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Conservation laws and associated Lie point symmetries admitted by the transient heat conduction problem for heat transfer in straight fins

Research Article

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Abstract: Some new conservation laws for the transient heat conduction problem for heat transfer in a straight fin are constructed. The thermal conductivity is given by a power law in one case and by a linear function of temperature in the other. Conservation laws are derived using the direct method when thermal conductivity is given by the power law and the multiplier method when thermal conductivity is given as a linear function of temperature. The heat transfer coefficient is assumed to be given by the power law function of temperature. Furthermore, we determine the Lie point symmetries associated with the conserved vectors for the model with power law thermal conductivity.

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1. Introduction

A very useful application of symmetry techniques is to construct conservation laws of the given differential equation (DE). A conservation law of a system of partial differential equations (PDEs) is a divergence expression which vanishes on solutions of a system of partial differential equations [1]. Conservation laws play a pivotal role in the study of DEs and in many applications. The mathematical idea of conservation laws comes from the formulation of physical laws such as for mass, energy and momentum. Furthermore, conservation laws have applications in the study of PDEs such as in showing existence and uniqueness of solutions for hyperbolic systems of conservation laws [1], and as well as in developing numerical methods such as finite element methods [2, 3].

There are nine approaches to construct the conservation laws for PDEs [4]. An elegant way of constructing conservation laws is by use of the Noether's theorem [5]. This theorem states that for Euler-Lagrange differential equations, to each Noether symmetry associated with a Lagrangian there corresponds a conservation law which can

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be determined explicitly by means of a formula. The application of Noether's theorem depends on the knowledge of a Lagrangian.

Here, we focus on deterministic models given in terms of PDEs describing heat transfer in extended surfaces (fins). Fins or extended surfaces are important elements for increasing the efficiency and effectiveness of heating systems. A fin is used to enhance heat dissipation from a hot surface through its convective, radiative, or convective-radiative surface [6]. In particular fins are used extensively in various industrial applications such as the cooling of computer processors, air conditioning and oil carrying pipe lines. A well documented review of heat transfer is presented by Kraus *et al.* [6]. Fin problems have attracted some interest from the symmetry analysts. Lie symmetry analysis of a nonlinear fin equation in which the thermal conductivity is an arbitrary function of the temperature and the heat transfer coefficient is an arbitrary function of a spatial variable was performed, for example, by [7–11]. Recently, the study of fins in boiling liquids has been increasing enormously and it has been found that the heat transfer coefficient may not only be given by a constant but also depends on the temperature distribution between the heated surface and its adjacent fluid [12], see also [13]. Thus, the resulting equations becomes highly nonlinear even in the simplest one-dimensional analysis [14].

Vaneeva *et al.* [10] constructed the conservation laws for a fin problem with spatial dependent heat transfer coefficient. As such, this study present yet a significant advancement in the understanding of heat conduction models for heat transfer in extended surfaces. Studies on conservation laws using both direct and multiplier methods (pioneered by Bluman and coauthors) may be found in [15–17]. Naz *et al* [4] compared different techniques for constructing conservation laws of some differential equation in fluid mechanics. Fin equation has no Lagrangian, thus we cannot apply this Noether's theorem. One may recall Noether's approach requires the equation to have a Lagragian. However, here we construct conservation laws using both the direct and the multiplier method. Some of the constructed conservation laws have never appeared in the literature. We also find Lie point symmetries associated with the conserved vectors for the model with power law thermal conductivity.

This paper is organized as follows; the mathematical background of the problem under consideration is described in Section 2. Construction of conservation laws for the nonlinear fin equation is provided in Section 3. We provide Lie point symmetries associated with the conserved vectors in Section 4. Lastly, we provide some discussions based on the results obtained in Section 5

2. Mathematical models

A physical system consisting of a longitudinal one dimensional fin of cross-sectional area A_c is shown in Figure 1. The perimeter of the fin is denoted by P and its length by L. The fin is attached to a fixed prime surface of temperature T_b and extends to an ambient fluid of temperature T_a. The fin thickness at the prime surface is given by δ_b and its profile is given by $F(X)$. We assume that the fin is initially at ambient temperature. At time $t = 0$, the temperature at the base of the fin is suddenly changed from T_a to T_b and the problem is to establish the temperature distribution in the fin for all $t \geq 0$. Based on the one dimensional heat conduction, the energy balance equation is then given by (see e.g. [6])

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial X}\left(\frac{\delta_b}{2}F(X)K(T)\frac{\partial T}{\partial X}\right) - \frac{P}{A_c}H(T)(T - T_a), \quad 0 \leq X \leq L. \tag{1}$$

where K and H are the non-uniform thermal conductivity and heat transfer coefficients depending on the temperature (see e.g. [6]), ρ is the density, c is the specific heat capacity, T is the temperature distribution, t is the time and X is the space variable. Assuming that the fin tip is adiabatic (insulated) and the base temperature is kept constant, then the boundary conditions are given by [6],

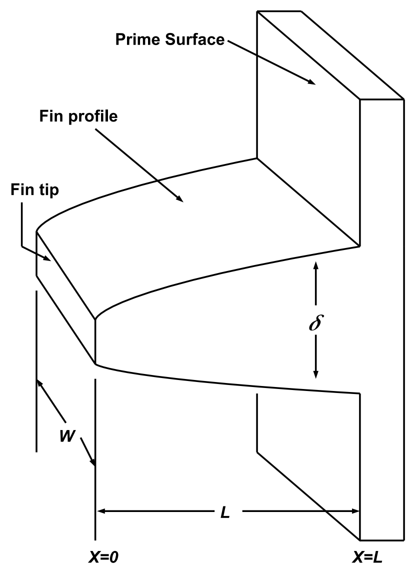

Figure 1. Schematic representation of a longitudinal fin of an unspecified profile.

$$T(t,L) = T_b \quad \text{and} \quad \left.\frac{\partial T}{\partial X}\right|_{X=0} = 0, \tag{2}$$

and initially the fin is kept at the ambient temperature,

$$T(0,X) = T_a. \tag{3}$$

Introducing the following dimensionless variables,

$$\begin{aligned} x &= \frac{X}{L},\ \tau = \frac{k_a t}{\rho c_v L^2},\ \theta = \frac{T - T_a}{T_b - T_a},\ h = \frac{H}{h_b}, \\ k &= \frac{K}{k_a},\ M^2 = \frac{P h_b L^2}{A_c k_a} \text{ and } f(x) = \frac{\delta_b}{2} F(X), \end{aligned} \tag{4}$$

reduces Eq. (1) to

$$\frac{\partial\theta}{\partial\tau} = \frac{\partial}{\partial x}\left(f(x)k(\theta)\frac{\partial\theta}{\partial x}\right) - M^2 h(\theta)\theta,\ 0 \le x \le 1 \tag{5}$$

and the prescribed initial and boundary conditions are given by

$$\theta(0,x) = 0,\ 0 \le x \le 1; \tag{6}$$

$$\left.\frac{\partial\theta}{\partial x}\right|_{x=0} = 0, \quad \theta(\tau,1) = 1. \tag{7}$$

The dimensionless variable M is the thermo-geometric fin parameter, θ is the dimensionless temperature, x is the dimensionless space variable, k is the dimensionless thermal conductivity, k_a is the thermal conductivity of the fin at ambient temperature, h_b is the heat transfer coefficient at the fin base. For most industrial application, the heat transfer coefficient maybe given as a power law [18, 19],

$$H(T) = h_b\left(\frac{T - T_a}{T_b - T_a}\right)^n, \tag{8}$$

where n and h_b are constants. The constant n may vary between -6.6 and 5. However, in most practical applications it lies between -3 and 3 [19]. The exponent n represents laminar film boiling or condensation when $n = -1/4$, laminar natural convection when $n = 1/4$, turbulent natural convection when $n = 1/3$, nucleate boiling when $n = 2$, radiation when $n = 3$ and $n = 0$ implies a constant heat transfer coefficient. Exact solutions may be constructed for the steady-state one-dimensional differential ODE describing temperature distribution in a straight fin when the thermal conductivity is a constant and $n = -1, 0, 1$ and 2 [19].

In dimensionless variables we have $h(\theta) = \theta^n$. We consider the thermal conductivity to be given by a power law function of temperature as follows;

$$K(T) = k_a\left(\frac{T - T_a}{T_b - T_a}\right)^m. \tag{9}$$

with m being a constant. The dimensionless thermal conductivity given by $k(\theta) = \theta^m$. Indeed, thermal conductivity of some material such as Gellium Nitride (GaN) and Aluminium Nitride (AlN) can be modeled by the power law (see e.g. [20]). Furthermore, the experimental data indicates that the exponent of the power law for these materials is positive for lower temperatures and negative for at higher temperatures [21–23]. The classical symmetry analysis of models transient heat transfer in rectangular longitudinal fin of power law thermal conductivity is performed by Mhlongo *et al.* [24].
On the other hand, thermal conductivity is given as a linear function of temperature in many Engineering applications (see e.g. [6, 25]). That is, we have

$$K(T) = k_a\left[1 + \lambda(T - T_a)\right], \tag{10}$$

where λ is the thermal conductivity gradient. The dimensionless thermal conductivity given by $k(\theta) = 1 + \beta\theta$, where thermal conductivity parameter $\beta = \lambda(T_b - T_a)$ is nonzero.

3. Conservation laws

In this section we construct conservation laws for the transient heat conduction equation. A conservation law of a given system of PDEs is a divergence expression that vanishes provided on all solutions of the system of PDEs. Consider a lth order single PDE,

$$F(\mathbf{x}, u, u_{(1)}, u_{(2)}, \ldots, u_{(l)}) = 0, \tag{11}$$

where $\mathbf{x}$ denotes r independent variables, u denotes the dependent variable and $u_{(i)}$ denote all the partial derivatives of order i. For an arbitrary PDE we write

$$D_i(T^i) = 0, \tag{12}$$

where T^i are differential functions of finite order and D_i is the total derivative defined by

$$D_i = \frac{\partial}{\partial x^i} + u_i\frac{\partial}{\partial u} + u_{ij}\frac{\partial}{\partial u_i} + \ldots + u_{ii_1 i_2 \ldots i_s}\frac{\partial}{\partial u_{i_1 i_2 \ldots i_r}} + \ldots, \quad i = 1, 2, \ldots r. \tag{13}$$

We define (12) as a conservation law for Eq. (11) if it satisfies the following equation.

$$D_i\big[T^i(\mathbf{x}, u, u_{(1)}, u_{(2)}, \ldots, u_{(l)})\big] = 0. \tag{14}$$

This can also be written as

$$D_i T^i\big|_{F=0} = 0. \tag{15}$$

The vector $T = (T^1, T^1, \ldots, T^r)$ is called a conserved vector. A Lie point symmetry generator

$$\Gamma = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta(x,u)\frac{\partial}{\partial u} + \zeta_i\frac{\partial}{\partial u_i} + \zeta_{ij}\frac{\partial}{\partial u_{ij}} + \ldots, \tag{16}$$

where

$$\zeta_{i_1 i_2 \ldots i_r} = D_{i_1 \ldots i_r}(W) + \zeta^i u_{i i_1 i_2 \ldots i_r}, \quad i = 1, 2, \ldots, r$$

and W is the Lie characteristic function defined by $W = \eta - \xi^i u_i$. The symmetry generator (16) is said to be associated with the conserved vector $T^i = (T^1, \ldots, T^r)$ for Eq. (11) if [26, 27]

$$\Gamma(T^i) + T^i D_l(\xi^l) - T^l D_l(\xi^i) = 0, \; i = 1, \ldots, r. \tag{17}$$

In Eq. (17), Γ is prolonged appropriately. The conserved vectors can be determined from Eq. (17) [26, 27]. We determine conservation laws using two techniques outlined below.

Direct method

The direct method gives all local conservation laws. Equation (12) is a conservation law. The direct method uses Eq. (12) subject to the Eq. (11) being satisfied as the determining equation for the conserved vectors. The components $T^1, \ldots, T^r$ are obtained by separating the resulting equation according to powers and products of the derivatives of u.

Multiplier method

This approach involves the variational derivative

$$D_i T^i = \Lambda^\alpha F, \tag{18}$$

where Λ^α are the characteristics. The characteristics are the multipliers which make the equation exact. This method is sometimes referred to this as the characteristic method.
Here we utilize both the direct and the multiplier method [15, 17] to construct conservation laws. Conservation laws for partial differential equations do not depend on the boundary conditions. Hence we consider two equations, namely,

(i) the nonlinear PDE with power law thermal conductivity

$$\frac{\partial\theta}{\partial\tau} = \frac{\partial}{\partial x}\left(\theta^m\frac{\partial\theta}{\partial x}\right) - M^2\theta^{n+1}, \tag{19}$$

(ii) the nonlinear PDE with linear thermal conductivity

$$\frac{\partial\theta}{\partial\tau} = \frac{\partial}{\partial x}\left((1+\beta\theta)\frac{\partial\theta}{\partial x}\right) - M^2\theta^{n+1}. \tag{20}$$

Here m and n are arbitrary constants. Equations (19) and (20) describe heat transfer in longitudinal fins of rectangular profile. Note that Eqs. (19) and (20) are subclasses of equations considered in [28].

3.1. Fin equation with power law thermal conductivity

In this section we employ the direct method for construction of conservation laws. A conservation law for Eq. (19) satisfies

$$D_1 T^1 + D_2 T^2\Big|_{\text{Eq. (19)}} = 0, \tag{21}$$

where D_1 and D_2 are the total derivatives defined in Eq. (13).
For simplicity and convenience we seek the conserved vectors of the form

$$T^1 = (\tau, x, \theta, \theta_x), \quad T^2 = (\tau, x, \theta, \theta_x). \tag{22}$$

Substituting Eq. (22) into Eq. (21) we obtain the following partial differential equation,

$$\left(\frac{\partial T^1}{\partial\tau} + \theta_\tau\frac{\partial T^1}{\partial\theta} + \theta_{x\tau}\frac{\partial T^1}{\partial\theta_x} + \frac{\partial T^2}{\partial x} + \theta_x\frac{\partial T^2}{\partial\theta} + \theta_{xx}\frac{\partial T^2}{\partial\theta_x}\right)\Bigg|_{\text{Eq. (19)}} = 0. \tag{23}$$

Expanding Eq. (23) we obtain,

$$\frac{\partial T^1}{\partial\tau} + \left[m\theta^{m-1}\left(\frac{\partial\theta}{\partial x}\right)^2 + \theta^m\frac{\partial^2\theta}{\partial x^2} - M^2\theta^{n+1}\right]\frac{\partial T^1}{\partial\theta} +$$

$$\theta_{x\tau}\frac{\partial T^1}{\partial\theta_x} + \frac{\partial T^2}{\partial x} + \theta_x\frac{\partial T^2}{\partial\theta} + \theta_{xx}\frac{\partial T^2}{\partial\theta_x} = 0. \tag{24}$$

Since T^1 and T^2 are independent of the second derivatives of θ, we can separate Eq. (24) by second derivatives of θ:

$$\theta_{xx} \quad : \quad \theta^m \frac{\partial T^1}{\partial \theta} + \frac{\partial T^2}{\partial \theta_x} = 0, \tag{25}$$

$$\theta_{x\tau} \quad : \quad \frac{\partial T^1}{\partial \theta_x} = 0, \tag{26}$$

$$\text{rem} \quad : \quad \frac{\partial T^1}{\partial \tau} + m\theta^{m-1}\theta_x^2 \frac{\partial T^1}{\partial \theta} + \frac{\partial T^2}{\partial x} + \theta_x \frac{\partial T^2}{\partial \theta} - M^2\theta^{n+1}\frac{\partial T^1}{\partial \theta} = 0. \tag{27}$$

Equation (26) implies that $T^1 = T^1(\tau, x, \theta)$. Thus, we can integrate Eq. (25) with respect to θ_x to obtain an explicit expression for T^2,

$$T^2 = -\theta^m \frac{\partial T^1}{\partial \theta}\theta_x + A(\tau, x, \theta). \tag{28}$$

Substituting T^2 into Eq. (27) and separating by θ_x:

$$\theta_x^2 \quad : \quad -\theta^m \frac{\partial^2 T^1}{\partial \theta^2} = 0, \tag{29}$$

$$\theta_x \quad : \quad -\theta^m \frac{\partial^2 T^1}{\partial \theta \partial x} + \frac{\partial A}{\partial \theta} = 0 \tag{30}$$

$$\text{rem} \quad : \quad \frac{\partial T^1}{\partial \tau} + \frac{\partial A}{\partial x} - M^2\theta^{n+1}\frac{\partial T^1}{\partial \theta} = 0. \tag{31}$$

Integrating Eq. (29) twice with respect to θ yields an explicit expression for T^1,

$$T^1 = B(\tau, x)\theta + C(\tau, x). \tag{32}$$

Equation (30) takes the form

$$-\theta^m \frac{\partial B}{\partial x} + \frac{\partial A}{\partial \theta} = 0. \tag{33}$$

Integrating Eq. (33) with respect to θ yields an explicit expression for A,

$$A = \frac{\theta^{m+1}}{m+1}\frac{\partial B}{\partial x} + D(\tau, x), \qquad m \neq -1. \tag{34}$$

Substituting A and T^1 into Eq. (31), we obtain,

$$\theta \frac{\partial B}{\partial \tau} + \frac{\partial C}{\partial \tau} + \frac{\theta^{m+1}}{m+1}\frac{\partial^2 B}{\partial x^2} + \frac{\partial D}{\partial x} - M^2\theta^{n+1}B = 0 \tag{35}$$

We now consider two cases to separate Eq. (35) by powers of θ.

Case $m \neq n \;\; (m \neq 0, n \neq 0, n \neq -1)$

$$\theta \quad : \quad \frac{\partial B}{\partial \tau} = 0, \tag{36}$$

$$\theta^{m+1} \quad : \quad \frac{1}{m+1}\frac{\partial^2 B}{\partial x^2} = 0, \tag{37}$$

$$\theta^{n+1} \quad : \quad -M^2 B = 0, \tag{38}$$

$$\text{rem} \quad : \quad \frac{\partial C}{\partial \tau} + \frac{\partial D}{\partial x} = 0. \tag{39}$$

Equation (36) implies that $B = B(x)$ and Eq. (37) implies that $B = ax + b$, where a and b are constants. Equation (38) implies that $M = 0$ or if $M \neq 0$ then $B = 0$ and we get trivial conservation laws. Note that Eq. (39) is directly satisfied, which implies that $D(\tau, x) = 0$, $C(\tau, x) = 0$. For non trivial conservation laws we get

$$A = \frac{\theta^{m+1}}{m+1}a, \tag{40}$$

where $\theta(\tau, x)$ is a solution of Eq. (19). Finally,

$$D_\tau\big[(ax+b)\theta\big] + D_x\left[-\theta^m(ax+b)\theta_x + a\frac{\theta^{m+1}}{m+1}\right] = 0. \tag{41}$$

A conserved vector of the Eq. (19) with $M = 0$ is therefore a linear combination of the two conserved vectors

$$T^1 = \theta, \quad T^2 = -\theta^m\theta_x, \tag{42}$$

$$T^1 = x\theta, \quad T^2 = -x\theta^m\theta_x + \frac{\theta^{m+1}}{m+1}. \tag{43}$$

Conserved vectors in (42) and (43) were obtained in [29, 30]. Note that the thermo-geometric fin parameter is proportional to the aspect ratio, that is $M = (Bi)^{1/2}E$, where $Bi = (h_b\delta_b)/k_a$ is the Biot number and $E = 2L/\delta_b$ is the aspect ratio or extension factor. As such, $M = 0$ is physically insignificant for fins but model other phenomena. The case $M \neq 0$ gives trivial conservation laws only. One may assume

$$T^i = T(\tau, x, \theta, \theta_x, \theta_\tau, \theta_{\tau x}, \theta_{\tau\tau}, \theta_{xx}) \tag{44}$$

which is computationally expensive. We now consider the second case when both thermal conductivity and the heat transfer coefficient are given by the same power law.

Case $m = n$

$$\theta^{m+1} \quad : \quad \frac{1}{m+1}\frac{\partial^2 B}{\partial x^2} - M^2 B = 0, \tag{45}$$

$$\theta \quad : \quad \frac{\partial B}{\partial \tau} = 0, \tag{46}$$

$$\text{rem} \quad : \quad \frac{\partial C}{\partial \tau} + \frac{\partial D}{\partial x} = 0. \tag{47}$$

We note that this case applies for $m = n \neq -1$ and $m = n \neq 0$. The general solution of Eq. (45) is given by,

$$B(x) = c_1 e^{\sqrt{1+m}Mx} + c_2 e^{-\sqrt{1+m}Mx}, \tag{48}$$

and thus,

$$A = \frac{\theta^{m+1}}{m+1}(c_1 \omega e^{\omega x} - c_2 \omega e^{-\omega x}), \tag{49}$$

where, $\omega = \sqrt{1+m}M$.

$$D_\tau\left[(c_1 e^{\omega x} + c_2 e^{-\omega x})\theta\right] + D_x\left[-\theta^m(c_1 e^{\omega x} + c_2 e^{-\omega x})\theta_x + \frac{\theta^{m+1}}{m+1}(c_1\omega e^{\omega x} - c_2 \omega e^{-\omega x})\right] = 0. \tag{50}$$

The non trivial conserved vectors are,

$$T^1 = \theta e^{\omega x}, \quad T^2 = e^{\omega x}\left(\frac{\theta^{m+1}}{m+1}\omega - \theta_x \theta^m\right), \tag{51}$$

$$T^1 = \theta e^{-\omega x}, \quad T^2 = -e^{-\omega x}\left(\frac{\theta^{m+1}}{m+1}\omega + \theta_x \theta^m\right). \tag{52}$$

The conservation laws (51) and (52) were derived in [28]. If consideration is performed over real field, then (51) and (52) are conservation laws for Eq. (19) with $n = m$ for only $m+1 > 0$. If $m+1 < 0$, then

$$B(x) = c_1 \sin \omega x + c_2 \cos \omega x,$$

where $\omega = \sqrt{|m+1|}|M|$ and the corresponding conservation laws have the form

$$T^1 = \theta \sin \omega x, \qquad T^2 = -\theta_x \theta^m \sin \omega x + \omega \frac{\theta^{m+1}}{m+1} \cos \omega x$$

and

$$T^1 = \theta \cos \omega x, \qquad T^2 = -\theta_x \theta^m \cos \omega x - \omega \frac{\theta^{m+1}}{m+1} \sin \omega x.$$

Equation (34) applies only for $m \neq -1$. A special case $m = -1$, results in extra conserved quantities as follows;

$$A = \ln \theta \frac{\partial B}{\partial x} + D(\tau, x). \tag{53}$$

A conserved vector of the PDE (19) with $m = -1$ is a linear combination of the two conserved vectors,

$$T^1 = \theta, \quad T^2 = -\theta^{-1}\theta_x, \tag{54}$$

$$T^1 = x\theta, \quad T^2 = -x\theta^{-1}\theta_x + \ln \theta. \tag{55}$$

We also observe that the separation of Eq. (35) requires $m \neq 0$, $n \neq 0$ and $n \neq -1$. Given $n = -1$ we obtain the following conserved vectors,

$$T^1 = \theta, \quad T^2 = -\theta^m \theta_x + M^2 x, \tag{56}$$

$$T^1 = x\theta, \quad T^2 = x\left(\frac{M^2 x}{2} - \theta^m \theta_x\right) + \frac{\theta^{m+1}}{m+1}. \tag{57}$$

The conservation laws (56) and (57) have not been recorded in the literature.

3.2. Fin equation with linear thermal conductivity

One may show that the conservation laws of Eq. (20) with $M = 0$ are given by

$$T^1 = \theta, \quad T^2 = -\theta_x(1 + \beta\theta), \tag{58}$$

$$T^1 = x\theta, \quad T^2 = -x\theta_x(1 + \beta\theta) + \theta + \beta\frac{\theta^2}{2}, \tag{59}$$

(see e.g. [29, 30]). In this section we utilize the method of multipliers. A multiplier Λ for Eq. (20) with $M \neq 0$ has the property that

$$\Lambda\left(\frac{\partial\theta}{\partial\tau} - \frac{\partial}{\partial x}\left((1+\beta\theta)\frac{\partial\theta}{\partial x}\right) + M^2\theta^{n+1}\right) = D_1 T^1 + D_2 T^2 \tag{60}$$

for all functions $\theta(\tau, x)$ and the determining equation is

$$E_\theta\left[\Lambda\left(\frac{\partial\theta}{\partial\tau} - \frac{\partial}{\partial x}\left((1+\beta\theta)\frac{\partial\theta}{\partial x}\right) + M^2\theta^{n+1}\right)\right] = 0, \tag{61}$$

where E_θ is the standard Euler operator given by

$$E_\theta = \frac{\partial}{\partial\theta} - D_x\frac{\partial}{\partial\theta_x} - D_\tau\frac{\partial}{\partial\theta_\tau} + D_x^2\frac{\partial}{\partial\theta_{xx}} + D_xD_\tau\frac{\partial}{\partial\theta_{x\tau}} + D_\tau^2\frac{\partial}{\partial\theta_{\tau\tau}} - \dots. \tag{62}$$

$$\theta_\tau\frac{\partial\Lambda}{\partial\theta} - \theta_{xx}\frac{\partial\Lambda}{\partial\theta} - \beta\theta_{xx}\Lambda - \beta\theta\theta_{xx}\frac{\partial\Lambda}{\partial\theta} - \beta\theta_x^2\frac{\partial\Lambda}{\partial\theta} + M^2(n+1)\theta^n\Lambda + M^2\theta^{n+1}\frac{\partial\Lambda}{\partial\theta} + 2\beta\theta_x\frac{\partial\Lambda}{\partial x} + 2\beta\theta_x^2\frac{\partial\Lambda}{\partial\theta} + 2\beta\theta_{xx}\Lambda - \frac{\partial\Lambda}{\partial\tau}$$
$$- \theta_\tau\frac{\partial\Lambda}{\partial\theta} - \frac{\partial^2\Lambda}{\partial x^2} - \theta_x\frac{\partial^2\Lambda}{\partial\theta\partial x} - \theta_x\frac{\partial^2\Lambda}{\partial\theta\partial x} - \theta_x^2\frac{\partial^2\Lambda}{\partial\theta^2} - \theta_{xx}\frac{\partial\Lambda}{\partial\theta} - \beta\theta\frac{\partial^2\Lambda}{\partial x^2} - \beta\theta_x\frac{\partial\Lambda}{\partial x} - \beta\theta_x\frac{\partial^2\Lambda}{\partial\theta\partial x} - \beta\theta_x\frac{\partial\Lambda}{\partial x} - \beta\theta_x^2\frac{\partial\Lambda}{\partial\theta} - \beta\theta_{xx}\Lambda$$
$$- \beta\theta\theta_x\frac{\partial^2\Lambda}{\partial\theta\partial x} - \beta\theta_x^2\frac{\partial\Lambda}{\partial\theta} - \beta\theta\theta_x^2\frac{\partial^2\Lambda}{\partial\theta^2} - \beta\theta\theta_{xx}\frac{\partial\Lambda}{\partial\theta} = 0. \tag{63}$$

Since Eq. (63) is satisfied for all functions $\theta(\tau, x)$ it can be separated by equating the coefficients of the partial derivatives of $\theta(\tau, x)$. The coefficients of θ_{xx} give

$$(1+\beta\theta)\frac{\partial\Lambda}{\partial\theta} = 0 \tag{64}$$

and $(1+\beta\theta) \neq 0$, thus $\Lambda = \Lambda(\tau, x)$. Eq. (63) reduces to

$$M^2(n+1)\theta^n\Lambda - \frac{\partial\Lambda}{\partial\tau} - \frac{\partial^2\Lambda}{\partial x^2} = 0. \tag{65}$$

Separating Eq. (65) by powers of θ gives

$$\theta^n \ : \ (n+1)M^2\Lambda = 0, \tag{66}$$
$$\theta^0 \ : \ \frac{\partial\Lambda}{\partial\tau} + \frac{\partial^2\Lambda}{\partial x^2} = 0. \tag{67}$$

We note that the separation requires $n \neq 0$. Equation (66) implies that $n = -1$ or $\Lambda = 0$. From Eq. (67), setting each term to zero we observe that $\Lambda = \Lambda(x)$ and

$$\Lambda(x) = c_1x + c_2 \tag{68}$$

where c_1 and c_2 are constants. From Eqs. (60) and (68) and by performing elementary manipulations,

$$(c_1 + c_2x)\left(\frac{\partial\theta}{\partial\tau} - \frac{\partial}{\partial x}\left((1+\beta\theta)\frac{\partial\theta}{\partial x}\right) + M^2\right) =$$
$$D_\tau[(c_1x + c_2)\theta] + D_x\Bigg[-(1+\beta\theta)(c_1x+c_2)\theta_x +$$
$$\left(\theta + \beta\frac{\theta^2}{2}\right)c_1 + M^2x\left(\frac{c_1x}{2} + c_2\right)\Bigg] \tag{69}$$

for all functions $\theta(\tau, x)$. Thus when $\theta(\tau, x)$ is a solution of (20)

$$D_\tau[(c_1x + c_2)\theta] + D_x\Bigg[-(1+\beta\theta)(c_1x+c_2)\theta_x +$$
$$\left(\theta + \beta\frac{\theta^2}{2}\right)c_1 + M^2x\left(\frac{c_1x}{2} + c_2\right)\Bigg] = 0. \tag{70}$$

A conserved vector of the PDE (20) with a multiplier of the form $\Lambda(t, x, u)$ is therefore a linear combination of the two conserved vectors

$$T^1 = \theta, \quad T^2 = -(1+\beta\theta)\theta_x + M^2x, \tag{71}$$

$$T^1 = x\theta, \quad T^2 = x\left(\frac{M^2x}{2} - \theta_x(1+\beta\theta)\right) + \theta + \beta\frac{\theta^2}{2}. \tag{72}$$

The conserved vectors (71) and (72) therefore form a basis of conserved vectors of the PDE (20) with multipliers of the form $\Lambda(\tau, x, \theta)$ and these are new.

4. Conserved vectors and associated point symmetries

We have derived conserved vectors for the two transient heat conduction equations. We notice that the conserved vectors obtained using the direct method for the transient equation with linear thermal conductivity were independent of the thermo-geometric fin parameter M. Interesting observations were made for fin equation with the power law thermal conductivity. In this case (given the power law thermal conductivity) we obtained conserved vectors that are independent of M for the general case where $m \neq n$

using the direct method, while the method of multipliers gave physically sound conservation laws. Hence we focus on finding Lie point symmetries associated with the conserved vectors for the fin equation with power law thermal conductivity only. The determining equation for the Lie point symmetries Γ associated with the conserved vector $T = (T^1, T^2)$ is given by

$$\Gamma(T^i) + T^i D_l(\xi^l) - T^l D_l(\xi^i) = 0. \tag{73}$$

Eq. (73) consists of two components

$$\Gamma(T^1) + T^1 D_2(\xi^2) - T^2 D_2(\xi^1) = 0, \tag{74}$$

$$\Gamma(T^2) + T^2 D_1(\xi^1) - T^1 D_1(\xi^2) = 0. \tag{75}$$

Note that the symmetry algebra associated with a conservation law is always a subalgebra of the Lie symmetry algebra of the equation in question. One may take the general linear combination of the basis operators of the Lie symmetry algebra of the initial equation (see e.g. [35]). However, the advantage of the procedure undertaken below (the Kara and Mahomed theorem [26]) is that one may obtain physical solutions by double reduction using the conserved vector and the associated Lie point symmetry [36, 37].

4.1. Case $m = n$ ($m \neq -1$, $m \neq 0$)

Substituting the elementary conserved vector (51) into (74) and (75) gives

$$\xi^2\theta\omega e^{\omega x} + \eta e^{\omega x} + \theta e^{\omega x}\frac{\partial \xi^2}{\partial x} + \theta\theta_x e^{\omega x}\frac{\partial \xi^2}{\partial \theta} - e^{\omega x}\theta^{m+1}\frac{\omega}{m+1}\frac{\partial \xi^1}{\partial x} - \frac{\omega}{m+1}\theta_x\theta^{m+1}e^{\omega x}\frac{\partial \xi^1}{\partial \theta} + \theta_x\theta^m e^{\omega x}\frac{\partial \xi^1}{\partial x} + \theta_x^2\theta^m e^{\omega x}\frac{\partial \xi^1}{\partial \theta} = 0. \tag{76}$$

and

$$\begin{aligned}&\xi^2\frac{\omega^2}{m+1}e^{\omega x}\theta^{m+1} - \xi^2\theta_x\theta^m\omega e^{\omega x} + \eta\omega e^{\omega x}\theta^m - \eta\theta_x m\theta^{m-1}e^{\omega x} - \theta^m e^{\omega x}\frac{\partial \eta}{\partial x} - \theta^m e^{\omega x}\theta_x\frac{\partial \eta}{\partial \theta}\\ &- \theta^m e^{\omega x}\left\{\frac{\partial \eta}{\partial x} + \left(\frac{\partial \eta}{\partial \theta} - \frac{\partial \xi^2}{\partial x}\right)\theta_x - \theta_x^2\frac{\partial \xi^2}{\partial \theta} - \theta_\tau\frac{\partial \xi^1}{\partial x} - \theta_x\theta_\tau\frac{\partial \xi^1}{\partial \theta}\right\} + \theta^m e^{\omega x}\theta_\tau\frac{\partial \xi^1}{\partial x} + \theta^m e^{\omega x}\theta_\tau\theta_x\frac{\partial \xi^1}{\partial \theta}\\ &+ \theta_x\theta^m e^{\omega x}\frac{\partial \xi^2}{\partial x} + \theta^m e^{\omega x}\theta_x^2\frac{\partial \xi^2}{\partial \theta} + \frac{\omega}{m+1}e^{\omega x}\theta^{m+1}\frac{\partial \xi^1}{\partial \tau} + \frac{\omega}{m+1}e^{\omega x}\theta^{m+1}\theta_\tau\frac{\partial \xi^1}{\partial \theta} - \theta_x\theta^m e^{\omega x}\frac{\partial \xi^1}{\partial \tau} - \theta_x\theta^m e^{\omega x}\theta_\tau\frac{\partial \xi^1}{\partial \theta}\\ &- \theta e^{\omega x}\frac{\partial \xi^2}{\partial \tau} - \theta e^{\omega x}\theta_\tau\frac{\partial \xi^2}{\partial \theta} = 0.\end{aligned} \tag{77}$$

Separating Eq. (76) by derivatives of θ we find

$$\theta_x^2 \;:\; \theta^m e^{\omega x}\frac{\partial \xi^1}{\partial \theta} = 0, \tag{78}$$

$$\theta_x \;:\; \theta e^{\omega x}\frac{\partial \xi^2}{\partial \theta} - \frac{\omega}{m+1}\theta^{m+1}e^{\omega x}\frac{\partial \xi^1}{\partial \theta} + \theta^m e^{\omega x}\frac{\partial \xi^1}{\partial x} = 0, \tag{79}$$

$$\text{rem} \;:\; \xi^2\theta\omega e^{\omega x} + \eta e^{\omega x} + \theta e^{\omega x}\frac{\partial \xi^2}{\partial x} - e^{\omega x}\theta^{m+1}\frac{\omega}{m+1}\frac{\partial \xi^1}{\partial x} = 0. \tag{80}$$

Separating Eq. (77) by derivatives of θ we find

$$\theta_\tau\theta_x \;:\; 2\theta^m e^{\omega x}\frac{\partial \xi^1}{\partial \theta} = 0, \tag{81}$$

$$\theta_x^2 \;:\; 2\theta^m e^{\omega x}\frac{\partial \xi^2}{\partial \theta} = 0, \tag{82}$$

$$\theta_x \;:\; -\xi^2\theta^m\omega e^{\omega x} - \eta m\theta^{m-1}e^{\omega x} - 2\theta^m e^{\omega x}\frac{\partial \eta}{\partial \theta} + \theta^m e^{\omega x}\frac{\partial \xi^2}{\partial x} + \theta^m e^{\omega x}\frac{\partial \xi^2}{\partial x} - \theta^m e^{\omega x}\frac{\partial \xi^1}{\partial \tau} = 0, \tag{83}$$

$$\theta_\tau \;:\; 2\theta^m e^{\omega x}\frac{\partial \xi^1}{\partial x} + \frac{\omega}{m+1}e^{\omega x}\theta^{m+1}\frac{\partial \xi^1}{\partial \theta} - \theta e^{\omega x}\frac{\partial \xi^2}{\partial \theta} = 0, \tag{84}$$

$$\text{rem} \;:\; \xi^2\frac{\omega^2}{m+1}e^{\omega x}\theta^{m+1} + \eta\omega e^{\omega x}\theta^m - 2\theta^m e^{\omega x}\frac{\partial \eta}{\partial x} + \frac{\omega}{m+1}e^{\omega x}\theta^{m+1}\frac{\partial \xi^1}{\partial \tau} - \theta e^{\omega x}\frac{\partial \xi^2}{\partial \tau} = 0. \tag{85}$$

From Eqs. (78), (79) and (82)

$$\xi^1 = \xi^1(\tau), \tag{86}$$

and

$$\xi^2 = \xi^2(\tau, x). \tag{87}$$

Note that the forms of the infinitesimals ξ^1 and ξ^2 can be obtained without calculations (see e.g. [38]). Equation (80) reduces to

$$\xi^2\theta\omega + \eta + \theta\frac{\partial \xi^2}{\partial x} = 0, \tag{88}$$

thus

$$\eta = -\xi^2\theta\omega - \theta\frac{\partial \xi^2}{\partial x}, \tag{89}$$

and

$$\frac{\partial \eta}{\partial x} = -\omega\theta\frac{\partial \xi^2}{\partial x} - \theta\frac{\partial^2 \xi^2}{\partial x^2}. \tag{90}$$

Substituting Eqs. (89) and (90) into Eq. (85) gives

$$\xi^2\frac{\omega^2}{m+1}\theta^{m+1} - \omega^2\theta^{m+1}\xi^2 - \omega\theta^{m+1}\frac{\partial \xi^2}{\partial x} + \omega\theta^{m+1}\frac{\partial \xi^2}{\partial x} + \theta^{m+1}\frac{\partial^2 \xi^2}{\partial x^2} + \frac{\omega}{m+1}\theta^{m+1}\frac{\partial \xi^1}{\partial \tau} - \theta\frac{\partial \xi^2}{\partial \tau} = 0. \tag{91}$$

Separating Eq. (91) by powers of θ gives

$$\theta \quad : \quad \frac{\partial \xi^2}{\partial \tau} = 0, \tag{92}$$

$$\theta^{m+1} \quad : \quad \xi^2\frac{\omega^2}{m+1} - \omega^2\xi^2 + \frac{\partial^2 \xi^2}{\partial x^2} + \frac{\omega}{m+1}\frac{\partial \xi^1}{\partial \tau} = 0. \tag{93}$$

Differentiating Eq. (93) with respect to τ and solving ξ^1 gives

$$\xi^1(\tau) = a_1\tau + a_2 \tag{94}$$

where a_1 and a_2 are arbitrary constants.
Substituting ξ^1 into Eq. (83) and solving for ξ^2 we obtain

$$\xi^2(x) = \frac{a_1}{m\omega} + a_2 e^{-\frac{\omega m x}{2+m}}, \tag{95}$$

where $m \neq -2$. The remaining determining equation requires $a_2 = 0$. Thus Eq. (89) gives

$$\eta = -\theta\frac{a_1}{m\omega}. \tag{96}$$

The Lie point symmetries associated with the conserved vectors (51) are given by

$$\Gamma_1 = \frac{\partial}{\partial \tau}, \tag{97}$$

$$\Gamma_2 = \tau\frac{\partial}{\partial \tau} + \frac{1}{m\omega}\frac{\partial}{\partial x} - \frac{\theta}{m}\frac{\partial}{\partial \theta}. \tag{98}$$

4.2. Case $m \neq n$ $(n = -1, m \neq -1, m \neq 0)$

The method of multipliers gave conserved vectors that depend on the M. Substituting the elementary conserved vector (56) into (74) and (75) gives

$$\eta + \theta\frac{\partial \xi^2}{\partial x} + \theta\theta_x\frac{\partial \xi^2}{\partial \theta} + \theta^m\theta_x\frac{\partial \xi^1}{\partial x} + \theta^m\theta_x^2\frac{\partial \xi^1}{\partial \theta} - xM^2\frac{\partial \xi^1}{\partial x} - xM^2\frac{\partial \xi^1}{\partial \theta} = 0. \tag{99}$$

and

$$\xi^2M^2 - m\eta\theta_x\theta^{m-1} - \theta^m\frac{\partial \eta}{\partial x} - \theta^m\theta_x\frac{\partial \eta}{\partial \theta} + \theta_\tau\theta^m\frac{\partial \xi^1}{\partial x} - \theta_\tau\theta_x\theta^m\frac{\partial \xi^1}{\partial \theta} + \theta_x\theta^m\frac{\partial \xi^2}{\partial x} + \theta^m\theta_x^2\frac{\partial \xi^2}{\partial \theta} - \theta^m\theta_x\frac{\partial \xi^1}{\partial \tau} - \theta_\tau\theta_x\theta^m\frac{\partial \xi^1}{\partial \theta} + xM^2\frac{\partial \xi^1}{\partial \tau} + xM^2\theta_\tau\frac{\partial \xi^1}{\partial \theta} - \theta\frac{\partial \xi^2}{\partial \tau} - \theta\theta_\tau\frac{\partial \xi^2}{\partial \theta} - \theta^m\left\{\frac{\partial \eta}{\partial x} + \left(\frac{\partial \eta}{\partial \theta} - \frac{\partial \xi^2}{\partial x}\right)\theta_x - \theta_x^2\frac{\partial \xi^2}{\partial \theta} - \theta_\tau\frac{\partial \xi^1}{\partial x} - \theta_x\theta_\tau\frac{\partial \xi^1}{\partial \theta}\right\} = 0. \tag{100}$$

Separating Eq. (99) by derivatives of θ we find

$$\theta_x^2 \quad : \quad \theta^m\frac{\partial \xi^1}{\partial \theta} = 0, \tag{101}$$

$$\theta_x \quad : \quad \theta\frac{\partial \xi^2}{\partial \theta} + \theta^m\frac{\partial \xi^1}{\partial x} - xM^2\frac{\partial \xi^1}{\partial \theta} = 0, \tag{102}$$

$$\text{rem} \quad : \quad \eta + \theta\frac{\partial \xi^2}{\partial x} - xM^2\frac{\partial \xi^1}{\partial x} = 0. \tag{103}$$

Separating Eq. (100) by derivatives of θ we find

$$\theta_x^2 \quad : \quad 2\theta^m\frac{\partial \xi^2}{\partial \theta} = 0, \tag{104}$$

$$\theta_x \quad : \quad -m\eta\theta^{m-1} - \theta^m\frac{\partial \eta}{\partial \theta} + \theta^m\frac{\partial \xi^2}{\partial x} - \theta^m\frac{\partial \xi^1}{\partial \tau} - \theta^m\left(\frac{\partial \eta}{\partial \theta} - \frac{\partial \xi^2}{\partial x}\right) = 0, \tag{105}$$

$$\theta_x\theta_\tau \quad : \quad 2\theta^m\frac{\xi^1}{\partial \theta} = 0, \tag{106}$$

$$\theta_\tau \quad : \quad \theta^m\frac{\partial \xi^1}{\partial x} + xM^2\frac{\partial \xi^1}{\partial \theta} - \theta\frac{\partial \xi^2}{\partial \theta} = 0, \tag{107}$$

$$\text{rem} \quad : \quad \xi^2M^2 - 2\theta^m\frac{\partial \eta}{\partial x} + xM^2\frac{\partial \xi^1}{\partial \tau} - \theta\frac{\partial \xi^2}{\partial \tau} = 0. \tag{108}$$

From Eqs. (101), (104) and (102)

$$\xi^1 = \xi^1(\tau), \tag{109}$$

and

$$\xi^2 = \xi^2(\tau, x). \tag{110}$$

Note that the forms of the infinitesimals ξ^1 and ξ^2 can be obtained without calculations (see e.g. [38]). Equation (103) reduces to

$$\eta + \theta \frac{\partial \xi^2}{\partial x} = 0, \tag{111}$$

thus

$$\eta = -\theta \frac{\partial \xi^2}{\partial x}, \tag{112}$$

and

$$\frac{\partial \eta}{\partial x} = -\theta \frac{\partial^2 \xi^2}{\partial x^2}. \tag{113}$$

Substituting Eqs. (112) and (113) into Eq. (108) gives

$$\xi^2 M^2 + \theta^{m+1} \frac{\partial \xi^2}{\partial x^2} + x M^2 \frac{\partial \xi^1}{\partial \tau} - \theta \frac{\partial \xi^2}{\partial \tau} = 0. \tag{114}$$

Separating Eq. (114) by powers of θ gives

$$\theta \quad : \quad \frac{\partial \xi^2}{\partial \tau} = 0, \tag{115}$$

$$\theta^{m+1} \quad : \quad \frac{\partial^2 \xi^2}{\partial x^2} = 0, \tag{116}$$

$$\theta^0 \quad : \quad \xi^2 M^2 + x M^2 \frac{\partial \xi^1}{\partial \tau} = 0. \tag{117}$$

From Eqs. (115) and (116) we get that

$$\xi^2(x) = a_1 x + a_2, \tag{118}$$

where a_1 and a_2 are arbitrary constants.
Substituting ξ^2 into Eq. (105) we obtain

$$(m+2)a_1 - \frac{\partial \xi^1}{\partial \tau} = 0. \tag{119}$$

Equation (119) implies that

$$\xi^1(\tau) = a_1(m+2)\tau + a_3. \tag{120}$$

Substituting the expression of ξ^2 into Eq. (112) we get the following expression for η

$$\eta = -a_1 \theta. \tag{121}$$

The Lie point symmetries associated with the conservation laws (56) are given by

$$\Gamma_1 = -\tau \frac{\partial}{\partial \tau} + x \frac{\partial}{\partial x} - \theta \frac{\partial}{\partial \theta}, \tag{122}$$

$$\Gamma_2 = \frac{\partial}{\partial \tau}, \tag{123}$$

provided $m = -3$. The case $m \neq -3$ leads to the time translation. The symmetry reductions for equations (19) and (20) have been presented in the literature and may be found in classical paper [34] and we therefore omit such analysis. the Lie point symmetries associated with the conservation laws span the subalgebra of the Lie algebra admitted by the original equation.

5. Concluding remarks

All local conservations laws of second order (1+1) dimensional evolution equations were described in [39]. Using corollary 2 in [39] we have found all local conservation laws for a heat conduction model for heat transfer in straight fins (19) and (20). Conservation laws are derived using the direct method when the thermal conductivity is given by the power law and by the method of multipliers when thermal conductivity is given as a linear function of temperature. One may comment here that both methods yield the same results for a given problem. Two conservation laws were derived for nonlinear heat equation with power law thermal conductivity, one of which was the elementary conservation law. Two cases were considered, (i) exponent of the thermal conductivity being the same as the exponent of the heat transfer coefficient, (ii) distinct exponents. Conserved vectors which depended on at most first order partial derivatives were considered.

Higher order conservation laws could be investigated by considering conserved vectors which depend on higher order spatial derivatives. The analysis will be computationally laborious and may best be done with the aid of computer programs. Furthermore, non trivial conservation laws may be constructed if the thermal conductivity is the same as the heat transfer coefficient. This is a very special case and may not always be true for heat transfer in fins.

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