Exact solutions of the Biswas-Milovic equation, the ZK(m,n,k) equation and the K(m,n) equation using the generalized Kudryashov method

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Abstract: In this article, we apply the generalized Kudryashov method for finding exact solutions of three nonlinear partial differential equations (PDEs), namely: the Biswas-Milovic equation with dual-power law nonlinearity; the Zakharov–Kuznetsov equation (ZK(m,n,k)); and the K(m,n) equation with the generalized evolution term. As a result, many analytical exact solutions are obtained including symmetrical Fibonacci function solutions, and hyperbolic function solutions. Physical explanations for certain solutions of the three nonlinear PDEs are obtained.

Keywords: Nonlinear PDEs; Generalized Kudryashov method; Exact solutions; Symmetrical hyperbolic Fibonacci function; The Biswas-Milovic equation; The Zakharov–Kuznetsov equation; The K(m,n) equation

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1 Introduction

The research area of nonlinear partial differential equations (PDEs) has been very active for the past few decades. There are many types of nonlinear PDEs that appear in various areas of the physical and mathematical sciences. Much effort has been expended on constructing exact solutions to these nonlinear PDEs, motivated by their important role in the study of nonlinear physical phenomena. Nonlinear phenomena appears in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics, and geochemistry. In recent years, a number of powerful and efficient methods for finding analytic solutions to nonlinear equations have drawn a lot of interest by a diverse group of scientists. These include, for example: Hirota’s bilinear transformation method [1, 2]; the tanh-function method [3, 4]; the \((G'/G)\)-expansion method [5–10]; the Exp-function method [11–14]; the multiple exp-function method [15–17]; the symmetry method [18, 19]; the modified simple equation method [20–22]; the improved \((G'/G)\)-expansion method [23]; a multiple extended trial equation method [24]; the Jacobi elliptic function expansion method [25, 26]; the Bäcklund transform method [27, 28]; the generalized Riccati equation method [29]; the modified extended Fan sub equation method [30]; the auxiliary equation method [31, 32]; the first integral method [33, 34]; the modified Kudryashov method [35–42], and the soliton ansatz method [43–70].

The objective of this paper is to apply the generalized Kudryashov method [42] to find the exact solutions of the Biswas-Milovic equation with dual-power law nonlinearity [71, 72], the ZK(m,n,k) [73, 74], and the K(m,n) equation with the generalized evolution term [75].

This paper is organized as follows: in Sec. 2, the description of the generalized Kudryashov method is given. In Sec. 3, we use this method to solve the three aforementioned nonlinear PDEs. In Sec. 4, physical explanations of certain results are presented, and conclusions are discussed in Sec. 5.

2 Description of the generalized Kudryashov method

Starting with a nonlinear PDE in the following form:

\[ F(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \ldots) = 0, \]  

where \( u = u(x, t) \) is an unknown function, \( F \) is a polynomial in \( u = u(x, t) \) and its partial derivatives, in which
the highest order derivatives and nonlinear terms are involved.

The main steps of the generalized Kudryashov method are described as follows:

**Step 1.** First, we use the wave transformation:

\[ u(x, t) = U(\zeta), \quad \zeta = kx + \lambda t, \quad (2.2) \]

where \( k \) and \( \lambda \) are arbitrary constants with \( k, \lambda \neq 0 \), in order to reduce equation (2.1) into a nonlinear ordinary differential equation (ODE) with respect to the variable \( \zeta \) of the form

\[ H(U, U', U'', U''') = 0, \quad (2.3) \]

where \( H \) is a polynomial in \( U(\zeta) \) and its total derivatives \( U', U'', U''' \), such that \( U' = \frac{dU}{d\zeta} \).

**Step 2.** We assume that the formal solution of the ODE (2.3) can be written in the following rational form:

\[ U(\zeta) = \frac{\sum_{i=0}^{n} a_i Q^i(\zeta)}{\sum_{j=0}^{m} b_j Q^j(\zeta)} = \frac{A[Q(\zeta)]}{B[Q(\zeta)]}, \quad (2.4) \]

where \( Q = \frac{1}{1 + \alpha \zeta} \), \( A[Q(\zeta)] = \sum_{i=0}^{n} a_i Q^i(\zeta) \) and \( B[Q(\zeta)] = \sum_{j=0}^{m} b_j Q^j(\zeta) \). The function \( Q(\zeta) \) is the solution of the equation

\[ Q' = Q(Q - 1) \ln(a), \quad 0 < a \neq 1. \quad (2.5) \]

Taking into consideration (2.4), we obtain

\[ U'(\zeta) = Q(Q - 1) \left[ A'B - A'B' \right] / B^2 \ln(a), \quad (2.6) \]

\[ U''(\zeta) = Q(Q - 1)(2Q - 1) \left[ A'B - A'B' \right] / B^2 \ln^2(a) + Q^2(Q - 1)^2 \left[ B(A'B - A'B') - 2A'B'B + 2A'B'^2 \right] / B^3 \ln^2(a), \quad (2.7) \]

\[ U'''(\zeta) = Q^3(Q - 1)^3 \ln^3(a) \times \left[ (A''B - A'B'' - 3A'B - 3A'B')B + 6B'(A'B' + A'B') ight] / B^3 
- 6A(B')^3 + 3Q^2(Q - 1)^2(2Q - 1) \]
\[ \times \left[ (A'B - A'B' - 2A'B')B + 2A(B')^2 \right] \ln^3(a) + Q(Q - 1)(6Q^2 - 6Q + 1) \left[ A'B - A'B' \right] / B^2 \ln^3(a), \quad (2.8) \]

and that similar solutions apply for higher order differentiation terms.

**Step 3.** Under the terms of the given method, we suppose that the solution of Eq. (2.3) can be written in the following form:

\[ U(\zeta) = a_0 + a_1 Q + a_2 Q^2 + + a_n Q^n \]
\[ b_0 + b_1 Q + b_2 Q^2 + + b_m Q^m. \quad (2.9) \]

To calculate the values of \( m \) and \( n \) in (2.9), i.e. the pole order for the general solution of Eq. (2.3), we progress as per the classical Kudryashov method by balancing the highest order nonlinear terms and the highest order derivatives of \( U(\zeta) \) in Eq. (2.3). This allows us to derive a formula for \( m \) and \( n \) and determine their values.

**Step 4.** We substitute (2.4) into Eq. (2.3) to get a polynomial \( R(Q) \) of \( Q \) and equate all the coefficients of \( Q^i \), \( i = 0, 1, 2, \ldots \) to zero, to yield a system of algebraic equations for \( a_i \) \( i = 0, 1, \ldots, n \) and \( b_j \) \( j = 0, 1, \ldots, m \).

**Step 5.** We solve the algebraic equations obtained in Step 4 using Mathematica or Maple, to get \( k, \lambda \) and the coefficients of \( a_i \) \( i = 0, 1, \ldots, n \) and \( b_j \) \( j = 0, 1, \ldots, m \). In this way, we attain the exact solutions to Eq. (2.3).

The obtained solutions depended on the symmetrical hyperbolic Fibonacci functions given in [76]. The symmetrical Fibonacci sine, cosine, tangent, and cotangent functions are, respectively, defined as:

\[ sFs(x) = \frac{a^x - a^{-x}}{\sqrt{5}}, \quad cFs(x) = \frac{a^x + a^{-x}}{\sqrt{5}}, \quad \tan Fs(x) = \frac{a^x - a^{-x}}{a^x + a^{-x}}, \quad \cot Fs(x) = \frac{a^x + a^{-x}}{a^x - a^{-x}}, \quad (2.10) \]

\[ sFs(x) = \frac{2}{\sqrt{5}}s\sh(x, \ln(a)), \quad cFs(x) = \frac{2}{\sqrt{5}}c\sh(x, \ln(a)), \quad \tan Fs(x) = s\coth(x, \ln(a)), \quad \cot Fs(x) = c\coth(x, \ln(a)). \quad (2.11) \]

### 3 Applications

In this section we construct the exact solutions in terms of the symmetrical hyperbolic Fibonacci functions of the following three nonlinear PDEs using the generalized Kudryashov method described in Sec. 2:

3.1 Example 1. The Biswas-Milovic equation with dual-power law nonlinearity

In this subsection, we study the Biswas-Milovic equation with dual-power law nonlinearity [71, 72]

\[ i(q^n)_t + a(q^n)_{xx} + b \left( |q|^{2n} + k |q|^{4n} \right) q^m = 0, \quad (3.1.1) \]
where \( a, b \) and \( k \) are constants, while \( m, n \) are positive integers.

If \( m = 1 \), then Eq. (3.1.1) becomes the nonlinear Schrödinger equation (NLSE) with dual-power law nonlinearity. In Eq. (3.1.1), the first term is the temporal evolution, while \( a \) is the coefficient of group-velocity dispersion (GVD), while \( b \) and \( k \) are the coefficients of the nonlinear terms. Let us now solve Eq. (3.1.1) using the method of Sec. 2. To this end, we use the wave transformation

\[
q(x, t) = U(\zeta) \exp(\i \theta), \quad \theta = \lambda x - \omega t, \quad \zeta = \mu(x - vt), \quad (3.1.2)
\]

where \( \lambda, \mu, \nu \) and \( \nu \) are constants.

Substituting (3.1.2) into (3.1.1), we obtain the nonlinear ODE:

\[
a \mu^2 (U''(m'))'' + i \nu (\nu + 2am\lambda) (U')(m'') + m(w - am\lambda^2) U'' + bU^{2m+1} + bkU^{4n+m} = 0.
\]

(3.13)

From Eq. (3.1.3), we deduce that

\[
\nu = 2am\lambda,
\]

(3.14)

and

\[
a \mu^2 (U''(m'))'' + m(w - am\lambda^2) U'' + bU^{2m+1} + bkU^{4n+m} = 0.
\]

(3.15)

Balancing \((U''(m'))''\) and \(U^{4n+m}\) in (3.15), the following relation is attained:

\[
mN + 2 = (4n + m)N \Rightarrow N = \frac{1}{2n}.
\]

(3.16)

Then we take into consideration the transformation

\[
U(\zeta) = [\nu(\zeta)]^{\frac{1}{2n}}.
\]

(3.17)

Substituting (3.1.7) into equation (3.1.5) we have the new equation

\[
2am\mu^2 v'' = am\nu^2 (m - 2n) (v')^2 + 4mn^2 (w - am\lambda^2)v^2 + 4bn^2 v^3 + 4bkn^2 v^4 = 0.
\]

(3.18)

Balancing the \(vv''\) and \(v^4\) in (3.18), then the following relation is obtained:

\[
(N - M) + (N - M) + 2 = 2(4N - M) \Rightarrow N = M + 1.
\]

(3.19)

If we choose \( M = 1 \) and \( N = 2 \), then the formal solution of Eq. (3.1.8) has the form:

\[
\nu(\zeta) = \frac{a_0 + a_1Q + a_2Q^2}{b_0 + b_1Q},
\]

(3.110)

and consequently,

\[
\nu' = \frac{Q(Q - 1)}{(b_0 + b_1Q)} 
\]

\[
\left[(a_1 + 2a_2Q)(b_0 + b_1Q) - b_1(a_0 + a_1Q + a_2Q^2)\right] \ln(a),
\]

(3.111)

\[
\nu' = \frac{Q(Q - 1)}{(b_0 + b_1Q)} \left[(a_1 + 2a_2Q)(b_0 + b_1Q) - b_1(a_0 + a_1Q + a_2Q^2)\right] \ln^2(a) + \frac{Q^2(Q - 1)^2 \ln^2(a)}{(b_0 + b_1Q)^3}
\]

\[
\times \left[2a_2(b_0 + b_1Q)^2 - 2b_1(a_1 + 2a_2Q)(b_0 + b_1Q)
\right.
\]

\[
+ 2b_1^2(a_0 + a_1Q + a_2Q^2)\right] .
\]

(3.112)

Substituting (3.10), (3.11) and (3.12) into (3.18), collecting the coefficients of each power of \( Q^n \), \( i = 0, 1, \ldots, 8 \) and setting each of the coefficients to zero, we obtain the following system of algebraic equations:

\[
Q^8 : am^2b_0^2b_1^2\theta^2 ln^2(a) + 2amnb_0^2b_1^2\theta^2 ln^2(a)
\]

\[
+ 4bkn^2a_0^2 = 0,
\]

\[
Q^7 : -2amna_0^2b_1^2 \theta^2 ln^2(a) + 16bkn^2a_1a_2
\]

\[
+ 4am^2b_0b_1\theta^2 ln^2(a) + 4amna_0^2b_0b_1\theta^2 ln^2(a)
\]

\[
+ 4amna_1b_0b_1\theta^2 ln^2(a) + 4amn^2b_0^2a_1
\]

\[
+ 4amn^2b_0^2a_1 \theta^2 ln^2(a) + 4amn^2b_0^2a_1 \theta^2 ln^2(a)
\]

\[
-2amna_1b_0b_1\theta^2 ln^2(a) + 4bkn^2a_1^2 \theta^2
\]

\[
+ 4amn^2b_0^2a_1 \theta^2 ln^2(a) + 4amn^2b_0^2a_1 \theta^2 ln^2(a)
\]

\[
+ am^2b_0^2b_1^2 \theta^2 ln^2(a) + 16bkn^2a_0a_1^2 + 4n^2mwa_2b_1
\]

\[
+ 8amna_0b_0b_1 \theta^2 ln^2(a) = 0
\]

\[
Q^6 : 12bn^2a_2^2b_1 + 4am^2b_0b_1^2b_2 \theta^2 ln^2(a)
\]

\[
- 2amna_0b_0b_1^2 \theta^2 ln^2(a) + 2am^2a_1b_0b_1 \theta^2 ln^2(a)
\]

\[
- 4n^2m^2a^2b_0^2 \theta^2 - 2am^2a_2b_0^2 \theta^2 ln^2(a)
\]

\[
- 8am^2b_0b_1 \theta^2 ln^2(a) + 24bkn^2a_1^2 \theta^2
\]

\[
+ 8amna_0b_0b_1 \theta^2 ln^2(a) + 4bkn^2a_1^2 \theta^2
\]

\[
- 6amna_2b_0b_1 \theta^2 ln^2(a) + 4amn^2b_0^2a_1 \theta^2 ln^2(a)
\]

\[
+ am^2b_2^2b_0^2 \theta^2 ln^2(a) + 16bkn^2a_0a_1^2 + 4n^2mwa_2b_1
\]

\[
+ 8amna_0b_0b_1 \theta^2 ln^2(a) = 0
\]

\[
Q^5 : 12bn^2a_2^2b_1 + 48bkn^2a_0a_1^2
\]

\[
+ 16amna_2b_0b_1a_0^2 \theta^2 ln^2(a) + 8n^2mwa_2b_0b_1
\]

\[
+ 12bn^2a_0a_1^2b_1 + 4am^2a_2b_0^2b_1 \theta^2 ln^2(a)
\]

\[
- 8amna_1b_0b_1a_2 \theta^2 ln^2(a) - 2amna_0b_0b_1a_2 \theta^2 ln^2(a)
\]

\[
+ 12bn^2a_1a_2^2b_0 - 4am^2a_1b_0b_1a_2 \theta^2 ln^2(a)
\]

\[
- 16amna_2^2b_0^2a_1 \theta^2 ln^2(a) + 4amn^2b_0^2a_2 \theta^2 ln^2(a)
\]

\[
- 4am^2a_2b_0b_1a_2 \theta^2 ln^2(a) + 8amna_2b_0b_1 \theta^2 ln^2(a)
\]

\[
+ 16bkn^2a_1^2b_1 + 8n^2m^2a_2b_0b_1 - 8n^2m^2a_2b_0b_1 \theta^2 ln^2(a)
\]

\[
- 4amna_2b_0b_1 \theta^2 ln^2(a) - 8amn^2b_0^2a_2 \theta^2 ln^2(a) = 0,
\]
\[ Q^4 : = -8n^2m^2a^2\alpha_0a_2b_1^2 - 2am_2a_2b_1^2a_0 + 16n^2mwa_1a_2b_0b_1 + 2am_2a_1b_0b_2b_1\ln^2(a) + 4bn^2a_1^2 + 8amn\alpha_1^2a_0\mu_2\ln^2(a) - 4n^2m^2a^2\alpha_2^2b_1^2 - 4n^2m^2a^2\alpha_2^2b_0^2 + 8n^2mwa_2a_1b_1 + 24bn^2a_0a_1b_1 + 48bn\alpha_1a_2a_1^2 + 4am_1^2b_0^2a_0^2\mu_2\ln^2(a) + 4bn^2a_1^2b_1 + 2amn\alpha_1^2b_0^2\ln^2(a) + 8am_2a_2b_0b_1a_0^2\mu_2\ln^2(a) - 2amn\alpha_1^2b_0^2\mu_2\ln^2(a) + 4n^2mwa_1b_1^2 + 4n^2mwa_2b_0^2 - 12bn^2a_0a_2b_0 - 16n^2m^2a^2_1a_2b_0b_1 + 12bn^2a_1^2a_2 + 4am_2^2a_0b_0^2\mu_2\ln^2(a) - 8am^2a_1b_0a_2^2\mu_2\ln^2(a) - 28amn\alpha_2b_0b_1a_0^2\mu_2\ln^2(a) - 10amn\alpha_3b_0^3a_1^2\mu_2\ln^2(a) + 24bn^2a_2^2a_0^2 - \alpha^2 + 2amn\alpha_1^2b_0^2\mu_2\ln^2(a) + 12amn\alpha_2b_0^3a_0\mu_2\ln^2(a) - 2amn\alpha_1^2b_0^2a_0^2\mu_2\ln^2(a) - 2am_2a_1b_0a_0^2\mu_2\ln^2(a) = 0, \]

\[ Q^3 : = -2am^2a_1^2b_0^3\mu_2\ln^2(a) - 2amn\alpha_1^2b_0^2\mu_2\ln^2(a) + 2amn\alpha_1^2b_0^2\mu_2\ln^2(a) + 24amn\alpha_1^2b_0^2\mu_2\ln^2(a) + 24amn\alpha_2b_0^3a_0\mu_2\ln^2(a) - 8n^2m^2a^2\alpha_2^2b_0^2 - 8n^2m^2a^2\alpha_1^2b_1^2 - 8n^2m^2a^2\alpha_1^2b_3^2 + 16bn^2a_0a_2^2 + 8n^2m^2a_0a_2^2b_1 + 8n^2mwa_0a_2b_1 + 24bn^2a_0a_2b_1 + 48bn^2a_0a_2b_0 - 4bn^2a_1^2b_0 + 12bn^2a_0^2b_1 + 12amn\alpha_2b_0^3a_0\mu_2\ln^2(a) - 2amn\alpha_1^2b_0^2a_0^2\mu_2\ln^2(a) - 2amn\alpha_1^2b_0^2a_0^2\mu_2\ln^2(a) - 2amn\alpha_1^2b_0^2a_0^2\mu_2\ln^2(a) - 2amn\alpha_1^2b_0^2a_0^2\mu_2\ln^2(a) = 0, \]

\[ Q^1 : = -8n^2m^2a^2\alpha_0a_1b_0^2 - 2amn\alpha_2b_0^3\mu_2\ln^2(a) - 8n^2m^2a^2\alpha_2b_0b_1 + 8n^2m^2wa_0a_1b_0^2 + 4bn^2a_0b_1 + 16bn^2a_0^2a_1 + 12bn^2a_0^2b_1 + 2amn\alpha_2b_0^3a_0\mu_2\ln^2(a) + 8n^2m^2wa_0b_0b_1 = 0, \]

\[ Q^0 : = -4n^2m^2a^2\alpha_0^2b_0^2 + 4bn^2a_0^2b_0 + 4n^2m^2wa_0b_0^2 = 0. \quad (3.1.13) \]

Solving the system of algebraic equations (3.1.13) by Maple or Mathematica, we obtain the following sets:

**Set 1:**

\[ \mu = \frac{1}{\ln(a)} \sqrt{-\frac{bn^2(2n+m)}{amn(k(n+m)^2)}}, \]

\[ w = amn^2 + \frac{2n(m+n)}{4(k(n+m)^2)}, \]

\[ a_0 = 0, \quad a_1 = \frac{b_0(2n+m)}{2k(n+m)}, \]

\[ a_2 = -\frac{b_1(2n+m)}{2k(n+m)}, \]

\[ b_0 = b_0, b_1 = b_1, \lambda = \lambda, m = m, n = n, \]

\[ k = k, a = a, b = b. \quad (3.1.14) \]

Substituting (3.1.14) into (3.1.10), we get the following solution:

\[ \nu(\zeta) = \frac{-\ln(2n+m)}{2k(n+m)(1 + \alpha^2)}. \quad (3.1.15) \]

From (3.1.7) and (3.1.15) we have

\[ U(\zeta) = \left[ -\frac{(2n+m)}{2k(n+m)(1 + \alpha^2)} \right]^{\frac{1}{2}}. \quad (3.1.16) \]

With the help of (2.10) and (2.11) the exact solution of Eq. (3.1.1) has the form:

\[ q(x, t) = -\frac{(2n+m)}{4k(n+m)} \times \left[ 1 - \tan \left( \frac{1}{2} \ln \left( \frac{-bn^2(2n+m)}{amn(k(n+m)^2)}(x - 2am\lambda t) \right) \right) \right]^{\frac{1}{2}}, \quad (3.1.17) \]

\[ = \left\{ \begin{array}{c}
\frac{-\ln(2n+m)}{4k(n+m)} \\
\left[ 1 - \tanh \left( \frac{1}{2} \ln \left( \frac{-bn^2(2n+m)^2}{amn(k(n+m)^2)}(x - 2am\lambda t) \right) \right) \right]^{\frac{1}{2}} \times \exp(i\theta),
\end{array} \right. \quad (3.1.18) \]
or
\[
q(x,t) = \left\{ -\frac{(2n+m)}{4k(n+m)} \right\} \times \left[ 1 - \csc Fs \left( \frac{1}{2} \ln(a) \sqrt{-\frac{bn^2(2n+m)}{amk(n+m)^2}(x-2am\lambda t)} \right) \right]^{\frac{1}{2n}} \times \exp(i\theta),
\]
\[
q(x,t) = \left\{ -\frac{(2n+m)}{4k(n+m)} \right\} \times \left[ 1 - \coth \left( \frac{1}{2} \ln(a) \sqrt{-\frac{bn^2(2n+m)}{amk(n+m)^2}(x-2am\lambda t)} \right) \right]^{\frac{1}{2n}} \times \exp(i\theta),
\]
where \( \theta = \lambda x - \left( am\lambda^2 + \frac{b(2n+m)}{4k(n+m)^2} \right) t \), provided that \( ab > 0 \) and \( k < 0 \).

**Set 2:**
\[
\mu = \frac{1}{\ln(a)} \sqrt{-\frac{bn^2(2n+m)}{amk(n+m)^2}},
\]
\[
w = am\lambda^2 + \frac{b(2n+m)}{4k(n+m)^2},
\]
\[
a_0 = a_1 = -\frac{b_1(2n+m)}{2k(n+m)},
\]
\[
a_2 = \frac{b_1(2n+m)}{2k(n+m)}, b_0 = b_1 = b_1,
\]
\[
\lambda = \lambda, m = m, n = n, k = k, a = a, b = b.
\]

Substituting (3.21) into (3.10), we get the following solution:
\[
v(\zeta) = -\frac{(2n+m)}{2k(n+m)} \frac{a^\zeta}{(a^\zeta \pm 1)}.
\]

From (3.17) and (3.112) we have
\[
U(\xi) = \left[ \frac{(2n+m)}{2k(n+m)} \frac{a^\xi}{(a^\xi \pm 1)} \right]^{\frac{1}{2n}}.
\]

With the help of (2.10) and (2.11) the exact solution of Eq. (3.11) has the form:
\[
q(x,t) = \left[ -\frac{(2n+m)}{4k(n+m)} (1 + \tanh \eta) \right]^{\frac{1}{2n}} \exp(i\theta),
\]
\[
q(x,t) = \left[ -\frac{(2n+m)}{4k(n+m)} (1 + \coth \eta) \right]^{\frac{1}{2n}} \exp(i\theta),
\]
where \( \eta = \frac{1}{2} \sqrt{-\frac{bn^2(2n+m)}{amk(n+m)^2}(x-2am\lambda t)}, \theta = \lambda x - \left( am\lambda^2 + \frac{b(2n+m)}{4k(n+m)^2} \right) t \), provided that \( ab > 0 \) and \( k < 0 \).

**Set 3:**
\[
\mu = \frac{1}{\ln(a)} \sqrt{-\frac{bn^2(2n+m)}{amk(n+m)^2}},
\]
\[
w = am\lambda^2 + \frac{b(2n+m)}{4k(n+m)^2}, a_1 = -\frac{(b_0 + b_1)(2n+m)}{2k(n+m)},
\]
\[
a_0 = a_2 = 0, b_0 = b_0, b_1 = b_1,
\]
\[
\lambda = \lambda, m = m, n = n, k = k, a = a, b = b.
\]

Consequently, we have the exact solutions of Eq. (3.11) in the form:
\[
q(x,t) = \left[ \frac{(b_0 + b_1)(2n+m)(1 - \tanh \eta)}{2k(n+m)[2b_0 + b_1(1 - \tanh \eta)]} \right]^{\frac{1}{2n}} \exp(i\theta),
\]
\[
q(x,t) = \left[ \frac{(b_0 + b_1)(2n+m)(1 - \coth \eta)}{2k(n+m)[2b_0 + b_1(1 - \coth \eta)]} \right]^{\frac{1}{2n}} \exp(i\theta),
\]
where \( \eta = \frac{1}{2} \sqrt{-\frac{bn^2(2n+m)}{amk(n+m)^2}(x-2am\lambda t)}, \theta = \lambda x - \left( am\lambda^2 + \frac{b(2n+m)}{4k(n+m)^2} \right) t \), provided that \( ab > 0 \) and \( k < 0 \).

On comparing our result (3.24), with the result (18) obtained in [71], we conclude that the two results are equivalent with \( \lambda = -\kappa \), while our results (3.18), (3.20), (3.25), (3.27) and (3.28) are new, and not discussed elsewhere.

### 3.2 Example 2. The Zakharov–Kuznetsov equation (ZK(m,n,k))

In this subsection, we apply the given method to solve the ZK(m,n,k) [73, 74]
\[
u_t + \lambda_0(u^m)_x + \lambda_1(u^n)_{xxx} + \lambda_2(u^k)_{xyy} = 0,
\]
where \( u = u(x, y, t) \) is a readily differentiable function, \( \lambda_0, \lambda_1 \) and \( \lambda_2 \) are arbitrary constants while \( m, n \) and \( k \) are positive integers. The function governs the behaviour of weakly nonlinear ion acoustic waves in a plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field [77]. Recently, Ma et al. [73] used the auxiliary equation method to find the solutions of the ZK(2,1) equation with \( \lambda_0 = \frac{1}{2}, \lambda_1 = \frac{1}{2}, \lambda_2 = \frac{2}{3} \).

This work is concerned with two cases of Eq. (3.21):

#### 3.2.1 Case 1: ZK(2,1)

In this case Eq. (3.21) becomes the form:
\[
u_t + \lambda_0(u^2)_x + \lambda_1 u_{xxx} + \lambda_2 u_{xyy} = 0.
\]
To seek travelling wave solutions, we use the wave transformation
\[ u(x, y, t) = U(\zeta), \quad \zeta = x + \beta y + \gamma t, \] (3.2.3)
where \( \beta \) and \( \gamma \) are arbitrary constants with \( \beta, \gamma \neq 0 \), to reduce equation (3.2.2) to the ODE:
\[ \gamma U' + \lambda_0 U^2 + (\lambda_1 + \lambda_2 \beta^2) U'' = 0. \] (3.2.4)
Integrating Eq. (3.2.4) with respect to \( \zeta \), with zero constant of integration, we get
\[ \gamma U + \lambda_0 U^2 + (\lambda_1 + \lambda_2 \beta^2) U'' = 0. \] (3.2.5)
Balancing the \( U' \) and \( U^2 \) in (3.2.5), then we have \( N = M + 2 \). If we choose \( M = 1 \) and \( N = 3 \), then the formal solution of Eq. (3.2.5) has the form:
\[ U(\zeta) = \frac{a_0 + a_1 Q + a_2 Q^2 + a_3 Q^3}{b_0 + b_1 Q}, \] (3.2.6)
and consequently,
\[ U'(\zeta) = \frac{Q (Q - 1)}{(b_0 + b_1 Q)^2} \left[ (a_1 + 2a_2 Q + 3a_3 Q^2) (b_0 + b_1 Q) - b_1 \left( a_0 + a_1 Q + a_2 Q^2 + a_3 Q^3 \right) \right] \ln(a), \] (3.2.7)
\[ U''(\zeta) = \frac{Q (Q - 1) (2Q - 1)}{(b_0 + b_1 Q)^3} \left[ (a_1 + 2a_2 Q + 3a_3 Q^2) (b_0 + b_1 Q) - b_1 \left( a_0 + a_1 Q + a_2 Q^2 + a_3 Q^3 \right) \right] \ln^2(a) + \frac{Q^2 (Q - 1)^2}{(b_0 + b_1 Q)^4} \left[ (b_0 + b_1 Q)^2 (2a_2 + 6a_3 Q) - 2b_1 \left( a_1 + 2a_2 Q + 3a_3 Q^2 \right) (b_0 + b_1 Q) + 2b_1^2 \left( a_0 + a_1 Q + a_2 Q^2 + a_3 Q^3 \right) \right] \ln(a). \] (3.2.8)
Substituting (3.2.6) and (3.2.8) into (3.2.5), and equating all the coefficients of \( Q^i, (i = 0, 1, \ldots, 7) \) to zero, we get a system of algebraic equations, which can be solved using Maple, to get the following sets:

**Set 1:**
\[ a_0 = 0, \]
\[ a_1 = \frac{6b_0 \left( \lambda_1 + \lambda_2 \beta^2 \right) \ln^2(a)}{\lambda_0}, \]
\[ a_2 = -\frac{6(b_0 - b_1) \left( \lambda_1 + \lambda_2 \beta^2 \right) \ln^2(a)}{\lambda_0}, \]
\[ a_3 = -\frac{6b_1 \left( \lambda_1 + \lambda_2 \beta^2 \right) \ln^2(a)}{\lambda_0}, \]
\[ b_0 = b_0, \quad b_1 = -2b_0, \quad \gamma = -\left( \lambda_1 + \lambda_2 \beta^2 \right) \ln^2(a), \]
\[ \beta = \beta, \quad \lambda_0 = \lambda_0, \quad \lambda_1 = \lambda_1, \quad \lambda_2 = \lambda_2. \] (3.2.9)

Substituting (3.2.9) into (3.1.10), we get the following solution of Eq. (3.2.6):
\[ U(\zeta) = \frac{6 \left( \lambda_1 + \lambda_2 \beta^2 \right) \ln^2(a)}{\lambda_0} \frac{a^\gamma}{(a^\gamma + 1)^2}. \] (3.2.10)

With the help of (3.1.10) and (3.2.11) the exact solution of Eq. (3.2.2) in the form:
\[ u(x, y, t) = \frac{3 \left( \lambda_1 + \lambda_2 \beta^2 \right) \ln^2(a)}{2\lambda_0} \text{sech} \eta \] (3.2.11)
or
\[ u(x, y, t) = -\frac{3 \left( \lambda_1 + \lambda_2 \beta^2 \right) \ln^2(a)}{2\lambda_0} \text{csch} \eta, \] (3.2.12)
where \( \eta = \frac{1}{2} \left[ x + \beta y - (\lambda_1 + \lambda_2 \beta^2) \ln^2(a) \right] t \ln(a) \).

**Set 2:**
\[ a_0 = -\frac{b_0 \left( \lambda_1 + \lambda_2 \beta^2 \right) \ln^2(a)}{\lambda_0}, \]
\[ a_1 = \frac{(6b_0 - b_1) \left( \lambda_1 + \lambda_2 \beta^2 \right) \ln^2(a)}{\lambda_0}, \]
\[ a_2 = -\frac{6 \left( \lambda_1 + \lambda_2 \beta^2 \right) \ln^2(a)}{\lambda_0}, \]
\[ a_3 = -\frac{6b_1 \left( \lambda_1 + \lambda_2 \beta^2 \right) \ln^2(a)}{\lambda_0}, \]
\[ \gamma = \left( \lambda_1 + \lambda_2 \beta^2 \right) \ln^2(a), \quad b_0 = b_0, b_1 = b_1, \]
\[ \beta = \beta, \quad \lambda_0 = \lambda_0, \quad \lambda_1 = \lambda_1, \quad \lambda_2 = \lambda_2. \] (3.2.13)

Consequently, we have the exact solutions of Eq. (3.2.2) in the form:
\[ u(x, y, t) = \frac{\left( \lambda_1 + \lambda_2 \beta^2 \right) \ln^2(a)}{2\lambda_0} \left[ 1 - 3 \tanh^2 \eta \right], \] (3.2.14)
or
\[ u(x, y, t) = \frac{\left( \lambda_1 + \lambda_2 \beta^2 \right) \ln^2(a)}{2\lambda_0} \left[ 1 - 3 \coth^2 \eta \right], \] (3.2.15)
where \( \eta = \frac{1}{2} \left[ x + \beta y + (\lambda_1 + \lambda_2 \beta^2) \ln^2(a) \right] t \ln(a) \).

If we choose \( \lambda_0 = \frac{1}{2}, \lambda_1 = \frac{1}{2}, \lambda_2 = \frac{1}{2}, a = e \) in our results (3.2.11),(3.2.12),(3.2.14) and (3.2.15) we have the well-known results \( u_1, u_2 \) of (10) and \( u_1, u_2 \) of (13) obtained in [73].

**3.2.2 Case 2: ZK(m,1,1), m \geq 3**

In this case Eq. (3.2.1) becomes the form:
\[ u_t + \lambda_0 (u^m)_x + \lambda_1 u_{xxx} + \lambda_2 u_{xyy} = 0. \] (3.2.16)
Using the same wave transformation (3.2.3), to reduce equation (3.2.16) to the ODE:

\[ \gamma U' + \lambda_0 U'' + \lambda_1 U + \lambda_2 U'' = 0. \]  

(3.2.17)

Integrating Eq. (3.2.17) with respect to \( \zeta \), with zero constant of integration, we get

\[ \gamma U + \lambda_0 U + (\lambda_1 + \lambda_2) U'' = 0. \]  

(3.2.18)

By balancing \( U'' \) with \( U'' \) we have \( N = \frac{2}{m-1} \), \( m \geq 3 \). Then we use the transformation

\[ U(\zeta) = [v(\zeta)]^\frac{2}{m-1}. \]  

(3.2.19)

Substituting (3.2.19) into equation (3.2.18) we have the new equation

\[
\begin{align*}
(m-1)^2 (\gamma v'^2 + \lambda_0 v^4) + (\lambda_1 + \lambda_2 v^2) = 0,
\end{align*}
\]

(3.2.20)

Balancing the \( v'v' \) and \( v^4 \) in (3.2.20), then we have \( N = M + 1 \). If we choose \( M = 1 \) and \( N = 2 \), then Eq. (3.2.20) has the same formal solution (3.1.10).

Substituting (3.1.10),(3.1.11) and (3.1.12) into (3.2.20), and equating all the coefficients of \( \zeta'^i(i = 0, 1, \ldots, 8) \) to zero, we get a system of algebraic equations, which can be solved using the aid of Maple or Mathematica, to get the following result:

\[
\begin{align*}
a_0 &= 0, \\
a_1 &= \pm \frac{2b_0 \ln(a)}{(m-1)} \sqrt{-\frac{2(\lambda_1 + \lambda_2)}{\lambda_0}}, \\
a_2 &= \frac{2b_0 \ln(a)}{(m-1)} - \frac{2(\lambda_1 + \lambda_2)}{\lambda_0}, \\
b_0 &= b_0, \\
b_1 &= -2b_0, \\
\gamma &= -\frac{4(\lambda_1 + \lambda_2) \ln^2(a)}{(m-1)^2}, \\
\beta &= \beta, \lambda_0 = \lambda_0, \lambda_1 = \lambda_1, \lambda_2 = \lambda_2. \\
\end{align*}
\]

(3.2.21)

provided that \( (\lambda_1 + \lambda_2) < 0 \).

Substituting (3.2.21) into (3.1.10), we get the following solution of Eq. (3.2.20):

\[ v(\zeta) = \pm \frac{2 \ln(a)}{(m-1)} \sqrt{-\frac{2(\lambda_1 + \lambda_2)}{\lambda_0}} \frac{a^\zeta}{(a^2 \zeta - 1)}. \]  

(3.2.22)

From (3.2.22) and (3.2.19) we have

\[ U(\zeta) = \left[ \pm \frac{2 \ln(a)}{(m-1)} \sqrt{-\frac{2(\lambda_1 + \lambda_2)}{\lambda_0}} \frac{a^\zeta}{(a^2 \zeta - 1)} \right]^\frac{1}{m-1}. \]  

(3.2.23)

With the help of (2.10) and (2.11) the exact solution of Eq. (3.2.16) in the form:

\[ u(x, y, t) = \left[ \pm \frac{\ln(a)}{(m-1)} \sqrt{-\frac{2(\lambda_1 + \lambda_2)}{\lambda_0}} \frac{\cosh(\eta)}{\eta} \right]^\frac{1}{m-1}, \]  

(3.2.24)

where \( \eta = (x + \beta y - \frac{4(\lambda_1 + \lambda_2)}{(m-1)^2} t) \ln(a) \).

### 3.3 Example 3. The K(m,n) equation

The K(m,n) equation with the generalized evolution term [75, 78, 79] is given by

\[ (q^m)_{l} + a q^n q_x + b (q^m)_{xxx} = 0, \]  

(3.3.1)

where, the first term is the generalized evolution term, the second term represents the nonlinearity, and the third term is the dispersion. Also, \( a, b \in R \) and are constants, while \( l, m, n \in Z^* \).

Eq. (3.3.1) is the generalized form of the KdV equation, where, in particular, the case \( l = m = n = 1 \) leads to the KdV equation. Eq. (3.3.1) appeared for the first time in [78] for \( l = 1 \). Thus, Eq. (3.3.1) reduces to the K(m,n) equation for \( l = 1 \). Therefore, for \( l = 1 \), K(1,1) is the KdV equation while K(2,1) is the mKdV equation. Eq. (3.3.1) has been discussed in [75] using the \((G'/G)\)-expansion method and its exact solutions have been found.

In order to solve Eq. (3.3.1) using the method of section 2, we introduce the wave transformation

\[ q(x, t) = U(\zeta), \quad \zeta = \mu(x - ct), \]  

(3.3.2)

where \( \mu \) and \( c \) are constants, to reduce Eq. (3.3.1) to the ODE:

\[ -\mu c U' + \mu a U + \mu^3 b U'' = 0. \]  

(3.3.3)

Integrating Eq. (3.3.3) with respect to \( \zeta \), with zero constant of integration, we have

\[ -\mu c U + \frac{\mu a}{m + 1} U^{m+1} + \mu^3 b U'' = 0. \]  

(3.3.4)

Let \( l = n \), balancing \( U^{m+1} \) and \( U'' \) in (3.3.4) yields \( N = \frac{2}{m+1-n} \), where \( m + 1 \neq n \). In order to obtain the closed form solutions, we use the transformation

\[ U(\zeta) = [v(\zeta)]^\frac{1}{m+n}, \]  

(3.3.5)

to reduce Eq. (3.3.4) into the ODE

\[ -c (m + 1 - n)^2 v^2 + \frac{a (m + 1 - n)^2 v^3}{m + 1} + b n u^2 (2n - m - 1) v^2 + b n u^2 (m + 1 - n)^2 v v'' = 0. \]  

(3.3.6)
Balancing the $v v^\nu$ and $v^3$ in (3.3.6), then we have $N = M + 2$. If we choose $M = 1$ and $N = 3$, then Eq. (3.3.6) has the same formal solution (3.2.6).

Substituting (3.2.6),(3.2.7) and (3.2.8) into (3.3.6), and equating all the coefficients of $Q^i$, ($i = 0, 1, \ldots, 10$) to zero, we get a system of algebraic equations, which can be solved using the aid of Maple or Mathematica, to get the following result:

$$
a_0 = 0, \quad a_1 = \frac{2cb_0 (m + 1)(m + 1 + n)}{na},
$$
$$
a_2 = \frac{-2c(b_0 - b_1)(m + 1)(m + 1 + n)}{na},
$$
$$
a_3 = \frac{-2cb_1 (m + 1)(m + 1 + n)}{na},
$$
$$
\mu = \pm \sqrt{\frac{c(m + 1 - n)}{b n \ln(a)}}, \quad b_0 = b_0, \quad b_1 = b_1, \quad c = c,
$$

(3.3.7)

provided that $bc > 0$.

Substituting (3.3.7) into (3.2.6), we get the following solution of Eq. (3.3.6):

$$
v(\zeta) = \pm \frac{2c(m + 1)(m + 1 + n)}{na} \frac{a^\zeta}{(a^\zeta \pm 1)}, \quad (3.3.8)
$$

From (3.3.8) and (3.3.5) we have

$$
U(\zeta) = \left[ \pm \frac{2c(m + 1)(m + 1 + n)}{na} \frac{a^\zeta}{(a^\zeta \pm 1)} \right]^{\frac{1}{m+n}}. \quad (3.3.9)
$$

With the help of (2.10) and (2.11) the exact solution of Eq. (3.3.1) in the form:

$$
u(x, y, t) = \left[ \frac{c(m + 1)(m + 1 + n)}{2na} \right]^{\frac{1}{m+n}} \operatorname{sech} \eta, \quad (3.3.10)
$$
or

$$
u(x, y, t) = \left[ - \frac{c(m + 1)(m + 1 + n)}{2na} \right]^{\frac{1}{m+n}} \operatorname{csch} \eta, \quad (3.3.11)
$$

where $\eta = \pm \sqrt{\frac{c}{b} \frac{(m+1-n)}{2n}} (x - ct)$.

4 Physical explanations for some of our solutions

In this section, we will illustrate the application of the results established above. Exact solutions of the results describe different nonlinear waves. The established exact solutions with symmetrical hyperbolic Fibonacci functions are special kinds of solitary waves.

5 Conclusions

We now examine Figures (1–4) which illustrate a selection of the results obtained above. Specific parameter values are selected, for example: in some of the solutions (3.1.24) and (3.1.25) of the Biswas-Milovic equation with dual-power law nonlinearity with $-10 < x, t < 10$, solutions (3.2.11) and (3.2.12) of the ZK(2,1,1) with $-10 < x, t < 10$, solution (3.2.24) of the ZK(m,1,1) with $-10 < x, t < 10$, solutions (3.3.10) and (3.3.11) of the K(m,n) equation with the generalized evolution with $-10 < x, t < 10$.

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4 Physical explanations for some of our solutions

In this section, we will illustrate the application of the results established above. Exact solutions of the results describe different nonlinear waves. The established exact solutions with symmetrical hyperbolic Fibonacci functions are special kinds of solitary waves.

5 Conclusions

In this paper we have shown that the symmetrical hyperbolic Fibonacci function solutions can be obtained for the general $\text{Exp}_a$-function by using generalized Kudryashov
method. We have successfully extended the generalized Kudryashov method to solve three nonlinear partial differential equations. In terms of practical applications, we have obtained many new symmetrical Fibonacci hyperbolic function solutions for the Biswas-Milovic equation with dual-power law nonlinearity, the ZK(m,n,k) and the K(m,n) equation with the generalized evolution term. This demonstrates that the generalized Kudryashov method is powerful, effective and convenient for solving nonlinear PDEs. The physical explanation for certain solutions of such equations have been presented. The generalized Kudryashov method provides a powerful mathematical tool to obtain more general exact analytical solutions of many nonlinear PDEs in mathematical physics.

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