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Rogue waves in Lugiato-Lefever equation with variable coefficients

Rapid Communication

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Abstract: In this paper, we theoretically investigate the generation of optical rogue waves from a Lugiato-Lefever equation with variable coefficients by using the nonlinear Schrödinger equation-based constructive method. Exact explicit rogue-wave solutions of the Lugiato-Lefever equation with constant dispersion, detuning and dissipation are derived and presented. The bright rogue wave, intermediate rogue wave and the dark rogue wave are obtained by changing the value of one parameter in the exact explicit solutions corresponding to the external pump power of a continuous-wave laser.

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1. Introduction

Extreme events involving rogue (or “freak”) waves, initially reported in ocean seafarer stories and later observed by satellite surveillance [1], have become a subject of increased interest in physical oceanography [2]. These waves, of much greater height and steepness than expected from the sea average state [3], appear both in deep ocean and in shallow water [4]. In contrast to tsunamis and storms associated with typhoons that can be predicted hours (sometimes days) in advance, the particular danger of oceanic rogue waves is that they develop spontaneously from nowhere and disappear without a trace. So far, this phenomenon has not been understood completely due to the difficult and restricted conditions of observation. Nevertheless, ongoing discussion [5] and work [6–9] is seeking to better understand the physical mechanisms of this type of wave.

Recently, the phenomena of rogue waves has also been observed and demonstrated in optics [10–12], in plasmas [13], in Bose-Einstein condensates [14], atmosphere [15], in superfluid helium [16], in capillary waves [17] and even in finance [18]. These discoveries indicate that rogue waves may be rather universal.

In optics for example, some experimental observations have shown that optical rogue waves produced in supercontinuum generation play a positive role in nonlinear optical

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fibers [10–12]. It is used to generate highly energetic optical pulses [10, 11] and [19–21]. Certainly, harnessing the occurrence of rogue waves in the ocean would be challenging due to their enormous destructiveness. The common features and differences among freak wave manifestations in their different physical contexts is a subject of intense discussion [4].

It is well known that the nonlinear Schrödinger equation (NLSE) can often be used to describe wave propagation in various physical systems such as optical fibers, deep water, and in atomic Bose-Einstein condensates. Based on this nonlinear equation, the rogue wave phenomena has been extensively studied [9, 22–24]. Literature includes rational solutions and their interactions, pulse splitting induced by higher-order modulation instability, and wave turbulence [25, 26]. Exact rogue waves solution have also been obtained analytically for some physical models such as the Kadomtsev-Petviashvili equation in shallow water [27] and the NLSE with variable coefficients [28], higher orders [29], or higher dimensions [30, 31]. However, to the best of our knowledge, the study of rogue waves in the Lugiato- Lefever equation (LLE) with variable coefficients remains unaddressed. The LLE is a paradigmatic equation for dissipative cavity solitons which is applicable to different types of cavities: in a ring cavity partially filled with a nonlinear medium; in a Fabry-Perot resonator containing a nonlinear medium; in a photonic crystal fiber resonator pumped by a coherent continuous-wave input beam [32, 33]; and in crystalline whispering-gallery mode disk resonators [34]. The LLE is a particular realization of the forced complex Ginzburg-Landau equation [35] at the (1:1) resonance [36], which is also considered as a variant of the NLSE that includes damping, driving and detuning. In this work, the analytical derivation of a family of exact rogue-wave solutions of the LLE with variable coefficients in an explicit and concise form is presented. Inspired by the importance of the recent high interest developments in the analysis of rogue waves in the NLSEs and their extensions with variable coefficients, we shall construct rogue wave solutions of the LLE with variable coefficients by using the NLSE-based constructive method proposed in Ref. [37, 38]. This method is inspired by the mapping method and the direct method of symmetry reduction.

The paper is structured as follows. In the following Section 2, we construct the exact solutions of the LLE with variable coefficients by using the NLSE-based constructive method. Then, in Section 3 we conclude our results.

2. Exact solutions of the LLE with variable coefficients

In the present study, the governing equation modeling the dissipative cavity solitons is described by the generalized LLE with variable coefficients, expressed as follows:

$$iu_\tau + \mu_1(\tau)u_{\xi\xi} + \mu_2(\tau)|u|^2u + (\mu_3(\tau) + i\mu_4(\tau))\,u = if(\xi,\tau). \tag{1}$$

The independent variables τ and ξ are the time and axial coordinate respectively, and $u = u(\tau,\xi)$ denotes the slowly varying envelope of electric field. The variable coefficients $\mu_1(\tau)$ and $\mu_2(\tau)$ denotes the dispersion and nonlinearity parameters respectively. $\mu_3(\tau)$ is the parameter representing detuning and $\mu_4(\tau)$ is the dissipative term. Finally, $f(\xi,\tau)$ is the external pumping.

Our objective is to transform Eq. (1) into the standard NLSE

$$iU_T + \frac{1}{2}U_{ZZ} + |U|^2U = 0, \tag{2}$$

substituting the ansatz (3) proposed in Refs. [37, 38], into Eq. (1)

$$u(\xi,\tau) = \{a_0(\tau) + a_1(\tau)U[Z(\xi,\tau),T(\tau)]\} \times \exp[i\varphi(\xi,\tau)], \tag{3}$$

where U satisfies Eq. (2) and u is the solution of Eq. (1). $a_0(\tau)$, $a_1(\tau)$, $Z(\xi,\tau)$, $T(\tau)$ and $\varphi(\xi,\tau)$ are real functions to be determined from the following differential equations

$$a_{1,\tau} + \mu_1(\tau)\varphi_{\xi\xi}a_1 + \mu_4(\tau)a_1 = 0, \quad Z_\tau + 2\mu_1(\tau)\varphi_\xi Z_\xi = 0,$$

$$\varphi_\tau + \mu_1(\tau)\varphi_\xi^2 - \mu_3(\tau) = 0, \quad 2\mu_1(\tau)Z_\xi^2 - T_\tau = 0, \quad Z_{\xi\xi} = 0, \tag{4}$$

$$\mu_2(\tau)a_1^2 - T_\tau = 0, \quad a_{0,\tau} + F(\tau) = 0$$
$$\text{and} \quad f(\xi,\tau) = F(\tau)\exp[i\varphi(\xi,\tau)].$$

Solving the set of Eq. (4), we obtain the following results:

$$a_0 = \int_0^\tau F(t)dt, \quad a_1(\tau) = \sqrt{c_1 f_1(\tau)}\exp\left[-\int_0^\tau \mu_4(t)dt\right]. \tag{5}$$

$$Z(\xi,\tau) = c_1 f_1(\tau)\xi + f_2(\tau), \quad T(\tau) = 2c_1^2\int_0^\tau \mu_1(t)f_1^2(t)dt. \tag{6}$$

$$\varphi(\xi,\tau) = -\frac{f_{1,\tau}}{4\mu_1(\tau)f_1(\tau)}\xi^2 - \frac{f_{2,\tau}}{2c_1\mu_1(\tau)f_1(\tau)}\xi + f_3(\tau). \tag{7}$$

where

$$\mu_2(\tau) = 2c_1\mu_1(\tau)f_1(\tau)\exp\left[2\int_0^\tau \mu_4(t)dt\right], \tag{8}$$

$$\mu_3(\tau) = f_{3,\tau} + \frac{f_{2,\tau}^2}{4c_1^2\mu_1(\tau)f_1^2(\tau)}. \tag{9}$$

According to the above transformation, the general form of the solution (3) is given by

$$u = \left[\int_0^\tau F(t)dt + \sqrt{c_1 f_1(\tau)}U\left[Z(\xi,\tau) \equiv c_1 f_1(\tau)\xi + f_2(\tau),\right.\right.$$
$$\left.T(\tau) \equiv 2c_1^2\int_0^\tau \mu_1(t)f_1^2(t)dt\right] \times \exp\left(-\int_0^\tau \mu_4(t)dt\right)\Big]$$
$$\times \exp(i\varphi(\xi,\tau)). \tag{10}$$

By choosing the arbitrary functions $f_1(\tau)$, $f_2(\tau)$ and coefficient c_1, the dispersion $\mu_1(\tau)$, the nonlinearity coefficient $\mu_2(\tau)$ and the detuning parameter $\mu_3(\tau)$ as well as the dissipative parameter $\mu_4(\tau)$ and the pumping term $F(\tau)$ can be designed and analytically determined. Therefore many shapes of solutions of Eq. (1) can be obtained. Since we are interested in the rogue wave solution, we assume that the seed solution $U(Z,T)$ of the transformation Eq. (3) is a rogue wave solution of the NLSE Eq. (2) (see Refs. [24, 39]). The first order rational solution of Eq. (1) can be derived as

$$u = \left[\int_0^\tau F(t)dt + \sqrt{c_1 f_1(\tau)}\left[1 - \right.\right.$$
$$\left. 4\frac{1 + 4ic_1^2\int_0^\tau \mu_1(\tau)f_1^2(t)dt}{1 + 4\left(2c_1^2\int_0^\tau \mu_1(\tau)f_1^2(t)dt\right)^2 + 4(c_1 f_1(\tau)\xi + f_2(\tau))^2}\right]$$
$$\times \exp\left(iT(\tau) - \int_0^\tau \mu_4(t)dt\right)\Big] \exp(i\varphi(\xi,\tau)). \tag{11}$$

Having described our analytical approach, it is important to note that in the literature, extreme events generally arise in the region of anomalous dispersion. In the case of optical fibers, Frisquet et *al.* [41] demonstrated that the multiples bright solitons, or breathers, observed in this region collide during propagation leads to the appearance of rogue waves. In whispering gallery mode resonators, Coillet et *al.* [40] theoretically showed that rogue waves can emerge due to the chaotic interplay between Kerr nonlinearity and anomalous group-velocity dispersion. Therefore in order to obtain a rogue wave solution to the LLE with variable coefficients, we proceed by presenting a specific case of Eq. (1) with variable nonlinearity term $\mu_2(\tau)$, constant dispersion, detuning and dissipation terms which

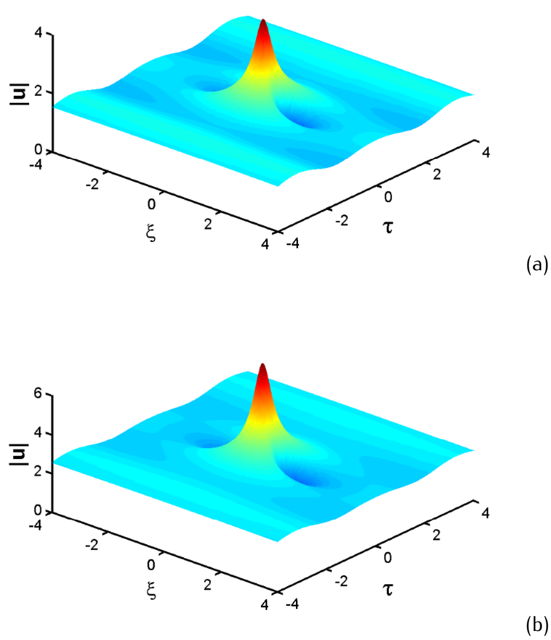

Figure 1. (Color online) The snapshots of the Peregrine breather solution (13), for $\gamma_0 = 1.5$ in (a) and $\gamma_0 = 2.5$ in (b). The values of parameters are $F_{00} = 0.15$ and $F_0 = \gamma = 0.01$.

correspond to choose:

$$f_1(\tau) = 1, \quad f_2(\tau) = 0, \quad f_3(\tau) = -\alpha\tau, \quad c_1 = 1,$$
$$\mu_1(\tau) = 1, \quad \mu_2(\tau) = 2\exp(2\gamma\tau + \gamma_0), \quad \mu_3(\tau) = -\alpha,$$
$$\mu_4(\tau) = \gamma \quad \text{and} \quad \int_0^\tau F(t)dt = F_0\tau + F_{00}. \tag{12}$$

Thus, the rogue wave solutions (11) of Eq. (1) take the form:

$$u(\xi,\tau) = \left[F_0\tau + F_{00} + \left[1 - \frac{4 + 16i\tau}{1 + 16\tau^2 + 4\xi^2}\right]\right.$$
$$\left.\times \exp[(2i - \gamma)\tau + \gamma_0]\right] \times \exp(-i\alpha)\tau, \tag{13}$$

where F_{00} and γ_0 are constants of integration representing the constant values of the pump and dissipative parameters used in many reference papers [40]. Based on the rational solution (13), we can study how the pumping parameters (F_0, F_{00}) of the laser and the coefficients (γ, γ_0) connected to the nonlinear parameter affect the rogue waves generation in the system. Considering lower values of F_0 and γ, we depict in Figs. 1 and 2 the modulus of the

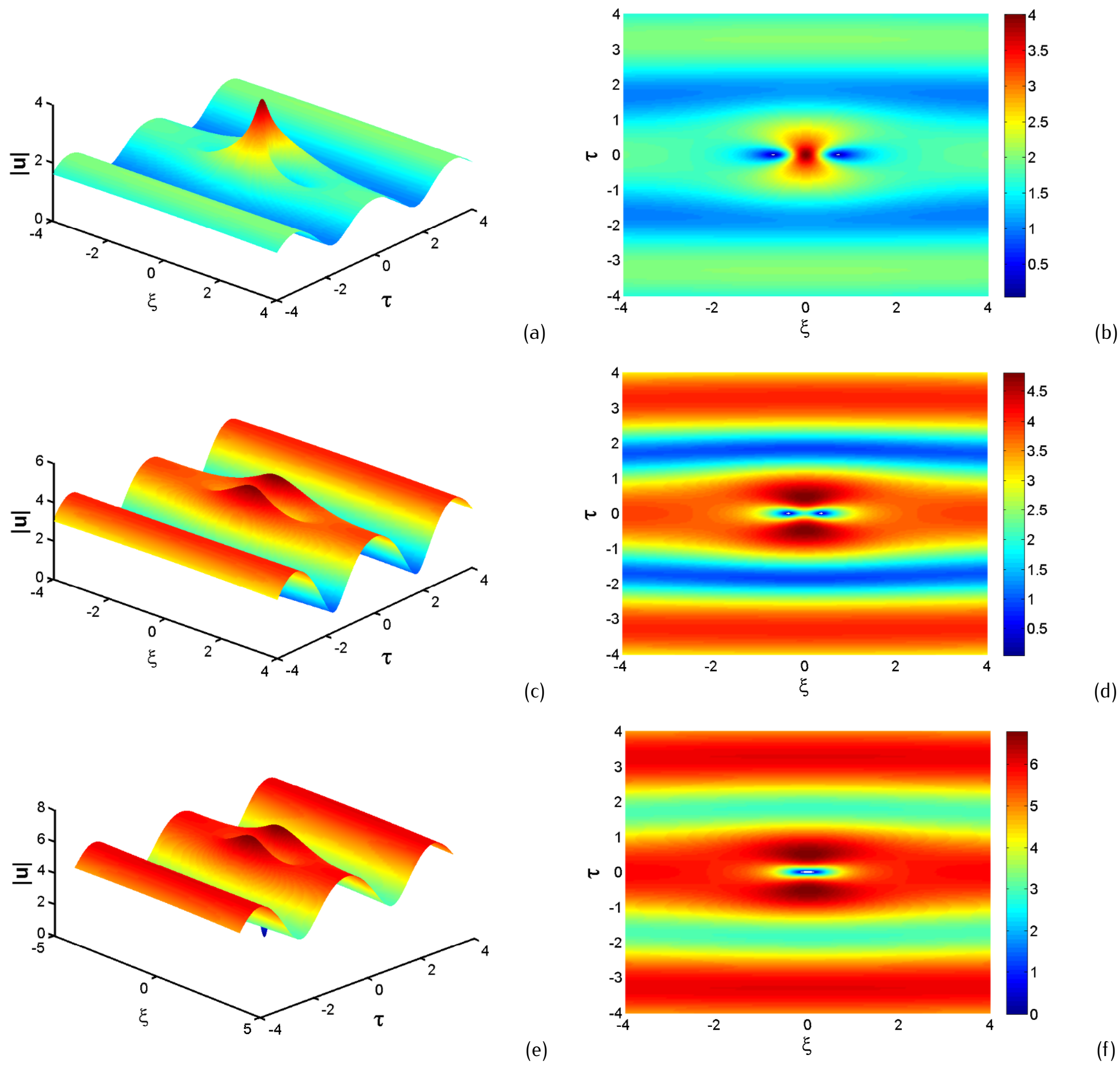

Figure 2. (Color online) The panels display the features of (a) bright-rogue wave for $F_{00} = 0.5$, (c) intermediate rogue wave for $F_{00} = 2.5$ and (e) dark-rogue wave for $F_{00} = 4.5$ and its density plot (b),(d) and (f) from solution (13). The values of parameters are $\gamma_0 = 1.5$ and $F_0 = \gamma = 0.01$.

solution (13) for different values of F_{00} and γ_0. For fixed values of $F_{00} = 0.15$, $F_0 = 0.01$ and $\gamma = 0.01$, we plot in Fig. 1 the modulus of the solution (13) for two exemplary values of γ_0.

It is clear that the amplitude of the rogue wave increases with the values of γ_0 as illustrated in Figs. 1, (a) and (b), for $\gamma_0 = 1.5$ and $\gamma_0 = 2.5$ respectively. This result demonstrates the influence of the nonlinear parameter $\mu_2(\tau)$ on the amplitude of the rogue wave. For a fixed value of $\gamma_0 = 1.5$, we represent in Fig. 2 the evolution of rogue waves for different values of F_{00}.

The features of bright-rogue wave, intermediate rogue wave and dark-rogue wave are represented by the solution (13) in Figs. 2 (a),(c) and (e), for $F_{00} = 0.5$, $F_{00} = 2.5$ and $F_{00} = 4.5$ respectively, and its corresponding density plots are shown in Figs. 2 (b),(d) and (f). It is well observed that these rogue waves exist on a periodic background. Amongst the recent investigations reported in the literature, in Ref. [40] the authors demonstrated that, as the continuous pump power is increased, rogue waves are observed in numerical solutions for whispering gallery mode resonators pumped with a continuous-wave laser. Therefore, we suggest that the results obtained in this work are in accordance with the numerical results proposed in [40]. Moreover we found that increasing the pump parameter induces different features of rogue waves.

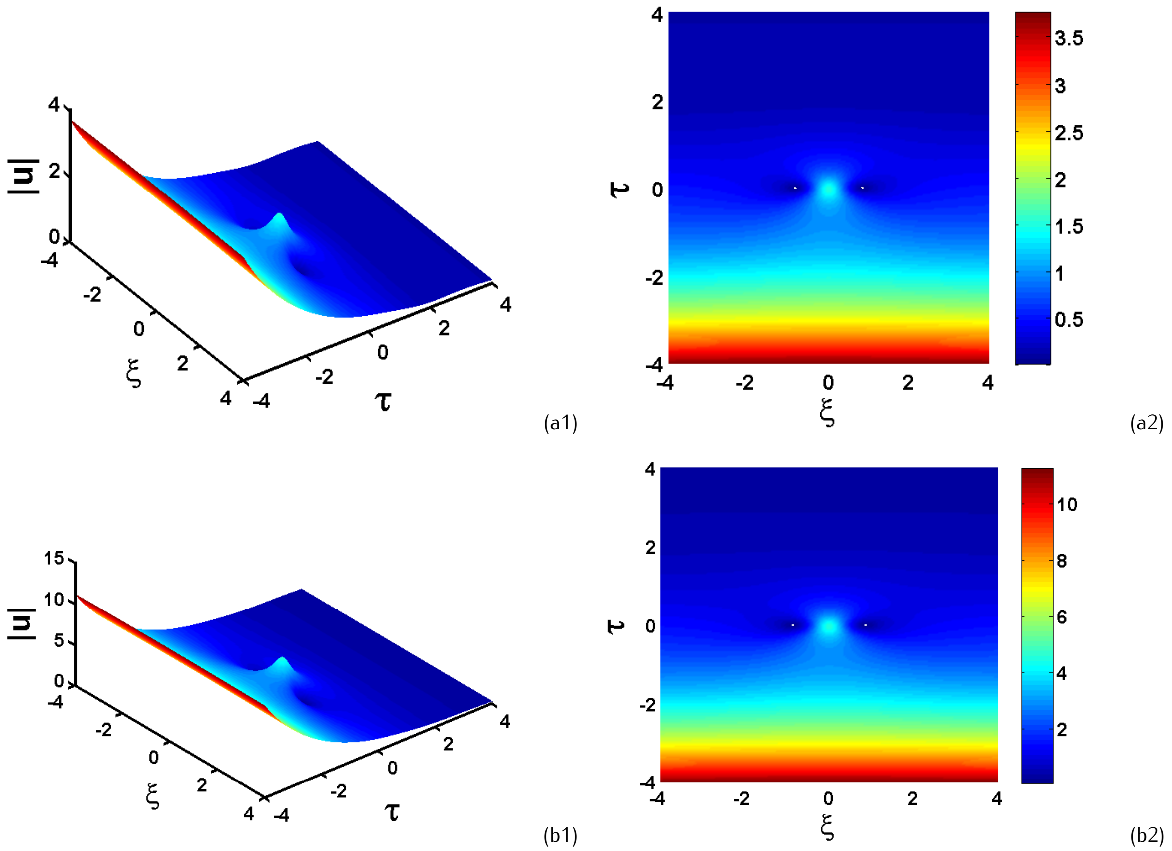

Figure 3. (Color online) In panel (1), we plot the snapshots of the rogue wave of solution (13) while in panel (2) we depict its density plot, for (a) $\gamma_0 = 0.5$ and (b) $\gamma_0 = 1.5$.

Considering now higher values of F_0 and γ, the profile of the rogue wave and its spatiotemporal distribution is illustrated in Fig. 3 for $F_0 = 0.5$, $\gamma = 0.5$, $F_{00} = 0.02$ and two exemplary values of γ_0.

From Fig. 3, one can see that the rogue wave exists on top of a monotonically decreasing background. By increasing the value of γ_0, we notice the increase in amplitude of the rogue wave profile as depicted in Figs. 3, (a) and (b), for $\gamma_0 = 0.5$ and $\gamma_0 = 1.5$ respectively. In further work (for which graphs are not shown) we considered the parameter values of Fig. 3, and also found that the system exhibits a bright-rogue wave, intermediate rogue wave and dark-rogue wave on top of a monotonically decreasing background, when the value of of F_{00} increases.

3. Conclusion

This work constituted a theoretical study of rogue wave solutions to the Lugiato-Lefever equation with variable coefficients. We focus on Lugiato-Lefever equation with constant dispersion, detuning and dissipation, which is relevant to the context of dissipative cavity solitons. The nonlinear Schrödinger equation-based constructive method provides an exact explicit rogue wave solutions, to the Lugiato-Lefever equation with variable coefficients where bright, intermediate and dark rogue wave solutions have been found. Transition between solutions is achieved by changing only one parameter, namely the external pump power of a continuous-wave laser. We anticipate these results providing a route to the experimental realization of rogue waves in crystalline whispering-gallery-mode disk resonators, or in a photonic crystal fiber resonator pumped by a coherent continuous-wave input beam.

References

[1] C. Kharif, E. Pelinovsky, Eur. J. Mech. B 22, 603 (2003)

[2] C. Kharif, E. Pelinovsky, A. Slunyaev, Rogue Waves

in the Ocean, (Springer, NY, 2009)
[3] K. Dysthe, H.E. Krogstad, P. Muller, Annu. Rev. Fluid Mech. 40, 287 (2008)
[4] N. Akhmediev, E. Pelinovsky, Eur. Phys. J. Special Topics 185, 1 (2010)
[5] V. Ruban et al., Eur. Phys. J. Special Topics 185, 5 (2010)
[6] P. K. Shukla, I. Kourakis, B. Eliasson, M. Marklund, L. Steno, Phys. Rev. Lett. 97, 094501 (2006)
[7] N. Akhmediev, A. Ankiewicz, J. M. Soto-Crespo, Phys. Rev. E 80, 026601 (2009)
[8] N. Akhmediev, J. M. Soto-Crespo, A. Ankiewicz, Phys. Rev. A 80, 043818 (2009)
[9] N. Akhmediev, A. Ankiewicz, M. Taki, Phys. Lett. A 373, 675 (2009)
[10] D.R. Solli, C. Ropers, P. Koonath, B. Jalali, Nature (London) 450, 1054 (2007)
[11] D.R. Solli, C. Ropers, B. Jalali, Phys. Rev. Lett. 101, 233902 (2008)
[12] B. Kibler et al., Nature Phys. 6, 1 (2010)
[13] W.M. Moslem, P.K. Shulkla, B. Eliasson, Europhys. Lett. 96, 25002 (2008)
[14] Y.V. Bludov, V.V. Konotop, N. Akhmediev, Phys. Rev. A 80, 033610 (2009)
[15] L. Stenflo, M. Marklund, J. Plasma Phys. 76, 293 (2010)
[16] A.N. Ganshin, V.B. Efimov, G.V. Kolmakov, L.P. Mezhov-Deglin, P.V.E. McClintock, Phys. Rev. Lett. 101, 065303 (2008)
[17] M. Shats, H. Punzmann, H. Xia, Phys. Rev. Lett. 104, 104503 (2010)
[18] Z.Y. Yan, Phys. Lett. A 375, 4274 (2011).
[19] J. Dudley, G. Genty, B. Eggelton, Opt. Express 16, 3644 (2008)
[20] A. Mussot, A. Kudlinski, M. Kolobov, E. Louvergneaux, M. Douay, M. Taki, Opt. Express 17, 17010 (2010)
[21] M. Taki, A. Mussot, A. Kudlinski, E. Louvergneaux, M. Kolobov, M. Douay, Phys. Lett. A 374, 691 (2010)
[22] A. Ankiewicz, J.M. Soto-Crespo, N. Akhmediev, Phys. Rev. E 81, 046602 (2010)
[23] Y.C. Ma, Stud. Appl. Math. 60, 43 (1979)
[24] D.H. Peregrine, J. Aust. Math. Soc. Ser. B, Appl. Math. 25, 16 (1983)
[25] M. Erkintalo et al. , Phys. Rev. Lett. 107, 253901 (2011)
[26] B. Kibler, K. Hammani, C. Michel, Ch. Finot, A. Picozzi, Phys. Lett. A 375, 3149 (2011)
[27] J.W. Miles, J. Fluid Mech. 79, 171 (1977)
[28] Z.Y. Yan, Phys. Lett. A 374, 672 (2010)
[29] A. Ankiewicz, N. Devine, N. Akhmediev, Phys. Lett. A 373, 3997 (2009)
[30] Z.Y. Yan, V.V. Konotop, N. Akhmediev, Phys. Rev. E 82, 036610 (2010)
[31] Z.Y. Yan, J. Math. Anal. Appl. 380, 689 (2011)
[32] M. Tlidi, A. Mussot, E. Louvergneaux, G. Kozyreff, A. G. Vladimirov, M. Taki, Opt. Lett. 32, 662 (2007)
[33] M. Tlidi, L. Gelens, Opt. Lett. 35, 306, (2010)
[34] A. Coillet et al., IEEE Photonics Journal 5, 6100409 (2013)
[35] C. Elphick, G. Iooss, E. Tirapegui, Phys. Lett. A 120, 459 (1987)
[36] L.A. Lugiato, R. Lefever, Phys. Rev. Lett. 58, 2209 (1987)
[37] C.Q. Dai, J.F. Zhang, Nonlinear Dyn. 73, 2049 (2013)
[38] H.J Jiang, J.J. Xiang, C.Q. Dai, Y.Y. Wang, Nonlinear Dyn. 75, 201 (2014)
[39] A. Ankiewicz, N. Devine, N. Akhmediev, Phys. Lett. A 373, 3997 (2009)
[40] A. Coillet, J. Dudley, G. Genty, L. Larger, Y.K. Chembo, Phys. Rev. A 89, 013835 (2014)
[41] B. Frisquet, B. Kibler, G. Millot, Phys. Rev. X 3, 041032 (2013)