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A q-deformed particle algebra with a unique d-dimensional representation

Research Article

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Abstract: We present an algebra generated by a single pair of creation and annihilation operators b and b^*. We prove that the algebra has a unique d-dimensional representation. Physically this algebra corresponds to a system where there are at most $d-1$ particles in a state with otherwise same quantum numbers.

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1. Introduction

The representation theory of quantum algebras with a single deformation (or quantization) parameter q has led to the development of q-deformed oscillator algebras [1–7]. Various q-deformations of the single fermion algebra and the single boson algebra have been investigated [8–13]. For undeformed oscillators the number operator $N = a^*a$ counts the possible number of particles in a given state [14]. For deformed oscillators this relation is usually replaced by $N_q = a^*a$ where the N_q is the deformed number operator.

In this paper we will be concerned with the particular deformation so that N_q has the eigenvalues

$$[n] = \frac{1-q^n}{1-q} \qquad n = 0, 1, 2, \cdots, d-1. \tag{1}$$

Since the algebra $\mathcal{A}_d(q)$ we will present has a unique d-dimensional representation it is related to the well-known orthofermion algebra $\mathcal{C}_p$ [15–19] with unique $p+1 = d$ dimensional representations. $\mathcal{C}_p$ is generated by 1, and p pairs of annihilation and creation operators c_i, c_i^* respectively which satisfy the following relations:

$$c_i c_j^* + \delta_{ij} \sum_{k=1}^{p} c_k^* c_k = \delta_{ij}, \tag{2}$$

$$c_i c_j = 0, \quad c_i^* c_j^* = 0, \quad i, j \in \{1, 2, \cdots, p\}, \tag{3}$$

where δ_{ij} stands for the Kronecker delta function. Thus $\mathcal{C}_1$ is the usual fermion algebra generated by a single pair of

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creation and annihilation operators c^* and c satisfying

$$cc^* + c^*c = 1, \quad c^2 = 0, \quad c^{*2} = 0.$$

One important difference is that $\mathcal{C}_p$ is generated by p pairs of annihilation and creation operators whereas the algebra $\mathcal{A}_d(q)$ we will describe is generated by a single pair of creation and annihilation operators. In [20] the isomorphism between $\mathcal{C}_p$ and $\mathcal{A}_d$ defined by

$$a^d = 0, \quad aa^* - a^*a = 1 - \frac{d}{(d-1)!}a^{*d-1}a^{d-1} \tag{4}$$

was established. In this paper we will introduce the algebra $\mathcal{A}_d(q)$

$$b^d = 0, \quad bb^* - qb^*b = 1 - \frac{[d]}{[d-1]!}b^{*d-1}b^{d-1}, \quad q \in \mathcal{R}, \quad q > 0, \tag{5}$$

which is the q-deformation of $\mathcal{A}_d$. Here the deformed numbers are defined as in (1) and $[d]! = [d][d-1]\cdots[1]$. In the case for $d = 2$ the algebra becomes undeformed fermion algebra. The fact that this is the undeformed fermion algebra is important since the single fermion algebra can not be deformed [21, 22]. Notwithstanding the orthofermion algebra is not deformed it is invariant under the quantum group $SU_q(d-1)$ for any deformation parameter q [23]. Its unique nontrivial $d = (p+1)$ dimensional irreducible representation has been given by $d \times d$ matrices with entries $[c_i]_{kl} = \delta_{k1}\delta_{li+1}$ and it has been shown that orthofermions are related to topological symmetries in [24].

In Section 2 we obtain $\mathcal{A}_d(q)$ (5) by starting from the orthofermion algebra $\mathcal{C}_p$ and explicitly constructing the creation and annihilation operators b and b^* in (5) in terms of c_i^*, c_i $(i = 1, 2, \cdots, p)$ in (2)-(3). Then we prove the isomorphism between $\mathcal{A}_d(q)$ and $\mathcal{C}_p$ by starting from (5) and explicitly constructing operators b_i, b_i^* $(i = 1, 2, \cdots, p)$ satisfying (2)-(3). Hence, under the isomorphism the elements b_i, b_i^* of $\mathcal{A}_d(q)$ are mapped to element c_i, c_i^* of $\mathcal{C}_p$.

In Section 3 we construct the unique representation of $\mathcal{A}_d(q)$ (5) by solely using the creation and annihilation operators b and b^* of $\mathcal{A}_d(q)$.

In Section 4 we present our conclusions.

2. The algebra $\mathcal{A}_d(q)$ and its relationship to the orthofermion algebra $\mathcal{C}_p$

We will exhibit that the algebra $\mathcal{A}_d(q)$ and $\mathcal{C}_p$ (2)-(3) are algebraically isomorphic. To prove this we exploit the fact that $\mathcal{A}_d$ and $\mathcal{C}_p$ are isomorphic which was shown in [20]. We know that the algebra $\mathcal{A}_d$ is generated by 1 and a single pair of creation and annihilation operators a^*, a satisfying (4). Then

$$N = a^*a = \Pi_1 + 2\Pi_2 + \cdots + (d-1)\Pi_{d-1} \tag{6}$$

is the number operator of $\mathcal{A}_d$, which has the spectrum $\{0, 1, \cdots, d-1\}$. Here

$$\Pi_k = \frac{(-1)^{d-1-k}}{k!(d-1-k)!}\prod_{\substack{j=0\\j\neq k}}^{d-1}(N-j), \quad k = 0, 1, \cdots, d-1 \tag{7}$$

are orthogonal projection operators such that

$$\begin{aligned}
&\Pi_0 + \Pi_1 + \cdots + \Pi_{d-1} = 1,\\
&(N-k)\Pi_k = \Pi_k(N-k) = 0, \quad k = 0, 1, 2, \cdots, d-1,\\
&\Pi_k\Pi_l = \delta_{kl}\Pi_k, \quad k, l = 1, 2, \cdots, d-1,\\
&a\Pi_k = \Pi_{k-1}a, \quad k = 1, 2, \cdots, d-1.
\end{aligned}$$

Furthermore a basis of the algebra is given explicitly by

$$\begin{aligned}
&\{x \in \mathcal{A}_d \mid x = \Pi_k,\ \Pi_k a^j,\ a^{*j}\Pi_k,\\
&k = 0, 1, \cdots, d-1,\\
&j = 1, 2, \cdots, d-1-k\}.
\end{aligned}$$

Because of the isomorphism between $\mathcal{C}_p$ and $\mathcal{A}_d$ we also have the equations

$$c_k^*c_k = \Pi_k, \quad c_kc_k^* = \Pi_0, \quad k = 1, 2, \cdots, p(= d-1). \tag{8}$$

The following steps are going to give us a new presentation of $\mathcal{C}_p$ and we will call this presentation as $\mathcal{A}_d(q)$ which is the q-deformation of $\mathcal{A}_d$.

Now we want a number operator for the orthofermion algebra $\mathcal{C}_p$, which has the spectrum $\{0, [1], \cdots, [d-1]\}$ where $[n] = \frac{1-q^n}{1-q}$ and $p = d-1$. To achieve this let us define

$$b = c_1 + \sqrt{[2]}c_1^*c_2 + \cdots + \sqrt{[p]}c_{p-1}^*c_p \tag{9}$$

in terms of the generators of the orthofermion algebra $\mathcal{C}_p$. Then

$$\begin{aligned}
b^*b &= c_1^*c_1 + [2]c_2^*c_2 + \cdots + [p]c_p^*c_p,\\
bb^* &= c_1c_1^* + [2]c_1^*c_1 + \cdots + [p]c_{p-1}^*c_{p-1}
\end{aligned} \tag{10}$$

and using (8)

$$b^*b = \Pi_1 + [2]\Pi_2 + \cdots + [p]\Pi_p,$$
$$bb^* = \Pi_0 + [2]\Pi_1 + \cdots + [p]\Pi_{p-1}. \quad (11)$$

Let us define

$$N_q = b^*b, \quad M_q = bb^* \quad (12)$$

and it can easily be seen that

$$N_q M_q = M_q N_q. \quad (13)$$

Manipulating the definition of N_q and using (7) we obtain Π_k's in terms of N_q such as

$$\Pi_k = \prod_{\substack{j=0\\ j\neq k}}^{p} \frac{N_q - [j]}{[k]-[j]}, \quad k = 0, 1, \cdots, p. \quad (14)$$

Then we get

$$\Pi_k M_q = M_q \Pi_k = [k+1]\Pi_k,$$
$$\Pi_p M_q = M_q \Pi_p = 0, \quad (15)$$

$$\Pi_k N_q = N_q \Pi_k = [k]\Pi_k. \quad (16)$$

For $q > 0$

$$M_q - qN_q - 1 = \sum_{k=1}^{p}[k]\Pi_{k-1} - q\sum_{k=0}^{p}[k]\Pi_k - \sum_{k=0}^{p}\Pi_k$$
$$= -[p+1]\Pi_p, \quad (17)$$

$$bN_q - qN_q b = \left(c_1 + \sum_{k=2}^{p}\sqrt{[k]}c_{k-1}^* c_k\right)\sum_{k=1}^{p}[k]c_k^* c_k$$
$$- q\sum_{k=1}^{p}[k]c_k^* c_k \left(c_1 + \sum_{k=2}^{p}\sqrt{[k]}c_{k-1}^* c_k\right)$$
$$= b, \quad (18)$$

$$qN_q b = b(N_q - 1), \quad (19)$$

$$b^{*p}b^p = (b^*)^{p-2}(b^* N_q b)b^{p-2}$$
$$= (b^*)^{p-2}(q^{-1}N_q(N_q-1))b^{p-2} = \cdots$$
$$= q^{-\frac{p(p-1)}{2}} N_q(N_q-1)(N_q-[2])\cdots(N_q-[p-1]). \quad (20)$$

Use (14) to obtain

$$b^{*p}b^p = [p]!\Pi_p. \quad (21)$$

From (6)

$$b^{p+1} = 0. \quad (22)$$

Let us consider relations (17), (21)-(22) which are satisfied by b, b^* for each $p = d-1$. We will use these relations to define an algebra , namely $\mathcal{A}_d(q)$; that is, generated by $1, b, b^*$ and satisfying

$$bb^* - qb^*b = 1 - \frac{[d]}{[d-1]!}b^{*d-1}b^{d-1},$$
$$b^d = 0, \quad q \in \mathcal{R}, \quad q > 0. \quad (23)$$

Conversely, starting from the algebra $\mathcal{A}_d(q)$ we turn back to $\mathcal{C}_p$, that is, there is an isomorphism from $\mathcal{A}_d(q)$ onto the orthofermion algebra $\mathcal{C}_p$. To show this first, let $N_q = b^*b$, $M_q = bb^*$ and multiply the first equation of (23) by b on the right then

$$bb^*b - qb^*bb = b, \quad (24)$$

$$b(q^{-1}(N_q - 1)) = N_q b, \quad (25)$$

$$b^{*k}b^k = q^{\frac{k(1-k)}{2}} N_q(N_q-1)(N_q-[2])\cdots(N_q-[k-1]),$$
$$k = 1, 2, \cdots, d-1, \quad (26)$$

$$0 = b^{*d}b^d = b^*\left(b^{*d-1}b^{d-1}\right)b$$
$$= q^{\frac{-d(d-1)}{2}} N_q(N_q-1)\cdots(N_q-[d-1]), \quad (27)$$

$$(N_q - [d-1])b^{*d-1}b^{d-1} = 0. \quad (28)$$

Now we will exhibit the basis elements of $\mathcal{A}_d(q)$ (23) as

$$\{x \in \mathcal{A}_d(q) \mid x = \Pi_k, \Pi_k b^j, b^{*j}\Pi_k,$$
$$k = 0, 1, \cdots, d-1, \quad j = 1, 2, \cdots, d-1-k\}, \quad (29)$$

where we now define

$$\Pi_k = \prod_{\substack{j=0\\ j\neq k}}^{d-1} \frac{N_q - [j]}{[k]-[j]}, \quad k = 0, 1, \cdots, d-1. \quad (30)$$

First, we will show that Π_k's are projection operators. From the definition of Π_k we get

$$\Pi_0 + \Pi_1 + \cdots + \Pi_{d-1} = 1, \quad (31)$$

$$N_q = \sum_{k=0}^{d-1} [k]\Pi_k. \quad (32)$$

Equation (32) for $k = d-1$ turns out to be

$$b^{*d-1} b^{d-1} = [d-1]!\Pi_{d-1}, \quad (33)$$

$$\begin{aligned} M_q &= qN_q + 1 - \frac{[d]}{[d-1]!} b^{*d-1} b^{d-1} \\ &= \sum_{k=0}^{d-1} q[k]\Pi_k + 1 - [d]\Pi_{d-1} = \sum_{k=0}^{d-2} [k+1]\Pi_k. \end{aligned} \quad (34)$$

The operator Π_k has all factors of $(N_q - [j])$, $j = 0, 1, 2, \cdots, d-1$, $j \neq k$ except that $(N_q - [k])$ and by (27)

$$(N_q - [k])\Pi_k = 0, \quad k = 0, 1, 2, \cdots, d-1, \quad (35)$$

$$\begin{aligned} (N_q - [j])\Pi_k &= N_q\Pi_k - [j]\Pi_k = ([k] - [j])\Pi_k, \\ j = 0, 1, \cdots, d-1, &\quad k = j, j+1, \cdots, d-1. \end{aligned} \quad (36)$$

Similar argument to this also give us

$$\Pi_k \Pi_l = 0, \quad k \neq l. \quad (37)$$

From (31), (37)

$$\Pi_k{}^2 = \Pi_k, \quad k = 0, 1, 2, \cdots, d-1 \quad (38)$$

and then

$$\Pi_k \Pi_l = \delta_{kl}\Pi_k, \quad k, l = 0, 1, 2, \cdots, d-1. \quad (39)$$

Rewrite (25)

$$\begin{aligned} &bN_q = (qN_q + 1)b, \\ &b \sum_{k=0}^{d-1} [k]\Pi_k = \left(\sum_{k=0}^{d-1} q[k]\Pi_k + \sum_{k=0}^{d-1} \Pi_k \right) b = \sum_{k=0}^{d-1} [k+1]\Pi_k b, \end{aligned} \quad (40)$$

from (33)

$$\Pi_{d-1} b = 0, \quad b\Pi_0 = 0, \quad (41)$$

$$\Pi_k b \Pi_l = 0, \quad l - k \neq 1, \quad k, l = 0, 1, \cdots, d-1, \quad (42)$$

$$b\Pi_k = \Pi_{k-1} b \Pi_k = \Pi_{k-1} b. \quad (43)$$

From (36), (26)

$$b^{*k} b^k \Pi_k = [k]!\Pi_k, \quad k = 1, 2, \cdots, d-1. \quad (44)$$

From (34), (43), (39)

$$b^k b^{*k} = [k]!\Pi_0, \quad k = 1, 2, \cdots, d-1. \quad (45)$$

Now we consider the elements of $\mathcal{A}_d(q)$

$$b_k = \frac{1}{\sqrt{[k]!}} \Pi_0 b^k, \quad k = 1, 2, \cdots, d-1 \quad (46)$$

and we claim that these elements with their stars generate the orthofermion algebra, hence $\mathcal{A}_d(q)$ is isomorphic to $\mathcal{C}_p$ for each $p = d-1$ and $q > 0$. We will show this:

$$\begin{aligned} b_k^* b_k &= \frac{1}{\sqrt{[k]!}} b^{*k} \Pi_0 \frac{1}{\sqrt{[k]!}} \Pi_0 b^k = \frac{1}{[k]!} b^{*k} \Pi_0 b^k \\ &= \frac{1}{[k]!} b^{*k} b^k \Pi_k = \Pi_k, \end{aligned} \quad (47)$$

$$\begin{aligned} b_k b_k^* &= \frac{1}{\sqrt{[k]!}} \Pi_0 b^k \frac{1}{\sqrt{[k]!}} b^{*k} \Pi_0 = \frac{1}{[k]!} b^k \Pi_k b^{*k} \Pi_0 \\ &= \frac{1}{[k]!} b^k b^{*k} \Pi_0 = \Pi_0 = 1 - \sum_{k=1}^{d-1} \Pi_k = 1 - \sum_{k=1}^{d-1} b_k^* b_k, \end{aligned} \quad (48)$$

$$\begin{aligned} b_i b_j^* &= \frac{1}{\sqrt{[i]!}} \Pi_0 b^i \frac{1}{\sqrt{[j]!}} b^{*j} \Pi_0 \\ &= \frac{1}{\sqrt{[i]![j]!}} b^i \Pi_i \Pi_j b^{*j} = 0, \quad i \neq j, \end{aligned} \quad (49)$$

$$\begin{aligned} b_i b_j &= \frac{1}{\sqrt{[i]!}} \Pi_0 b^i \frac{1}{\sqrt{[j]!}} \Pi_0 b^j \\ &= \frac{1}{\sqrt{[i]![j]!}} \Pi_0 b^{i-1} (b\Pi_0) b^i = 0, \end{aligned} \quad (50)$$

similarly,

$$b_i^* b_j^* = 0, \quad i, j, k = 1, 2, \cdots, d-1. \quad (51)$$

Take $j > i$

$$\begin{aligned} b_i^* b_j &= \frac{1}{\sqrt{[i]!}} b^{*i} \Pi_0 \frac{1}{\sqrt{[j]!}} \Pi_0 b^j = \frac{1}{\sqrt{[i]![j]!}} b^{*i} \Pi_0 b^j \\ &= \frac{1}{\sqrt{[i]![j]!}} b^{*i} b^i \Pi_i b^r = \sqrt{\frac{[i]!}{[j]!}} \Pi_i b^r, \quad r = j - i, \end{aligned} \quad (52)$$

and then

$$b_j^* b_i = \frac{\sqrt{[i]!}}{\sqrt{[j]!}} b^{*r} \Pi_i, \quad j = 1, 2, \cdots, d-1, \quad i = j+1, j+2, \cdots, d-1. \tag{53}$$

All above shows that b_i, b_i^* satisfy (2)-(3) and we have d^2 linearly independent elements in terms of $\Pi_k b^l$, $b^{*l}\Pi_k$, $k = 0, \cdots, d-1,\ \ l = 0, 1, \cdots, d-1-k \ \ (\Pi_k = \Pi_k^*)$.

3. Representations of the algebra $\mathcal{A}_d(q)$

The creation and annihilation operators of $\mathcal{A}_d(q)$ for each d satisfy

$$bb^*b - qb^*bb = b, \quad b^d = 0. \tag{54}$$

Considering the first equation of (54) and then multiplying it by b^* on the right gives us

$$M_q(M_q - qN_q - 1) = 0. \tag{55}$$

Since N_q and M_q commute they have simultaneous eigenvectors, namely $|\nu, \mu\rangle$ for which ν and μ eigenvalues of N_q and M_q respectively. Hence (55) implies

$$\mu(\mu - q\nu - 1) = 0. \tag{56}$$

In the case $\mu \neq 0, \ \ \mu = q\nu + 1$, the relations

$$N_q b = q^{-1} b(N_q - 1), \tag{57}$$

$$N_q b^* = b^*(qN_q + 1) \tag{58}$$

give us that

$$\nu = [k] \tag{59}$$

for some $k \in \mathcal{N}$. This fact is exactly the same as the representation of the q-deformed boson algebra

$$bb^* - qb^*b = 1. \tag{60}$$

However due to the second equation of (54) $b^d = 0$ this can not happen for all k.
In the case $\mu = 0$ the similar way performed in [20] is applied. Starting with the vector $|d-1, 0\rangle$ we will construct a finite dimensional representation. For $d = 1$, $b|0,0\rangle = 0$ and $b^*|0,0\rangle = 0$ hence we have the only vector $|0,0\rangle$ and one dimensional trivial representation.

$$b^*|d-1, 0\rangle = 0 \tag{61}$$

which immediately follows from the fact that the norm of this vector is zero since $|d-1, 0\rangle$ is an eigenvector of $M_q = bb^*$ with eigenvalue zero. Since we have (59) the equations

$$N_q b|d-1, 0\rangle = [d-2] b|d-1, 0\rangle, \tag{62}$$

$$M_q b|d-1, 0\rangle = [d-1] b|d-1, 0\rangle \tag{63}$$

suggest that

$$b|d-1, 0\rangle = \sqrt{[d-1]}|d-2, d-1\rangle, \tag{64}$$

$$b^*|d-2, d-1\rangle = \sqrt{[d-1]}|d-1, 0\rangle. \tag{65}$$

Similarly, we have

$$b|k-1, k\rangle = \sqrt{[k-1]}|k-2, k-1\rangle, \quad k = d-1, d-2, \cdots 2, \tag{66}$$

$$b|0, 1\rangle = 0, \tag{67}$$

$$b^*|k-1, k\rangle = \sqrt{[k]}|k, k+1\rangle, \quad k = d-2, d-3, \cdots 1. \tag{68}$$

Thus by this method we end up with the unique d-dimensional representations.

4. Conclusion

We proved that using the creation and annihilation operators of the q-deformed algebra $\mathcal{A}_d(q)$ defined by (5), the creation and annihilation operators of $\mathcal{C}_p$ (2)-(3) can uniquely be constructed. Then we have a unique $d = p+1$ dimensional representation of q-deformed algebra. Two aspects of this construction are important. The first is that although $\mathcal{A}_d(q)$ has one pair of creation and annihilation operators, $\mathcal{C}_p$ has p pairs of creation and annihilation operators. The second important point is that although $\mathcal{A}_d(q)$ is deformed, $\mathcal{C}_p$ is undeformed. Representations of orthofermion algebra have states $c_n^*|0\rangle$ corresponding to a state containing the nth orthofermion, whereas in representation of $\mathcal{A}_d(q)$ this corresponds to an n-particle state. The q-oscillator (60) has a multidimensional generalization with important physical and mathematical properties [25, 26]. Such generalization of the particle algebra considered in this paper should be interesting and useful.

It will also be interesting and useful to investigate the two parameter generalization $\mathcal{A}_d(q_1, q_2)$. These would be similar to two parameter [27, 28] generalizations of the q-oscillator which turn out to be important for thermodynamics [29, 30].

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