The rows and columns of projection matrices are marked by the values of quantum numbers j (and indirectly, also l), allowed for a given supermultiplet st. The mentioned channel potentials $V^{jst}(r)$, as usual, can be defined by averaging Eq. (2) with the angular and spin-isospin functions of the defined channel, *i.e.* $\Psi_{jst}(\Omega)$,

$$\int \Psi^{+}_{jst}(\Omega)\, V(r\Omega)\, \Psi_{jst}(\Omega)\, d\Omega = \sum_{p} v_p(r) \int \Psi^{+}_{jst}(\Omega)\, O_p(\Omega)\, \Psi_{jst}(\Omega)\, d\Omega, \qquad (30)$$

or in short-hand notation,

$$V^{jst}(r) = \sum_{p} v_p(r)\, O^{jst}_{p}. \qquad (31)$$

The channel functions are matrices columns of the first or second order (for the coupled channel) composed of partial waves functions

$$\Psi_{jlst}(\Omega) = \left\{ Y_l(\theta\varphi) \otimes \alpha_s\left(\sigma_i\sigma_j\right)\right\}_j \alpha_t\left(\tau_i\tau_j\right). \qquad (32)$$

The momenta projection quantum numbers m, m_t are not written here, because they do not appear in the final expressions due to the scalarity of the Hamiltonian operator.

The expression of the channel potential can be obtained by applying the above supermultiplet potentials and mean values of the operators

$$V^{jst}(r) = V^{st}_{c}(r)\langle \mathbf{1}\rangle^{jst} + V^{st}_{ls}(r)\langle \mathbf{L}\cdot\mathbf{S}\rangle^{jst} + V^{st}_{t}(r)\left\langle \mathbf{S}_{ij}\right\rangle^{jst}. \qquad (33)$$

Every supermultiplet contains an infinite number of NN channels; hence, all matrices of the operators present are infinite. As mentioned, in coupled channels like $1^{+}0$ second order matrices appear because the wave-function of this channel consists of two partial waves (for the mentioned channel these are S- and D- waves). Projection operators of infinite order cannot be constructed by applying these matrices; however, the finite order parts of these matrices can be used while building projectors of finite order. The main goal of our method is to construct these projectors and to obtain the number of projectors that can be built by applying a restricted set of operators. In the supermultiplets $st = 00$ and $st = 01$ only a scalar part of the potential is present; hence the phase shifts describable in these supermultiplets are $jst = 100$ $\left({}^1P_1\right)$ and $jst = 001$ $\left({}^1S_0\right)$ respectively. No more projectors than the first order ones can be built applying the unity matrix, corresponding to the central part of NN potential. In channels of supermultiplet $st = 10$, *i.e.* $jst = 110$ and $jst = 210$ the potential matrices

$$V^{110}(r) = \begin{pmatrix} V^{10}_{c}(r) & 2\sqrt{2}V^{10}_{t}(r) \\ 2\sqrt{2}V^{10}_{t}(r) & V^{10}_{c}(r) - 3V^{10}_{ls}(r) - 2V^{10}_{t}(r) \end{pmatrix} \qquad (34)$$

and

$$V^{210}(r) \equiv V^{2210}(r) = V^{10}_{c}(r) - V^{10}_{ls}(r) + V^{10}_{t}(r) \qquad (35)$$

are obtained. In channels of supermultiplet $st = 11$, *i.e.* $jst = 011$, $jst = 111$ and the P-wave of channel $jst = 211$, there are again only three independent radial potentials; hence, they can describe only three phase shifts in P-states. The potentials are defined as

$$V^{011}(r) \equiv V^{0111}(r) = V^{11}_{c}(r) - 2V^{11}_{ls}(r) - 4V^{11}_{t}(r), \qquad (36)$$

$$V^{111}(r) \equiv V^{1111}(r) = V^{11}_{c}(r) - V^{11}_{ls}(r) + 2V^{11}_{t}(r), \qquad (37)$$

and

$$V^{211,1}(r) \equiv V^{2111}(r) = V^{11}_{c}(r) + V^{11}_{ls}(r) - \frac{2}{5}V^{11}_{t}(r). \qquad (38)$$

Now let us formulate these results in terms of orthogonal projectors, starting with the last supermultiplet. Obviously, the third order matrices of operators present in Eq. (33) are (rows and columns of matrices are marked by partial waves of this supermultiplet, given in Tab. 1)

$$\langle \mathbf{1} \rangle = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \langle \mathbf{L}\cdot\mathbf{S} \rangle = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \langle \mathbf{S}_{ij} \rangle = \begin{pmatrix} -4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2/5 \end{pmatrix}. \tag{39}$$

The projectors corresponding to states of supermultiplet $st = 11$ can be expressed in terms of these operators,

$$\mathbf{P}_{0[11]} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{9}\langle \mathbf{1} \rangle - \frac{1}{6}\langle \mathbf{L}\cdot\mathbf{S} \rangle - \frac{5}{36}\langle \mathbf{S}_{ij} \rangle, \tag{40}$$

$$\mathbf{P}_{1[11]} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{3}\langle \mathbf{1} \rangle - \frac{1}{4}\langle \mathbf{L}\cdot\mathbf{S} \rangle + \frac{5}{24}\langle \mathbf{S}_{ij} \rangle, \tag{41}$$

$$\mathbf{P}_{2[11]} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{5}{9}\langle \mathbf{1} \rangle + \frac{5}{12}\langle \mathbf{L}\cdot\mathbf{S} \rangle - \frac{5}{72}\langle \mathbf{S}_{ij} \rangle. \tag{42}$$

The expressions for operators in terms of projectors are

$$\langle \mathbf{1} \rangle = \mathbf{P}_{0[11]} + \mathbf{P}_{1[11]} + \mathbf{P}_{2[11]}, \tag{43}$$

$$\langle \mathbf{L}\cdot\mathbf{S} \rangle = -2\mathbf{P}_{0[11]} - \mathbf{P}_{1[11]} + \mathbf{P}_{2[11]}, \tag{44}$$

$$\langle \mathbf{S}_{ij} \rangle = -4\mathbf{P}_{0[11]} + 2\mathbf{P}_{1[11]} - 2/5\mathbf{P}_{2[11]}. \tag{45}$$

Inserting these operators into the potential (Eq. (33)), the required expression of this supermultiplet potential in terms of projectors is obtained

$$V^{11}(r) = \sum_{j=0}^{2} V^{j11}(r)\,\mathbf{P}_{j[11]}, \tag{46}$$

where the channel potentials present in front of the projectors are as given in Eqs. (36) – (38).

Slightly different results can be obtained by analyzing the projectors to channels of supermultiplet $st = 10$. The submatrices of the basic operators, corresponding to the first three partial waves of this supermultiplet, *i.e.* 1010 $\left({}^3S_1\right)$, 1210 $\left({}^3D_1\right)$ and 2210 $\left({}^3D_2\right)$ are

$$\langle \mathbf{1} \rangle = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \langle \mathbf{L}\cdot\mathbf{S} \rangle = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \langle \mathbf{S}_{ij} \rangle = \begin{pmatrix} 0 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{47}$$

However, combining these matrices one can build only two orthogonal second order projectors. The projector to the last partial wave – ${}^3D_2$ cannot be constructed. Thus, this set of basic operators cannot ensure an independent description of the corresponding phase shift. Moreover, for two orthogonal projectors of this supermultiplet construction, the spin-orbit operator is not necessary. The projectors are

$$\mathbf{P}_{1[10],1} \equiv \begin{pmatrix} 1/3 & -\sqrt{2}/3 \\ -\sqrt{2}/3 & 2/3 \end{pmatrix} = \frac{1}{3}\langle \mathbf{1} \rangle - \frac{1}{6}\langle \mathbf{S}_{ij} \rangle. \tag{48}$$

$$\mathbf{P}_{1[10],2} \equiv \begin{pmatrix} 2/3 & \sqrt{2}/3 \\ \sqrt{2}/3 & 1/3 \end{pmatrix} = \frac{2}{3}\langle \mathbf{1} \rangle + \frac{1}{6}\langle \mathbf{S}_{ij} \rangle. \tag{49}$$

Expressions for the basic operators in terms of projectors can be written

$$\langle \mathbf{1} \rangle = \mathbf{P}_{1[10],1} + \mathbf{P}_{1[10],2}, \tag{50}$$

$$\langle \mathbf{S}_{ij} \rangle = 2\mathbf{P}_{1[10],2} - 4\mathbf{P}_{1[10],1}. \tag{51}$$

In this case, the potential matrix is

$$V^{110}(r) = V^{110,1}(r)\,\mathbf{P}_{1[10],1} + V^{110,2}(r)\,\mathbf{P}_{1[10],2}, \tag{52}$$

where

$$V^{110,1}(r) = V_c^{10}(r) - 4V_t^{10}(r), \quad V^{110,2}(r) = V_c^{10}(r) + 2V_t^{10}(r). \tag{53}$$

So, the complete potential operator matrix for the 1^+0 (or 110, or ${}^3S_1 - {}^3D_1$) channel is

$$V^{110}(r) = \begin{pmatrix} V_c^{10}(r) & 2\sqrt{2}V_t^{10}(r) \\ 2\sqrt{2}V_t^{10}(r) & V_c^{10}(r) - 2V_t^{10}(r) \end{pmatrix}. \tag{54}$$

This expression is not equal to the usual potential matrix of this channel definition, given in Eq. (34). We will discus this result later.

5. Character of expression in orthogonal projectors

The complete, *i.e.* infinite, expression of an arbitrary local operator can be written in terms of orthogonal projectors,

$$\mathbf{W}(r\Omega) = \sum_{\alpha=0}^{\infty} w_\alpha(r)\,\mathbf{P}_\alpha(\Omega). \tag{55}$$

Let us call $w_\alpha(r)$ channel potentials and $\mathbf{P}_\alpha(\Omega)$ projectors to the corresponding channels. The orthogonality of the projectors means that

$$\int d\Omega \mathbf{P}_\alpha(\Omega)\,\mathbf{P}_{\alpha'}(\Omega) = \delta_{\alpha,\alpha'}. \tag{56}$$

Obviously, when we apply the matrix representation of projectors, integration converts to the trace of the direct product of the corresponding matrices.
In practical applications it is only possible to take a finite number of projectors; thus, one always has a finite number of terms,

$$\mathbf{W}_N(r\Omega) = \sum_{\alpha=0}^{N} z_\alpha(r)\,\mathbf{P}_\alpha(\Omega). \tag{57}$$

The question is, what functions $z_\alpha(r)$ (channel potentials in our case) are optimal while applying this restricted set of projectors?
The best criterion is the minimal norm of the difference of exact and restricted operators,

$$\|\mathbf{W}-\mathbf{W}_N\|^2 = \int\int |\mathbf{W}(r\Omega)-\mathbf{W}_N(r\Omega)|^2\, r^2 dr d\Omega. \tag{58}$$

Inserting the above operator expressions and applying the orthogonality condition, Eq. (56), we obtain

$$\begin{aligned}\|\mathbf{W}-\mathbf{W}_N\|^2 &= \sum_{\alpha=N+1}^{\infty}\int |w_\alpha(r)|^2 r^2 dr \\ &+ \sum_{\alpha=1}^{N}\int |w_\alpha(r)-z_\alpha(r)|^2 r^2 dr \\ &\geq \sum_{\alpha=N+1}^{\infty}\int |w_\alpha(r)|^2 r^2 dr.\end{aligned} \tag{59}$$

Thus, the optimal values of "channel potentials" $z_\alpha(r)$ for a restricted expression of NN potential can be obtained using the same procedure as for the orthogonal projectors,

$$w_\alpha(r) = \int \mathbf{W}(r\Omega)\,\mathbf{P}_\alpha(\Omega)\,d\Omega. \tag{60}$$

Thus, the "channel potentials" present in front of the projection operators are independent and give the best approximation for any operator with any finite number of terms in the expansion. This result is completely analogous to the independence of Fourier series coefficients when basic functions compose the orthogonal system. Taking the potential expression in terms of basic operators (Eq. (2)) and performing the necessary integrations, one can clearly see the correlations of different parts of the potential $v_p(r)$.

6. Conclusions

The results obtained are not very surprising. We started with the assumption that the NN potential is a completely phenomenological structure and obtained a corresponding result: eight independent potentials, present in Eq. (6), ensure the description of only five phase shifts for the ${}^1S_0$, ${}^1P_1$ and ${}^3P_{0,1,2}$ partial waves and the phase shifts and mixing parameter ϵ_1 of channel ${}^3S_1 - {}^3D_1$. This conclusion is supported by experience accumulated over many years of work on constructing channel potentials. It is well-known that the $\mathbf{L}^2$ is necessary for the description of phase shifts in channels ${}^1D_2$ and ${}^1F_3$ of supermultiplets $st = 01$ and 00 respectively together with the basic channels ${}^1S_0$ and ${}^1P_1$. The description of D channels of supermultiplet $st = 10$ is also a problem; its solution is found only after introduction of a quadratic spin-orbit potential. The description of the coupled channel ${}^3P_2 - {}^3F_2$ again is a serious problem, requiring careful composition of different parts of the potential.
As a consequence, our results state that a set of eight basic operators allows potential construction only for the mentioned channels. The only way to describe more channels is to introduce additional potential operators. More operators ensures larger dimensions of the projection operators and therefore, an independent description of a larger number of NN channels. The orthogonal projector method allows the prediction of the set of these channels. The really nontrivial results obtained while applying the orthogonal projectors are the following:

1. The set of eight basic operators is not enough for a phase shift and hence, for the description of the NN potential in a ${}^3D_2$ channel;
2. With this restricted set of operators, the spin-orbit part of the NN potential is not necessary for the best description of phase shifts in the ${}^3S_1 - {}^3D_1$ channel, and the deuteron properties.

In an old publication, the authors of one of the first papers introducing a spin-orbit potential, Signell and Mar-

shak [10], state that the spin-orbit potential has to be sufficiently short-ranged to be masked by the centrifugal barrier for the deuteron D-state. The same picture emerges when studying recent realistic potentials. Thus, the spin-orbit part of the NN potential is not necessary for the description of deuteron. The dominant force controlling the deuteron structure is one-pion-exchange, which has only central and tensor interactions. Moreover, the spin-orbit terms are important in differentiating between ${}^3P_{0,1,2}$ partial waves. The fitting of deuteron properties without the spin-orbit potential opens more possibilities for the description of these phases.

The tensorial potential, constructed taking into account only channels ${}^3S_1 - {}^3D_1$, without worrying about the ${}^3D_2$ phase description, can be better fitted, keeping in mind the problems with the p-shell nuclei description and the saturation of the NN force.

These conclusions can help us to reconsider and improve the NN potential definition. Considering the NN potential as a purely phenomenological structure and applying techniques of orthogonal projectors, one can independently fit different parts of the potential by applying phase shift information step by step. Our suggestion is to construct the central part of the NN potential, consisting of three independent numerical functions $V_c^{00}(r)$, $V_c^{01}(r)$ and $V_c^{10}(r)$ using phase shifts in the partial waves ${}^1P_1$, ${}^1S_0$ and ${}^3S_1$ respectively. The tensorial part of the potential $V_t^{10}(r)$ together with the previously-defined $V_c^{10}(r)$ must serve for description of deuteron. The remaining tensorial part of the potential, $V_t^{11}(r)$, together with the function $V_c^{11}(r)$ of the central part and the function $V_{ls}^{11}(r)$ of the spin-orbit part of the potential, can be applied for the P phases of supermultiplet $st = 11$ fitting. The function $V_{ls}^{10}(r)$ of the spin-orbit potential is not necessary and can be applied for further fitting of partial waves together with the undefined radial functions present in front of the operators ($\mathbf{L}^2$, $(\mathbf{L}\cdot\mathbf{S})^2$ and may be other) in the phenomenological NN potential expansion.

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