Research Article

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Distributed containment control of heterogeneous fractional-order multi-agent systems with communication delays

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Abstract: Because many networked systems can only be characterized with fractional-order dynamics in complex environments, fractional-order calculus has been studied deeply recently. When diverse individual features are shown in different agents of networked systems, heterogeneous fractional-order dynamics will be used to describe the complex systems. Based on the distinguishing properties of agents, heterogeneous fractional-order multi-agent systems (FOMAS) are presented. With the supposition of multiple leader agents in FOMAS, distributed containment control of FOMAS is studied in directed weighted topologies. By applying Laplace transformation and frequency domain theory of the fractional-order operator, an upper bound of delays is obtained to ensure containment consensus of delayed heterogeneous FOMAS. Consensus results of delayed FOMAS in this paper can be extended to systems with integer-order models. Finally, numerical examples are used to verify our results.

Keywords: Heterogeneous fractional order, Distributed containment control, Multi-agent systems, Communication delays

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1 Introduction

Recently, the problems of fractional calculus have been paid more and more attention since fractional operators have been broadly applied in more and more scientific fields, for examples: in mechanics, physics, material science, informatics and engineering [1–7]. Not only can fractional order controllers be used to add robustness of control systems, but controlled more easily than that of integer-order systems [8, 9].

With the development of communication technology, distributed coordination of networked systems has been studied in details [10–14]. Cooperative control of multiple agent systems has become a hot topic in the fields of automation, computer science, etc. [15–19]. It has been applied in both military and civilian territories, such as formation control of mobile robots, cooperative control of unmanned spacecraft, and attitude adjustment of satellites, etc. As a kind of distributed consensus control problems for distributed systems with multiple leaders, containment control will regulate followers eventually converging to a target geometric area (convex hull formed by the leaders) by designing a communication protocol [20–24].

In the actual situation, many dynamical systems can not be depicted with the integer-order differential equation and only be characterized with the fractional-order differentials [1, 2, 8]. For example, flocking movement by means of the individual secretions, food searching of microorganisms in the complex environments, formation of submersible robots in the seabed with a mass number of microorganisms [3, 9]. The coordination of fractional-order multi-agent systems (FOMAS) were studied by Cao and Ren in [4, 5], and the relationship between the agents’ number and the fractional order is obtained for a stable system. The distributed coordination of FOMAS with communication delays for continue-time and discrete-time systems were studied by Yang et.al. [6, 7]. Motivated by the broad application of FOMAS, distributed containment control of FOMAS will be studied in this paper.

There is a lot of researche on containment control problems mainly focused on integer-order systems [20–25]. In [20], containment control for first-order multi-agent systems with the undirected connected topology was investigated, and the effectiveness of control strategy was

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proven by using partial equation method. In [21], second-order multi-agent systems with multiple leaders were studied, the necessary and sufficient conditions for containment control of multi-agent systems were obtained. Two asymptotic containment controls of continuous-time systems and discrete-time systems were proposed in [22], and the constraint condition for control gain and sampling period was obtained. By considering factors such as external disturbance and parameter uncertainty in [23], the attitude containment control of nonlinear systems was studied in directed network. Supposing multi-agent systems consist of first-order and second-order integrator agents with multiple leaders, the containment control of distributed systems was studied in [25]. Containment control of dynamical systems in a noisy communication environment was studied in [26] and [27].

Since coordination networks apply the sensors or other communication devices to transfer the information, communication delays have a great impact on the networks’ behaviors. Now, the influences of communication delays on multi-agent systems have also been paid more attentions [11, 16–19, 24, 28] where these research activities on the coordination problem are mainly concentrated on integer-order multi-agent systems. In [24], containment control of distributed systems with time delays was studied in fixed topology, and two cases of multiple dynamic leaders and multiple stationary leaders were discussed, respectively. To the best of Authors’ knowledge, few researches have been done on the containment control of fractional order multiple leader systems with time delays.

In this paper, we assume that the heterogeneous dynamics of fractional-order systems is shown in a complex environment, and containment control of chiasmal order FOMAS with multiple leaders and communication delays is analyzed. It is different for the research of this paper with that in Reference [6, 29, 30], where consensus of FOMAS without leader is easier than containment control of FOMAS with multiple leaders in this paper. The rest of the paper is organized as follows. In Section 2, we introduce some essential conceptions about fractional calculus. In Section 3, some preliminaries about graph theory are shown, and heterogenous fractional order coordination model of multi-agent systems is presented. Containment control of fractional coordination algorithm for multi-agent systems with communication delays is studied in Section 4. In Section 5, the theoretical results are verified by numerical simulations. Finally, conclusions are drawn in Section 6.

## 2 Fractional calculus

Fractional calculus plays an important role in science research. There are two widely used fractional-order calculus system: Caputo and Riemann-Liouville fractional-order operators [8]. In physical systems, Caputo fractional-order calculus is more practical since Caputo fractional algorithm does not need an initial value. Therefore, Caputo fractional-order calculus will be used to model the system dynamical equation in this paper. In general, Caputo fractional-order calculus includes the Caputo fractional-order integral and the Caputo fractional-order derivative. The Caputo fractional-order integral is defined as

\[
aD^p_{a+} f(t) = \frac{1}{\Gamma(p)} \int_{a}^{t} \frac{f(\theta)}{(t-\theta)^{1-p}} d\theta,
\]

where the integral order \( p \in (0, 1) \), \( \Gamma(\bullet) \) is the Gamma function, and real number \( a \) is the lower limit of the integral. On account of the Caputo fractional-order integral, the Caputo fractional-order derivative is defined as

\[
aD^p_{a+} f(t) = aD^p_{a+} \left[ \frac{d^{[a]+1}}{d[t^{a+1}]} f(t) \right],
\]

where \( a \) is a nonnegative real number, \( p = [a] + 1 - a \in (0, 1) \) and \([a]\) is the integer part of \( a \). When \( a \) is an integer, \( p = 1 \) and the Caputo fractional-order derivative will be equivalent to the integer-order derivative. In this paper, a simple notation \( f^{(a)}(t) \) is used instead of \( L[f(t)] \). Let \( \mathcal{L} \) express the Laplace transformation of a function, the Laplace transformation of the Caputo derivative is shown as

\[
\mathcal{L}(f^{(a)}(t)) = s^a F(s) - \sum_{k=1}^{[a]+1} s^{a-k} f^{(k-1)}(0),
\]

where \( F(s) = \mathcal{L}(f(t)) = \int_{0}^{\infty} e^{-st} f(t) dt \) is the Laplace transformation of function \( f(t) \), \( f^{(k)}(0) = \lim_{\xi \to 0} f^{(k)}(\xi) \) and \( f^{(0)}(0) = f(0) = \lim_{\xi \to 0} f(\xi) \).

## 3 Containment control of heterogenous FOMAS

### 3.1 Problem description

In this paper, a directed graph \( G = (V, E) \) will be used to describe the relationships among agents. In the graph \( G \), an arc \((i, j)\) from agent \(i\) to agent \(j\) represents an information exchange link from agent \(j\) to agent \(i\), that is, agent \(i\)
receives the information from agent $j$. The neighboring set of the $i$th agent is defined as $N_i = \{j | j \in V, (i, j) \in E\}$. Let the adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ where $a_{ij} > 0$ if $(i, j) \in V$, otherwise $a_{ij} = 0$. Assume $G$ without self-loops, i.e., $a_{ii} = 0$. A path in $G$ from one node $i$ to another node $j$, i.e., $i$ is reachable from $j$. It is called globally reachable if a node is reachable from every other node in the graph. The Laplacian matrix of the digraph $G$ is illustrated as $L = D - A$, where matrix $D = \text{diag}(d_1, d_2, ..., d_n)$ is the out-degree matrix with the elements $d_i = \sum_{j=1}^{n} a_{ij}$.

**Lemma 1.** [12]. 0 is a simple eigenvalue of Laplacian matrix $L$, and $X_0 = C[1, 1, ..., 1]^T$ is corresponding right eigenvector, i.e., $LX_0 = 0$, if and only if the digraph $G = (V, E, A)$ has a globally reachable node.

**Lemma 2.** [18]. Matrix $L + B$ is a positive definite matrix, where $L$ is a Laplacian matrix of the digraph $G = (V, E, A)$ with a globally reachable node, and $B = \text{diag}(b_1, ..., b_n)$ with $b_1 \geq 0$ and at least there is one element $b_i > 0$.

**Definition 1.** The convex hull $Co\{x_1, ..., x_m\}$ is the minimal convex set including all points $x_i$, $i = 1, ..., m$. More specifically, $Co\{x_1, ..., x_m\} = \{\sum_{i=1}^{m} v_i x_i | v_i > 0, \sum_{i=1}^{m} v_i = 1\}$.

In this paper, we assume there are two groups of multi-agent systems with heterogeneous fractional-order dynamics in the complex environment. The heterogeneous fractional-order dynamical equations with Caputo fractional derivative are described as:

$$
\begin{align*}
X_i^{(a)}(t) &= u_i(t), i = 1, ..., m, \\
X_i^{(b)}(t) &= u_i(t), i = m + 1, ..., n,
\end{align*}
$$

where $X_i(t) \in \mathbb{R}$ and $u_i(t) \in \mathbb{R}$ represent the $i$-th agent’s state and control input respectively, $X_i^{(a)}$ represents the $\alpha$ order Caputo derivative for the state $x_i(t)$, $X_i^{(b)}$ represents the $\beta$ order Caputo derivative, and $1 > \alpha > \beta > 0$. The following control protocols are presented in multi-agent systems:

$$
\begin{align*}
u_i(t) &= -\gamma \sum_{k \in N_i} a_{ik} [x_i(t) - x_k(t)],
\end{align*}
$$

where $\gamma > 0$ is control gain, $a_{ik}$ represents the $(i, k)$ elements of adjacency matrix $A$, $N_i$ is the neighbors’ set of the $i$-th agent.

### 3.2 Containment control of heterogenous FOMAS

In this paper, suppose the multi-agent system consisting of $m_1$ following agents and $m_2$ leader agents with $\alpha$ order Caputo derivatives, where $m_1 + m_2 = m$. Then, the compounded fractional-order dynamical equations of multi-agent systems can be rewritten as

$$
\begin{align*}
x_i^{(a)}(t) &= -\gamma \sum_{k \in N_i} a_{ik} [x_i(t) - x_k(t)], \\
x_i^{(b)}(t) &= 0, i = m_1 + 1, ..., m,
\end{align*}
$$

where $x_i(t), i = m_1 + 1, ..., m$ are used to identify the states of the leaders whose states is stationary. The recombination of the systems (5) is

$$
\begin{bmatrix}
x_1^{(a)}(t) \\
x_2^{(a)}(t) \\
x_3^{(a)}(t)
\end{bmatrix} = -\gamma
\begin{bmatrix}
L_{11} & L_{12} & L_{13} \\
0 & 0 & 0 \\
L_{31} & L_{32} & L_{33}
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t) \\
x_3(t)
\end{bmatrix}
$$

where $X(t) = [X_1(t), X_2(t), X_3(t)]^T$, $X_1(t) = [x_1(t), x_2(t), ..., x_{m_1}(t)]^T$, $X_2(t) = [x_{m_1+1}(t), ..., x_{m}(t)]^T$, $X_3(t) = [x_{m+1}(t), ..., x_n(t)]^T$, $X_1(t)$ and $X_3(t)$ are the set of the followers with different fractional order, and $X_2(t)$ is the set of the leaders. Matrix $L_{11} = (l_{ik}) \in R^{m_1 \times m_1}$, $L_{33} = (l_{ik}) \in R^{(n-m) \times (n-m)}$, whose elements satisfy

$$
l_{ik} = \begin{cases} 
\sum_{k=1}^{n} a_{ik} - a_{ii}, i = k, \\
-a_{ik}, i \neq k.
\end{cases}
$$

Matrix $L_{12} = (l_{ik}) \in R^{m_1 \times m_2}$, $L_{13} = (l_{ik}) \in R^{m_1 \times (n-m)}$, $L_{31} = (l_{ik}) \in R^{(n-m) \times m_1}$, $L_{32} = (l_{ik}) \in R^{(n-m) \times m_2}$, whose elements satisfy

$$
l_{ik} = -a_{ik}, i = 1, 2, ..., m_1, m_1 + 1, ..., n; \\
k = 1, 2, ..., m_1, m, m + 1, ..., n.
$$

By applying permutation matrix, heterogeneous FOMAS can be rewritten as

$$
\begin{bmatrix}
x_1^{(a)}(t) \\
x_2^{(a)}(t) \\
x_3^{(a)}(t)
\end{bmatrix} = -\gamma
\begin{bmatrix}
L_{11} & L_{12} & L_{13} \\
L_{31} & L_{32} & L_{33} \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t) \\
x_3(t)
\end{bmatrix}
$$

Let

$$
\Delta = \begin{bmatrix} L_{11} & L_{13} \\
L_{31} & L_{33} \end{bmatrix}.
$$

Assume the collection formed by leaders is regarded as a virtual node, if one follower agent can connect to any one leader, then the follower is connected to the virtual node.
Definition 2. The containment control is realized for the system (3) with control protocol (4), if the moving paths of the followers are asymptotically converged to the convex hull constructed by the leaders.

Assumption 1. For any one follower, there is a directed connected path to the virtual node formed by leaders.

Lemma 3. With Assumption 1, matrix $\Delta$ is positive definite, and $-\Delta^{-1} \begin{bmatrix} L_{12} \\ L_{32} \end{bmatrix}$ is a non-negative matrix whose entries sum in every row equals to 1.

Proof. From Lemma 2, matrix $\Delta$ is positive definite matrix. Let $\Delta = dL_{n-m_2} - \overline{Q}$ where $d$ is a positive number which is large enough and matrix $\overline{Q}$ is a non-negative matrix. We have

$$\Delta^{-1} = (dL_{n-m_2} - \overline{Q})^{-1} = d^{-1}(L_{n-m_2} + d^{-1}\overline{Q} + (d^{-1}\overline{Q})^2 + \ldots).$$

Then, we obtain $\Delta^{-1}$ is a non-negative matrix, and $-\Delta^{-1} \begin{bmatrix} L_{12} \\ L_{32} \end{bmatrix}$ is a non-negative matrix.

From Lemma 1, Laplacian matrix $L$ will be satisfied with $LX_0 = 0$, where $X_0 = [1, 1, \ldots, 1]^T \in \mathbb{R}^{n-1}$. Then we have

$$\begin{bmatrix} L_{11} & L_{13} & L_{12} \\ L_{31} & L_{33} & L_{32} \end{bmatrix} X_0 = 0.$$

It has

$$\Delta X_{01} + \begin{bmatrix} L_{12} \\ L_{32} \end{bmatrix} X_{02} = 0,$$

where $X_{01} = [1, 1, \ldots, 1]^T \in \mathbb{R}^{(n-m_2)-1}$ and $X_{02} = [1, 1, \ldots, 1]^T \in \mathbb{R}^{m_1}$. Since $\Delta$ is a positive definite matrix from Assumption 1 and Lemma 2, it has

$$X_{01} = -\Delta^{-1} \begin{bmatrix} L_{12} \\ L_{32} \end{bmatrix} X_{02}.$$

Therefore, $-\Delta \begin{bmatrix} L_{12} \\ L_{32} \end{bmatrix}$ is a stochastic matrix with entries sum in every row equaling to 1.

Theorem 1. Consider directed dynamic systems (3) with diverse fractional-order systems of $n - m_2$ followers and $m_2$ leaders, whose dynamic topologies are satisfied with Assumption 1. Then containment control is realized for the heterogenous FOMAS under control protocol (4).

Proof. Based on the system (7), we have

$$\begin{bmatrix} X^{(o)}_1(t) \\ X^{(b)}_1(t) \\ X^{(o)}_2(t) \end{bmatrix} = -\gamma \begin{bmatrix} L_{11} & L_{13} & L_{12} \\ L_{31} & L_{33} & L_{32} \end{bmatrix} X_3(t) + \begin{bmatrix} L_{12} \\ L_{32} \end{bmatrix} X_2(t), \quad X^{(o)}_2(t) = 0. \quad (8)$$

Let $[X_1(t), X_3(t)]^T = [X_1(t), X_3(t)]^T + \Delta^{-1}[L_{12} \ L_{32}]^T X_2(t)$, system (8) can be rewritten as

$$\begin{bmatrix} X^{(o)}_1(t) \\ X^{(b)}_1(t) \end{bmatrix} = -\gamma \begin{bmatrix} L_{11} & L_{13} \\ L_{31} & L_{33} \end{bmatrix} \begin{bmatrix} X^{(o)}_1(t) \\ X^{(b)}_1(t) \end{bmatrix}, \quad (9)$$

where $[X_1(t), X_3(t)]^T = [X_1(t), X_3(t)]^T + \Delta^{-1}[L_{12} \ L_{32}]^T X_2(t)$.

Since matrix $-\Delta \begin{bmatrix} L_{12} \\ L_{32} \end{bmatrix}$ is a stochastic matrix, the states of the followers will asymptotically converge to the convex hull formed by the leaders with Definition 1. Then, based on Definition 2, containment control is realized for the system (3) with the control protocol (4).

4 Containment control of heterogenous FOMAS with communication delays

In this section, we assumes that there are communication delays in the dynamical systems, and the containment control of the heterogenous FOMAS with communication delays is studied. Under the influence of communication delays, we can get the following algorithms:

$$X^{(o)}_i(t) = u_i(t - \tau), \quad i = 1, \ldots, m, \quad (10)$$

$$X^{(b)}_i(t) = u_i(t - \tau), \quad i = m + 1, \ldots, n,$$

where $\tau > 0$ is the communication delay of FOMAS. Through a simple change, we can get

$$\begin{bmatrix} X^{(o)}_1(t) \\ X^{(b)}_1(t) \\ X^{(o)}_2(t) \end{bmatrix} = -\gamma \begin{bmatrix} L_{11} & L_{13} & L_{12} \\ L_{31} & L_{33} & L_{32} \end{bmatrix} X_3(t - \tau) - X_3(t - \tau) + \begin{bmatrix} L_{12} \\ L_{32} \end{bmatrix} X_2(t - \tau). \quad (11)$$
Theorem 2. Suppose that multi-agent systems are composed of \( n \) independent agents with \( n - m_2 \) followers and \( m_2 \) leaders, where \( n - m_1 \) followers are with heterogeneous fractional-order dynamics whose connection network topology is undirected with Assumption 1. Then heterogeneous fractional-order multi-agent system (11) with time delays can asymptotically achieve containment control, if

\[
\tau < \frac{(2 - \alpha)\pi}{2(\lambda_{\text{max}} \gamma)}/1/\beta, \tag{12}
\]

where \( \lambda_{\text{max}} = \max\{\lambda_i, i \in I\} \), \( \lambda_i \) are the eigenvalues of matrix \( \Delta \).

Proof. Let \([\bar{X}_1(t), \bar{X}_3(t)]^T = [X_1(t), X_3(t)]^T + \Delta^{-1} [L_{12} \ L_{32}]^T X_2(t)\), Eq. (11) can be rewritten as

\[
[\bar{X}_1(\tau), \bar{X}_3(\tau)]^T = -\gamma \begin{pmatrix}
L_{11} & L_{13} \\
L_{31} & L_{33}
\end{pmatrix} \begin{pmatrix}
\bar{X}_1(t) \\
\bar{X}_3(t)
\end{pmatrix} + \Delta^{-1} \begin{pmatrix}
X_2(\tau) \\
0
\end{pmatrix},
\]

where \( \Delta \) is a positive definite matrix because the system has a globally reachable node from Assumption 1 and Lemma 2. Due to \( \alpha > \beta > 0 \), then \( s = 0 \) is not a characteristic root of the characteristic equation (14).

When \( s \neq 0 \), we have

\[
F(s) = \det (I_{n-m_1} + \gamma e^{-ts} \begin{pmatrix}
s^{-\alpha}I_{m_1} \\
1 & s^{-\beta}I_{n-m}
\end{pmatrix} \Delta ),
\]

the characteristic equation (14) is equivalent to \( F(s) = 0 \). Next, we prove that all zero points of \( F(s) = 0 \) have negative real parts. Let

\[
G(s) = \gamma e^{-ts} \begin{pmatrix}
s^{-\alpha}I_{m_1} \\
1 & s^{-\beta}I_{n-m}
\end{pmatrix} \Delta,
\]

according to the generalized Nyquist criterion, if for \( s = j\omega \), where \( j \) is complex number unit, point \(-1 + j0\) is not surrounded by the Nyquist curve of \( G(j\omega) \)'s eigenvalues, then all zero points of \( F(s) \) have negative real parts. Let \( s = j\omega \), then we can obtain

\[
G(j\omega) = H(j\omega) \gamma \Delta,
\]

where

\[
H(j\omega) = e^{-j\omega} \begin{pmatrix}
\omega^{-\alpha}e^{-j\alpha \pi/2}I_{m_1} \\
\omega^{-\beta}e^{-j\beta \pi/2}I_{n-m}
\end{pmatrix}.
\]

Let \( Q = \text{diag}\{Q_i, i = 1, \ldots, n - m_2\} \) where \( Q_i = ((2 - \alpha)\pi/(2\tau)) \) \( (i = 1, \ldots, m_1) \) and \( Q_i = ((2 - \beta)\pi/(2\tau)) \) \( (l = m_1 + 1, \ldots, n - m_2) \). Matrix \( QH(j\omega) = \text{diag}\{Q_iH_i(j\omega), i = 1, \ldots, n - m_2\} \) is a diagonal matrix where the Nyquist curve of its diagonal elements passes over point \(-1 + j0\). Since

\[
\det(\lambda I_{n-m_2} - QH(j\omega)) = \det(\lambda I_{n-m_2} - QH(j\omega) \gamma Q^{-1/2} \Delta Q^{-1/2}),
\]

suppose \( \lambda(G(j\omega)) \) is the eigenvalue of matrix \( G(j\omega) \), we have

\[
\lambda(G(j\omega)) = \lambda \left( QH(j\omega) \gamma Q^{-1/2} \Delta Q^{-1/2} \right)
\]

\[
\in \rho \left( \gamma Q^{-1/2} \Delta Q^{-1/2} \right)
\]

\[
\times \text{Co} \left( 0 \cup \{Q_iH_i(j\omega), i = 1, \ldots, n - m_2\} \right),
\]

where \( \rho() \) is used to represent the spectral radius of a matrix and \( \text{Co}(\xi) \) is the convex hull of \( \xi \). Because \( Q_iH_i(j\omega) \) will pass over point \(-1 + j0\), point \(-1 + j0\) is at the margin of convex hull \( \text{Co}(0 \cup \{Q_iH_i(j\omega), i = 1, \ldots, n - m_2\}) \). Since the matrix \( \Delta \) is symmetrical positive, there are orthogonal matrices \( P \) satisfying \( \Delta = P \Lambda P^{-1} \), where \( \Lambda = \text{diag}\{\lambda_1, \ldots, \lambda_{n-m_2}\} \) with \( \lambda_i > 0 \). Then, we have

\[
\rho \left( \gamma Q^{-1/2} \Delta Q^{-1/2} \right) = \rho \left( \gamma Q^{-1/2} \Lambda Q^{-1/2} \right)
\]

\[
= \max \{|\gamma Q_i^{-1} \lambda_i|, i = 1, \ldots, n - m_2\},
\]

where \( \lambda_i \) is the eigenvalue of matrix \( \Delta \). According to the hypothesis condition \( \tau < (2 - \alpha)\pi/(2(\lambda_{\text{max}} \gamma)1/\beta) \), we can get

\[
\rho \left( \gamma Q^{-1/2} \Delta Q^{-1/2} \right) < 1.
\]

Therefore, point \(-1 + j0\) is outside of the convex hull \( \rho \left( \gamma Q^{-1/2} \Delta Q^{-1/2} \right) \times \text{Co}(0 \cup \{Q_iH_i(j\omega), i = 1, \ldots, n - m_2\}) \). That is, point \(-1 + j0\) is not included in the Nyquist curve of the eigenvalue of \( G(j\omega) \). According to generalized Nyquist theorem, the zero points of the characteristic equation have negative real parts. Therefore the moving of multi-agent systems can asymptotically reach stability. We obtain

\[
\lim_{t \to \infty} [X_1(t) = 0 \quad \text{and} \quad \lim_{t \to \infty} X_3(t) = 0], \text{ i.e.,}
\]

\[
\lim_{t \to \infty} \begin{pmatrix}
X_1(t) \\
X_3(t)
\end{pmatrix} = -\Delta^{-1} \begin{pmatrix}
L_{12} \\
L_{32}
\end{pmatrix} X_2(t).
\]

Since matrix \( -\Delta \begin{pmatrix}
L_{12} \\
L_{32}
\end{pmatrix} \) is stochastic matrix, the moving paths of the followers will asymptotically converge to the convex hull constructed by the leaders from Definition 1.

\[\square\]

Corollary 1. Suppose heterogeneous multi-agent systems, composed of \( n \) independent agents with \( n - m_2 \) followers and \( m_2 \) leaders, have directed symmetrical network topology
with a global reachable node. Then heterogenous fractional-order multi-agent system (11) with time delays can asymptotically achieve containment consensus with \( \alpha = \beta = 1 \), if

\[
\gamma \tau < \pi / (2 \lambda_{\text{max}}),
\]

where \( \lambda_{\text{max}} \) is the maximum eigenvalue of matrix \( \Delta \).

5 Examples and simulation results

Consider the dynamic topology with 5 followers and 3 leaders (illustrated as A1, A2, A3) shown in Figure 1, where the connection weight of each edge is 1. Suppose the fractional orders of the first agent and the second agent are \( \alpha = 0.9 \) (the fractional order of Leaders is 0.9) and others are \( \beta = 0.8 \).

From the communication topology of FOMAS, the system matrix can be obtained:

\[
\Delta = \begin{bmatrix}
3 & -1 & 0 & 0 & -1 \\
-1 & 2 & 0 & 0 & 0 \\
0 & 0 & 2 & -1 & 0 \\
0 & 0 & -1 & 3 & -1 \\
-1 & 0 & 0 & -1 & 3
\end{bmatrix}
\]

Let the control parameter of system be \( \gamma = 1.0 \). The initial positions of followers are taken as \( x_1(0) = (1, 1), x_2(0) = (1, 2), x_3(0) = (2, 1), x_4(0) = (2, 3), x_5(0) = (3, 2) \), respectively. The initial positions of leaders are taken as \( A_1(0) = (4, 4), A_2(0) = (4, 6), A_3(0) = (6, 4) \). Figure 2 shows the state trajectories of FOMAS without time delays, where the followers have moved into the convex hull formed by the leaders.

Next, we will verify the results of FOMAS with time delays. The maximum eigenvalue of \( \Delta \) is 4.618 by calculation. According to the result of Theorem 2, the relationship between the control gain and the upper bound of communication delays is illustrated in Figure 3. With the help of Figure 3, the control gain can be decided according to the communication delays of the system. Conversely, if we know the control gain of the system, the enable communication delays can be obtained in Figure 3, such that FOMAS meets the condition of containment consensus. In the simulation, we suppose the system control gain \( \gamma = 1 \), the upper bound of communication delays is 0.255s from Figure 3.

Let time delays of multi-agent systems \( \tau = 0.15s \). The initial parameters in the experiments are the same as those built in front of the simulation. Figure 4 shows the state
domain, and the containment control condition for heterogeneous delayed FOMAS is obtained. The relationship between the upper bound of time delays and the control gain of FOMAS is derived. The extended conclusion of the results is the same as that of integer order systems when the orders of the FOMAS are equal to 1. In the future work, a research on the robust stability of containment control for compounded FOMAS will be carried out.

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References