Research Article

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Estimation for coefficient of variation of an extension of the exponential distribution under type-II censoring scheme

https://doi.org/10.1515/phys-2017-0065
Received Jan 13, 2017; accepted Apr 06, 2017

Abstract: The coefficient of variation [CV] has several applications in applied statistics. So in this paper, we adopt Bayesian and non-Bayesian approaches for the estimation of CV under type-II censored data from extension exponential distribution [EED]. The point and interval estimate of the CV are obtained for each of the maximum likelihood and parametric bootstrap techniques. Also the Bayesian approach with the help of MCMC method is presented. A real dataset is presented and analyzed, hence the obtained results are used to assess the obtained theoretical results.

Keywords: Coefficient of variation; Extension exponential distribution; Type-II censoring scheme; Bayesian estimation; Bootstrap confidence interval; Maximum likelihood estimation; MCMC.

PACS: 02.50.Ga; 02.50.Ng; 02.50.Tt

1 Introduction

In statistical analysis, especially for the lifetime data the model with monotone risk functions such as gamma and Weibull distributions is preferred. Recently modified extensions of the exponential distributions have monotone risk functions to be suitable for life time data analysis. Gupta and Kundu [1, 2] proposed and discussed the generalized exponential distribution [GED] under probability density function [PDF] given by

\[ f(x) = \frac{\alpha \beta}{\alpha + \beta} \left(1 + \beta x\right) \exp \left[-\alpha x\right], \]

\[ a, \beta > 0, \ x > 0, \]

and

\[ F(x) = 1 - \left[1 + \frac{\alpha \beta x}{\alpha + \beta}\right] \exp \left[-ax\right]. \]

Also the reliability and hazard rate functions given respectively by

\[ R(t) = \left[1 + \frac{\alpha \beta t}{\alpha + \beta}\right] \exp \left[-at\right], \]

and

\[ h(t) = \frac{\alpha^2 (1 + \beta t)}{\alpha + \beta + \alpha \beta t}. \]

The EED satisfies the following properties see (Gemez et al. [4]).

1: The population mean \( \mu \) of \( X \)

\[ \mu = \frac{\alpha + 2 \beta}{\alpha (\alpha + \beta)}, \]
2. The population standard deviation \( \sigma \) of \( X \)

\[
\sigma = \sqrt{V(x)} = \sqrt{\frac{\alpha^3 + 5\alpha^2\beta + 6\alpha\beta^2 + 2\beta^3}{\alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \alpha^2\beta^2}}.
\] (8)

The population CV is defined as the ratio of the standard deviation to the population mean. The CV used to measure the stability or uncertainty, also the relative dispersion of population data to the population mean can be presented. Different authors addressed the problems of CV (Pang et al. [8]). For applications of CV in, medical sciences, engineering, physics, chemistry and telecommunications see, Miller and Karson [9], Hamer et al. [10], Reh and Scheffler [11], Ahn [12] and Gong and Li [13]. Several authors studied CV in normal phenomena but the works in non normal phenomena are rare, see Soliman et al. [14] and Abd-Elmougod et al. [15].

The main aim of this paper is to study CV of EED when the data obtained from type-II censoring experiment and also the maximum likelihood estimation is presented. In Section 3, parametric bootstrap confidence intervals are calculated. The posterior empirical distribution of CV with the help of MCMC method are obtained. The obtaining results are discussed with real data set.

This article is organized as follows. In Section 2, a maximum likelihood estimation is presented. In Section 3, parametric bootstrap confidence intervals are discussed. In Section 4, the point and interval estimation of CV under posterior empirical distribution are discussed. A real life data set applied to illustrate our proposed method is presented in Section 5. Conclusion can be found in Section 6.

2 Maximum likelihood estimation

Let \( X_1, X_2, \ldots, X_r \) be a Type-II censoring data from a EED obtained from a data with \( n \) size, From (3), (4) and (9) the joint likelihood function written as

\[
L(x_1, x_2, \ldots, x_r) = \frac{n!}{(n-r)!} \prod_{i=1}^{r} f(x_i) [1 - F(x_i)]^{(n-r)},
\] (9)

where

\[
0 < x_1, x_2, \ldots, x_r < \infty,
\]

we will execute maximum likelihood estimation with the an extension of the exponential distribution. Also the parametric bootstrap confidence intervals are calculated. The posterior empirical distribution of CV with the help of MCMC method are obtained. The obtaining results are discussed with real data set.

The likelihood equations obtained by taking the first derivatives of (11) with respect to parameters \( \alpha \) and \( \beta \), hence equating each of them to zero, we obtain

\[
\frac{2r}{\alpha} - \frac{r}{\alpha + \beta} - \sum_{i=1}^{r} x_i + \frac{(n-r)\beta^2 x_r}{(\alpha + \beta)(\alpha + \beta + a\beta x_r)} = 0.
\] (12)

\[
-\frac{r}{\alpha + \beta} + \sum_{i=1}^{r} \frac{x_i}{1 + \beta x_i} + \frac{(n-r)\alpha^2 x_r}{(\alpha + \beta)(\alpha + \beta + a\beta x_r)} = 0.
\] (13)

The equations (12) and (13) cannot be solved analytically for \( \hat{\alpha} \) and \( \hat{\beta} \), these are can be solved numerical with the Newton’s method with the initial parameters values obtained by the method of moments as

\[
E(X) = \bar{X},
\] (14)

\[
E(X^2) = \bar{X}^2.
\] (15)

Therefore, the ML estimation of \( \hat{\alpha} \) and \( \hat{\beta} \) is obtained and hence ML estimation of CV is given by

\[
CV_{\text{ML}} = \frac{\hat{\sigma}}{\hat{\mu}},
\] (16)

where \( \hat{\mu} \) and \( \hat{\sigma} \) are obtained by replacing \( \alpha \) and \( \beta \) by \( \hat{\alpha} \) and \( \hat{\beta} \) in (7) and (8).

3 Bootstrap confidence intervals

To obtain interval estimations of the parameters or any function of the parameters, we can use the Bootstrap technique which builds confidence intervals or widely improves estimators other than asymptotic results. Parametric and nonparametric bootstrap methods, are discussed in [16] and [17]. More details about parametric bootstrap percentile technique which is used in this section to build a bootstrap confidence interval of CV are also discussed in...
To obtain for estimation of CV of EED we consider the following algorithm under type-II bootstrap sample.

**Algorithm 1**

1. For given data set \( x = x_1, x_2, \ldots, x_r \), compute the MLE \( \hat{a} \) and \( \hat{\beta} \) from equation (12) and (13), \( CV_{ML} \) from (16).
2. The estimate values of \( \hat{a} \) and \( \hat{\beta} \) are used for generation of bootstrap data \( \hat{x} \) under the fixed values of \( n \), and \( r \).
3. Based on \( \hat{x} \) as given in step 1, we built the bootstrap samples estimation of \( \hat{a} \), \( \hat{\beta} \) and \( CV_{ML} \) are computed.
4. Steps 2 and 3 are repeated \( s \) times, the \( s \) bootstrap samples of \( \hat{a} \), \( \hat{\beta} \) and \( CV_{ML}^s \) are obtained based on \( s \) different data \( x^s \).
5. The ordering bootstrap samples \( (CV_{ML}^{1[1]}, CV_{ML}^{2[1]}, \ldots, CV_{ML}^{s[1]}) \) are obtained by arrange all \( CV_{ML}^{j} \) in an ascending order.
6. The bootstrap empirical distribution of \( CV_{ML}^* \), say \( \Psi(t) = P(CV_{ML}^* \leq t) \) define \( CV_{ML}^{* [boot]} = \Psi^{-1}(t) \) for given \( t \). Then the corresponding bootstrap 100(1 - 2\( \gamma \))% confidence interval of \( CV_{ML}^* \) given by

\[
[CV_{ML}^{* [boot]}(\gamma), CV_{ML}^{* [boot]}(1 - \gamma)].
\]

### 4 Bayesian estimation

Bayes estimation of CV of EED under squared error loss (SEL) function can be achieved under independent gamma prior for each parameters \( a \) and \( \beta \). So that the prior density of \( a \) and \( \beta \) can be represented as

\[
\psi_1(a) \propto a^{a-1} \exp(-ba), \quad a > 0, \quad a, \quad b > 0.
\]

and

\[
\psi_2(\beta) \propto \beta^{c-1} \exp(-d\beta), \quad \beta > 0, \quad c, \quad d > 0.
\]

The joint posterior density function [JPDF] \( h(a, \beta|x) \) respected to parameters \( a \) and \( \beta \) for given data represented as

\[
h(a, \beta|x) = \int_0^{\infty} \int_0^{\infty} \psi_1(a)\psi_2(\beta)L(a, \beta|x) \, da \, d\beta.
\]

under SEL, the estimation of any function \( \varphi(a, \beta) \) say \( [a, \beta \text{ or } CV] \), given by

\[
\hat{\varphi}_B = E_{a,\beta|x}[ \varphi(a, \beta)] = \int_0^{\infty} \int_0^{\infty} \varphi(a, \beta) \psi_1(a)\psi_2(\beta)L(a, \beta|x) \, da \, d\beta,
\]

In several cases the ratio in (21) of two integrals does not have a closed form. So numerical integration or Lindely approximation can be used for this problem. Another important method can be used for generating samples from the posterior distributions called MCMC methods, which present estimation of \( \varphi(a, \beta) \) under the SEL.

**MCMC approach:** From equations (20) and (21), it is not possible to compute analytically form in this case. Different approaches can be used here, the important and more accurate method is Markov Chain Monte Carlo (MCMC) which can be used not only to approximate the point estimation of parameters and any function of the parameters such as a CV. It has the advantage of interval estimation of parameters and any function of the parameters such as a CV. From equations (10), (18), (19) and (20) the PDF of \( a \) and \( \beta \) given data can be obtained as

\[
h(a, \beta|x) \propto a^{a+2r-1} \beta^{c-1} \exp \left( -r \log[a + \beta] - ba \right)
\]

\[
- d\beta - a \sum_{i=1}^{r} x_i + \sum_{i=1}^{r} \log(1 + \beta x_i)
\]

\[
- a(n - r)x_r + (n - r) \log \left[ 1 + \frac{a \beta x_r}{a + \beta} \right].
\]

The full conditional distribution of \( a \) and \( \beta \) written by

\[
h_1(a|\beta, x) \propto a^{a+2r-1} \exp \left( -r \log[a + \beta] - ba \right)
\]

\[
- a \sum_{i=1}^{r} x_i - a(n - r)x_r
\]

\[
+ (n - r) \log \left[ 1 + \frac{a \beta x_r}{a + \beta} \right].
\]

and

\[
h_2(\beta|a, x) \propto \beta^{c-1} \exp \left( -r \log[a + \beta] \right)
\]

\[
+ \sum_{i=1}^{r} \log(1 + \beta x_i) - d\beta
\]

\[
+ (n - r) \log \left[ 1 + \frac{a \beta x_r}{a + \beta} \right].
\]

A more variety of MCMC schemes are existing. The important one of MCMC methods is Gibbs sampling. The property of two distributions of (23) and (24) shows that each are similar to normal distribution, so Metropolis-within-Gibbs samplers are employed. We use the MH method Metropolis et al. [19] with normal proposal distribution as follows.
Table 1: A real type-II censoring sample of size 40 from EED

<table>
<thead>
<tr>
<th></th>
<th>0.0251</th>
<th>0.0886</th>
<th>0.0891</th>
<th>0.2501</th>
<th>0.3113</th>
<th>0.3451</th>
<th>0.4763</th>
<th>0.5650</th>
<th>0.5671</th>
<th>0.6566</th>
</tr>
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<tr>
<td></td>
<td>0.6748</td>
<td>0.6751</td>
<td>0.6753</td>
<td>0.7696</td>
<td>0.8375</td>
<td>0.8391</td>
<td>0.8425</td>
<td>0.8645</td>
<td>0.8851</td>
<td>0.9113</td>
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<tr>
<td></td>
<td>0.9120</td>
<td>0.9836</td>
<td>1.0483</td>
<td>1.0596</td>
<td>1.0773</td>
<td>1.1733</td>
<td>1.2570</td>
<td>1.2766</td>
<td>1.3211</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.3503</td>
<td>1.3551</td>
<td>1.4595</td>
<td>1.4880</td>
<td>1.5728</td>
<td>1.5733</td>
<td>1.7083</td>
<td>1.7263</td>
<td>1.7460</td>
<td>1.7630</td>
</tr>
</tbody>
</table>

Table 2: The point and interval estimates of MLE and Bootstrap of $\beta$ and $CV$

<table>
<thead>
<tr>
<th>Parameter</th>
<th>(•)ML</th>
<th>(•)Boot</th>
<th>(95%)ML</th>
<th>Leng.</th>
<th>(95%)Boot</th>
<th>Leng.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>0.8988</td>
<td>0.9012</td>
<td>(0.6401, 1.1575)</td>
<td>0.5174</td>
<td>(0.7458, 2.0145)</td>
<td>1.2687</td>
</tr>
<tr>
<td>$\beta$</td>
<td>5.1425</td>
<td>5.4215</td>
<td>(0.1234, 7.2136)</td>
<td>7.0902</td>
<td>(0.8216, 9.3265)</td>
<td>8.5049</td>
</tr>
<tr>
<td>CV</td>
<td>0.7597</td>
<td>0.8013</td>
<td>(0.7482, 0.9584)</td>
<td>0.2102</td>
<td>(0.7482, 0.9584)</td>
<td>0.2102</td>
</tr>
</tbody>
</table>

MH under Gibbs sampling algorithm:

Algorithm 2

1: Put the indicator $I = 1$ and ML estimates as the initial values of the parameters $\alpha$ and $\beta$.

2: By using MH algorithm generate $\alpha^{(l)}_1$ and $\beta^{(l)}_1$ from (23) and (24) with normal proposal distributions $N(\alpha^{(l)}_1, \sigma_1)$ and $N(\beta^{(l)}_1, \sigma_2)$ respectively where $\sigma_1$ and $\sigma_2$ can be obtained from approximate information matrix.

3: Compute $\varphi^{(l)}(\alpha, \beta) = \varphi(\alpha^{(l)}_1, \beta^{(l)}_1)$

4: Put $I = I + 1$ and.

5: Repeat steps from 2 and 3, $N^*$ times.

6: The Bayes estimation of $\varphi(\alpha, \beta)$ under the MCMC methods as

$$E(\varphi(\alpha, \beta)|z) = \frac{1}{N^* - M^*} \sum_{l=M^*+1}^{N^*} \varphi^{(l)}, \quad (25)$$

where $M^*$ is the number of iterations, need to the stationary distribution.

1. Put the values $\varphi^{(1)}, \varphi^{(2)}, \ldots, \varphi^{(N^*-M^*)}$ in ordered forms as $\varphi^{(1)}, \varphi^{(2)}, \ldots, \varphi^{(N^*-M^*)}$, so the symmetric credible interval 100$(1 - 2\gamma)$% given by

$$\left(\varphi((\gamma/2)\,[N^*-M^*]), \varphi((1-\gamma)/2\,[N^*-M^*])\right). \quad (26)$$

5 Data analysis

Let us consider the real data set presented by[4], which fits to EED. The data presented the life of fatigue fracture of Kevlar 373/epoxy that are subject to constant pressure at the 90% stress level until all had failed, for more surveys about this data see Andrews and Herzberg [20] and Barlow et al. [21]. Let us consider that type-II censoring sample with censoring schemes $n = 76$ and $r = 40$. The obtained type-II censoring data is presented in Table 1. The classical point estimation of parameters $\alpha$ and $\beta$ and $CV$ using each of maximum likelihood and bootstrap methods as well as the approximate percentile bootstrap confidence intervals are given in Table 2. In Bayesian approach based on the MCMC samples of size 11000 with 1000 as ‘burn-in’, the Bayes point estimates and 95% credible intervals for $\alpha$, $\beta$ and $CV$ are computed and presented in Table 3. Figures (1-
Table 3: The MCMC point and interval estimates of $\alpha$, $\beta$ and CV values

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Mean</th>
<th>Median</th>
<th>Mode</th>
<th>SD</th>
<th>95% Probability interval</th>
<th>Leng.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>0.7606</td>
<td>0.7648</td>
<td>0.7732</td>
<td>0.1247</td>
<td>(0.5056, 0.9931)</td>
<td>0.4875</td>
</tr>
<tr>
<td>$\beta$</td>
<td>1.8935</td>
<td>1.6586</td>
<td>1.1888</td>
<td>1.2620</td>
<td>(0.2229, 5.0842)</td>
<td>4.8614</td>
</tr>
<tr>
<td>CV</td>
<td>0.8291</td>
<td>0.8200</td>
<td>0.8020</td>
<td>0.0484</td>
<td>(0.7593, 0.9469)</td>
<td>0.1876</td>
</tr>
</tbody>
</table>

Figure 3: The MCMC Simulation number of $\beta$

Figure 4: The corresponding MCMC Histogram of $\beta$

Figure 5: The MCMC Simulation number of CV

Figure 6: The corresponding MCMC Histogram of CV

6) show the simulation number of $\alpha$, $\beta$ and CV generated by MCMC method and the corresponding histogram. The figures (1-6) plots the first 11000 steps for the Gibbs sampler and shown the convergence in the empirical posterior results. The corresponding histogram given by figures (2), (4) and (6) show that the empirical posterior distribution of each parameter and CV. Figures (2) and (4) show that the empirical posterior distribution of parameter $\alpha$ and $\beta$ is similar to normal distribution, so the posterior mean is used to get a better approximation for the posterior estimation of parameters $\alpha$ and $\beta$. Also the histogram in figure (6) it is quite skewed for a skewed distribution, so the posterior mode is adopted as the best estimation for the CV more details on the CV see Upadhyay and Peshwani [22].

6 Conclusion

In practice, the theoretical sampling distribution interpreted in CV is more difficult to be represented mathematically. Therefore, the statistical inference about the CV in several applications need to obtain an interval estimation for the CV. In this article we proposed Bayesian and non-Bayesian approach for estimation results of CV. The proposed method for obtaining a point estimate as well as an interval estimate for the CV. The results show an effective MCMC method therefore our method used in this arti-
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can be easily extended for another extension exponential distribution discussed in this article.

Acknowledgement: This project was funded by the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, under grant no. (G-541-558-1437). The authors, therefore, acknowledge with thanks DSR for technical and financial support.

References