Research Article

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A comparison study of steady-state vibrations with single fractional-order and distributed-order derivatives

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Abstract: We conduct a detailed study and comparison for the one-degree-of-freedom steady-state vibrations under harmonic driving with a single fractional-order derivative and a distributed-order derivative. For each of the two vibration systems, we consider the stiffness contribution factor and damping contribution factor of the term of fractional derivatives, the amplitude and the phase difference for the response. The effects of driving frequency on these response quantities are discussed. Also the influences of the order $\alpha$ of the fractional derivative and the parameter $\gamma$ parameterizing the weight function in the distributed-order derivative are analyzed. Two cases display similar response behaviors, but the stiffness contribution factor and damping contribution factor of the distributed-order derivative are almost monotonic change with the parameter $\gamma$, not exactly like the case of single fractional-order derivative for the order $\alpha$. The case of the distributed-order derivative provides us more options for the weight function and parameters.

Keywords: fractional calculus; vibration; distributed-order derivative; excitation; response

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1 Introduction

In recent decades, fractional calculus has been applied to describe memory phenomena and hereditary properties of various materials and processes in different fields of science and engineering. These areas of application cover viscoelasticity theory, non-Newtonian flow, anomalous diffusion, control theory, image processing, capacitor theory, fitting of experimental data, and so on [1–4]. In recent years, even fractional-order models of happiness [5] and love [6] have been developed, and they are claimed to give a better representation than the integer-order dynamical systems approach.

Viscoelasticity theory is one of the earliest and most active application areas of fractional calculus. The wide use of polymers in fields of engineering has promoted the development of this subject.

In fact, the introducing of the fractional calculus to viscoelastic theory is very natural [7]. Scott-Blair [8–10] suggested a fractional constitutive relation for a viscoelastic body, which reflects mechanical properties between elastic solids and viscous fluids. Such fractional model was called as the Scott-Blair model [3, 11]. In [12], the terminology "spring-pot" was introduced for the Scott-Blair model.

Fractional oscillator and related models and equations were presented and investigated by Caputo [13], Bagley and Torvik [14], Beyer and Kempfle [15], Mainardi [16], Gorenflo and Mainardi [17], and others [18–25]. Achar et al. [18] considered response feature for fractional oscillator. Lim et al. [19] considered the relationship between fractional vibration and fractional Brownian movement. Lim and Teo [20] proposed fractional process by using a stochastic differential equation of fractional-order. Li et al. [21] investigated impulse response and system stability for a type of fractional oscillation. Shen et al. [22, 23], Huang and Duan [24] and Duan et al. [25] deliberated the dynamics and resonance phenomenon for fractional-order vibration. A fractional dynamical system and its stability problem and chaotic behaviors were analyzed by Li et al. [26], Zhang et al. [27], Wang and Hu [28], Huang et al. [29] and Wu et al. [30]. Li and Ma [31] presented a linearization and stability theorem for nonlinear fractional system.

Let us recall the basic definitions of fractional calculus. Suppose $f(t)$ is piecewise continuous on $(b, +\infty)$ and integrable on any subinterval $(b, t)$. The Riemann-Liouville fractional integral of $f(t)$ of order $\alpha$ is defined
as [1–4]:

\[ b^\alpha_t f(t) := \int_b^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds, \quad \alpha > 0, \quad (1) \]

where \( \Gamma(\cdot) \) is the Euler’s gamma function and is defined as the infinite integral

\[ \Gamma(z) = \int_0^{+\infty} e^{-x} x^{z-1} \, dx, \quad z > 0. \quad (2) \]

Next we write down the expression for the Riemann-Liouville fractional derivative of \( f(t) \) of order \( \alpha \),

\[ \mathcal{D}_b^\alpha f(t) := \frac{d^n}{dt^n} \left( b^\alpha t^n f(t) \right), \quad n - 1 < \alpha < n, \quad (3) \]

Let \( f^{(n)}(t) \) be piecewise continuous on \((b, +\infty)\) and integrable on any subinterval \((b, t)\). The Caputo fractional derivative of \( f(t) \) of order \( \alpha \) is defined as

\[ C^\alpha_t f(t) := b^\alpha t^n f^{(n)}(t), \quad n - 1 < \alpha < n, \quad (4) \]

We note that existence of solution of a fractional differential equation was investigated by Podlubny [1], Baleanu et al. [32] and Baleanu and Mustafa [33]. Analytical and numerical methods for solving fractional differential equations were presented in references [1, 2, 4, 16, 17, 34–36].

In most of the above-mentioned articles, the lower limit \( b \) of integration in fractional integral and derivative was taken as zero. So the initial conditions are necessary for solving of fractional-order differential equations. In general, this kind of problem does not admit any periodic solution [16, 17, 37–40].

Distributed-order derivatives, which are defined by means of integration with respect to the order of fractional derivatives, were presented and discussed in terms of filtering by Caputo [41]. In [42], distributed-order differential equations were introduced to constitutive relations of dielectric media and to a diffusion model. In [43, 44], distributed-order differential equations were solved by introducing a generalized Taylor series representation of solutions. In [45, 46], the relations of viscoelastic bodies were discussed by using distributed-order derivatives.

For the sake of convenient reference, we recall the classic integer-order one-degree-of-freedom vibration subject to a harmonic excitation,

\[ \ddot{x} + kx + c \dot{x} = F e^{j\omega t}, \quad (5) \]

where \( k, c, F, \omega \) are positive constants and \( i \) is the imaginary unit. The steady response is

\[ x(t) = \frac{Fe^{j\omega t}}{k - \omega^2 + j\omega c} = \frac{Fe^{j(\omega t - \varphi)}}{\sqrt{(k - \omega^2)^2 + (\omega c)^2}}, \quad (6) \]

where \( \varphi \) is the phase difference between the excitation and the response and is determined by the following equation,

\[ \varphi = \begin{cases} \arctan \frac{\omega c}{k - \omega^2}, & k - \omega^2 > 0, \\ \frac{\pi}{2}, & k - \omega^2 = 0, \\ \pi + \arctan \frac{\omega c}{k - \omega^2}, & k - \omega^2 < 0. \end{cases} \quad (7) \]

For vibration responses, we are usually more interested in the steady-state response. We remark that if the lower limits \( b \) in the two definitions of fractional derivatives are taken as \(-\infty\), then (3) and (4) are identical. We denote them as \(-\infty \mathcal{D}_b^\alpha f(t)\) uniformly.

In this work, we conduct a comparative study for fractional vibration systems with single fractional-order derivative and distributed-order derivative. We consider the steady-state responses to the harmonic excitation by using the fractional-order derivative operator \(-\infty \mathcal{D}_b^\alpha\). In Section 2, we consider the steady-state vibration of a system with single fractional-order derivative. In Section 3, the system with distributed-order derivative is taken into account. In Section 4, two special cases belonging to two different systems in Section 2 and Section 3, respectively, are compared.

2 Steady-state vibration with single fractional-order derivative

Suppose a mass block \( m \) slides on a frictionless surface. It is attached to a spring of stiffness factor \( k \) in parallel with a dashpot of damping coefficient \( c \) and a viscoelastic rod of length \( l \), whose constitutive relation obeys Scott-Blair law. We consider the steady-state vibration of the block under the harmonic driving \( Fe^{j\omega t} \).

The one-degree-of-freedom vibration system is written as

\[ m\ddot{x} + c \dot{x} + k \dot{x} + G \tau^a \cdot -\infty \mathcal{D}_b^\alpha x = Fe^{j\omega t}, \quad (8) \]

where \( m, k, c, G, \tau, F, \omega \) are positive real constants and \( 0 < \alpha \leq 1 \). If \( \alpha = 0 \) or 1, \(-\infty \mathcal{D}_b^\alpha x\) denotes \( x \) or \( \dot{x} \), respectively.

Introducing the dimensionless variables and parameters

\[ \frac{t}{\tau} \rightarrow t, \quad x \rightarrow x, \quad \frac{\tau^2 k}{m} \rightarrow k, \quad \frac{\tau c}{m} \rightarrow c, \quad \frac{\tau^2 G}{m}\rightarrow G, \quad \frac{\tau^2 F}{ml} \rightarrow F, \quad \tau \omega \rightarrow \omega, \]

we derive the dimensionless form for the vibration system

\[ \ddot{x} + kx + c \dot{x} + G \cdot -\infty \mathcal{D}_b^\alpha x = Fe^{j\omega t}. \quad (9) \]
We look for the steady-state same frequency response of the system as

\[ x(t) = X e^{i\omega t}, \]  

(10)

where \( X \) is independent of \( t \) and is called as the complex amplitude.

Substituting the first- and second-order derivatives of \( x(t) \) and the fractional derivative

\[ -\infty \left\{ D_i^\alpha x \right\} (t) = X (i\omega)^\alpha e^{i\omega t}, \]  

(11)

into Eq. (9), we achieve the equation

\[ X \left[ k - \omega^2 + i\omega c + G (i\omega)^\alpha \right] = F, \]  

(12)

By the identity \( i^\alpha = \cos \frac{\pi \alpha}{2} + i \sin \frac{\pi \alpha}{2} \), Eq. (12) is written as

\[ X \left[ k + G \omega^\alpha \cos \frac{\pi \alpha}{2} - \omega^2 \right.\]
\[ \left. + i\omega \left( c + G \omega^{\alpha-1} \sin \frac{\pi \alpha}{2} \right) \right] = F. \]  

(13)

Inserting Eq. (10) results in the steady-state response

\[ x(t) = Fe^{i\omega t} / \left[ k + G \omega^\alpha \cos \frac{\pi \alpha}{2} - \omega^2 + \right.\]
\[ \left. i\omega \left( c + G \omega^{\alpha-1} \sin \frac{\pi \alpha}{2} \right) \right]. \]  

(14)

By comparison with the results of the integer-order case in Eq. (6), we derive the equivalent stiffness coefficient and the equivalent damping coefficient and denote them as

\[ \tilde{k} = k + G \omega^\alpha \cos \frac{\pi \alpha}{2}, \tilde{c} = c + G \omega^{\alpha-1} \sin \frac{\pi \alpha}{2}. \]  

(15)

So the steady-state response is expressed as

\[ x(t) = \frac{Fe^{i\omega t}}{\tilde{k} - \omega^2 + i\omega \tilde{c}}. \]  

(16)

Making use of the exponential representation of complex number, we rewrite the steady-state response as

\[ x(t) = \frac{Fe^{i(\omega t - \varphi)}}{\sqrt{\left( \tilde{k} - \omega^2 \right)^2 + \left( \omega \tilde{c} \right)^2}}, \]  

(17)

where \( \varphi \) is the phase difference between the excitation and the response and is determined by the following

\[ \varphi = \begin{cases} \arctan \frac{\omega \tilde{c}}{\tilde{k} - \omega^2}, & \tilde{k} - \omega^2 > 0, \\ \pi / 2, & \tilde{k} - \omega^2 = 0, \\ \pi + \arctan \frac{\omega \tilde{c}}{\tilde{k} - \omega^2}, & \tilde{k} - \omega^2 < 0. \end{cases} \]  

(18)

2.1 The stiffness contribution factor and damping contribution factor

The equivalent stiffness factor and damping factor in Eq. (15) mean that the steady-state response of the fractional vibration system (9) is equivalent to that of the integer-order system

\[ \ddot{x} + \tilde{k} x + \tilde{c} \dot{x} = Fe^{i\omega t}. \]  

(19)

We call

\[ \tilde{k} - k = G \omega^\alpha \cos \left( \frac{\pi \alpha}{2} \right), \tilde{c} - c = G \omega^{\alpha-1} \sin \left( \frac{\pi \alpha}{2} \right), \]  

(20)

as the stiffness contribution factor and damping contribution factor of the fractional derivative term \( G \left\{ D_i^\alpha x \right\} \), respectively. They are all positive. The stiffness contribution factor \( \tilde{k} - k \) is an increasing function of the driving frequency \( \omega \), while the damping contribution factor \( \tilde{c} - c \) is an decreasing function of the driving frequency \( \omega \).

Figures 1 and 2 show, respectively, the stiffness contribution factor \( \tilde{k} - k \) and damping contribution factor \( \tilde{c} - c \) versus \( \omega \) for \( G = 1 \) and \( \alpha = 0.25, 0.5, 0.75 \).
In order to investigate the dependence of \( \tilde{k} - k \) and \( \tilde{c} - c \) on the order \( \alpha \), we calculate the derivatives
\[
\frac{d(k - k)}{d\alpha} = G\omega^\alpha \cos\left(\frac{\pi\alpha}{2}\right) \left(\ln(\omega) - \frac{\pi}{2} \tan\left(\frac{\pi\alpha}{2}\right)\right),
\]
and
\[
\frac{d(\tilde{c} - c)}{d\alpha} = G\omega^{\alpha - 1} \sin\left(\frac{\pi\alpha}{2}\right) \left(\ln(\omega) + \frac{\pi}{2} \cot\left(\frac{\pi\alpha}{2}\right)\right).
\]

It follows from Eq. (21) that when \( 0 < \omega \leq 1 \), \( \tilde{k} - k \) monotonically decreases as \( \alpha \) increases, while when \( \omega > 1 \), \( \tilde{k} - k \) firstly increases then decreases on the interval \( 0 \leq \alpha \leq 1 \); see Figure 3, where the curves of \( \tilde{k} - k \) versus \( \alpha \) are shown for \( G = 1 \) and different values of \( \omega \). From Eq. (22) we deduce that if \( 0 < \omega < 1 \), \( \tilde{c} - c \) increases followed by decreasing on the interval \( 0 \leq \alpha \leq 1 \), while if \( \omega \geq 1 \), \( \tilde{c} - c \) is a monotonically increasing function of \( \alpha \); see Figure 4, where the curves of \( \tilde{c} - c \) versus \( \alpha \) are shown for \( G = 1 \) and different values of \( \omega \).

### 2.2 Amplitude and phase difference

The amplitude of the response is obtained from Eqs. (15) and (17) as
\[
|x(t)| = F \left[ \left( k + G\omega^\alpha \cos\frac{\pi\alpha}{2} - \omega^2 \right)^2 + \left( \omega c + G\omega^\alpha \sin\frac{\pi\alpha}{2} \right)^2 \right]^{1/2}.
\]

We take \( k = 1.2, c = 0.1, F = G = 1 \), the curves of amplitudes \( |x(t)| \) versus the driving frequency \( \omega \) are shown in Figure 5 for different values of the order \( \alpha \). The curves of amplitude \( |x(t)| \) versus the order \( \alpha \) are shown in Figure 6 for different values of the driving frequency \( \omega \). We observe that as \( \alpha \) increases, resonance peaks and resonance frequencies decrease. Increasing and decreasing of the amplitude with the order \( \alpha \) depend on the frequency region.

In Figures 7 and 8, the curves of phase-difference \( \phi \) in Eq. (18) versus the frequency \( \omega \) and versus the order \( \alpha \) are shown for the same parameter values as in Figures 5 and 6, respectively. As the driving frequency \( \omega \) increases, the phase-difference \( \phi \) increases from 0 to \( \pi \), and the variation is more abrupt for small \( \alpha \). The changing of the phase-difference with the order \( \alpha \) relies on the frequency region.

### 3 Steady-state vibration with distributed-order derivative

We suppose the acting force of the viscoelastic rod in the vibration system (8) in Section 2 is expressed as an integration of the Scott-Blair model with respect to the order,
The weight distribution function of the order $p(\lambda)$ can be given in infinitely many ways. Here we take it as an exponential function with the constant base $\mu$ as a parameter,

$$p(\lambda) = C(\mu) \mu^{|\lambda|}, \quad 0 < \mu < +\infty,$$

where the coefficient is given as

$$C(\mu) = \begin{cases} \frac{\ln \mu}{\mu - 1}, & \mu \neq 1, \\ 1, & \mu = 1. \end{cases}$$

It is easy to check that $p(\lambda)$ in Eq. (26) satisfies the conditions in (24). We remark that such selection of the weight distribution function is for the sake of calculation. Also we note that if we take the weight distribution function $p(\lambda)$ as the Dirac delta distribution $\delta(\lambda - a)$, Eq. (9) is derived from the system with the distributed-order (25).

Similar to the last section, with the assumption

$$x(t) = X e^{i \omega t},$$

we derive the relation for the complex amplitude $X$ as

$$X \left[ k - \omega^2 + i \omega c + G C(\mu) \int_0^1 (\mu \omega)^{\lambda} d\lambda \right] = F.$$  \hspace{1cm} (29)

Calculating the integration yields

$$X \left[ k + 4 C(\mu) \frac{2 \mu \omega - 4 \ln(\mu \omega)}{4 \ln^2(\mu \omega) + \pi^2} \omega^2 + i \omega \left( c + G C(\mu) \frac{4 \mu \omega \ln(\mu \omega) + 2 \pi}{4 \omega \ln^2(\mu \omega) + \pi^2 \omega} \right) \right] = F.$$  \hspace{1cm} (30)

On substitution into Eq. (28) we obtain

$$x(t) = F e^{i \omega t} \left[ k + 2 G C(\mu) \frac{2 \mu \omega - 2 \ln(\mu \omega)}{4 \ln^2(\mu \omega) + \pi^2} \right] - \omega^2 + i \omega \left( c + 2 G C(\mu) \frac{4 \mu \omega \ln(\mu \omega) + 2 \pi}{4 \omega \ln^2(\mu \omega) + \pi^2 \omega} \right).$$  \hspace{1cm} (31)

The equivalent stiffness factor and damping factor are derived as

$$\tilde{k} = k + 2 G C(\mu) \frac{\pi \mu \omega - 2 \ln(\mu \omega)}{4 \ln^2(\mu \omega) + \pi^2},$$  \hspace{1cm} (32)

and

$$\tilde{c} = c + 2 G C(\mu) \frac{2 \mu \omega \ln(\mu \omega) + \pi}{4 \omega \ln^2(\mu \omega) + \pi^2 \omega}.$$  \hspace{1cm} (33)

Accordingly, the steady-state response of the system is

$$x(t) = \frac{F e^{i \omega t}}{k - \omega^2 + i \omega \tilde{c}} = \frac{F e^{i(\omega t - \varphi)}}{\sqrt{(\tilde{k} - \omega^2)^2 + (\omega \tilde{c})^2}},$$  \hspace{1cm} (34)
where the phase difference between the excitation and response is determined as
\[ \varphi = \begin{cases} 
\arctan \frac{\omega \mu}{k - \omega^2}, & k - \omega^2 > 0, \\
\pi/2, & k - \omega^2 = 0, \\
\pi + \arctan \frac{\omega \mu}{k - \omega^2}, & k - \omega^2 < 0. 
\end{cases} \] (35)

For the sake of comparison and valuation, we apply the replacement
\[ \mu = \tan \frac{\pi \gamma}{2}. \] (36)

So we convert the infinite interval of \( \mu: 0 < \mu < +\infty \) into the finite interval of \( \gamma: 0 < \gamma < 1 \). Since \( \mu \) is a constant in the above derivation, such replacement is allowable.

We remark that there are an infinite number of ways to perform such replacement as in Eq. (36).

The weight function parameterized by \( \gamma \) is
\[ p(\lambda) = p(\lambda; \gamma) = C(\tan \frac{\pi \gamma}{2}) \left( \tan \frac{\pi \gamma}{2} \right)^{\lambda}, \] (37)

where \( 0 < \gamma < 1 \). In Figure 9, the curves of the weight functions \( p(\lambda) \) are shown for different parameters \( \gamma \).

**3.1 The stiffness contribution factor and damping contribution factor**

It follows from Eqs. (32) and (33) that the stiffness contribution factor and damping contribution factor of the distributed-order derivative term are
\[ \hat{k} - k = 2GC(\mu) \frac{\pi \mu \omega - 2 \ln(\mu \omega)}{4 \ln^2(\mu \omega) + \pi^2}, \] (38)

and
\[ \hat{c} - c = 2GC(\mu) \frac{2 \mu \omega \ln(\mu \omega) + \pi}{4 \omega \ln^2(\mu \omega) + \pi^2 \omega}. \] (39)

respectively.

It is easy to prove that
\[ \pi y - 2 \ln y > 0, \ 2y \ln y + \pi > 0, \ \text{for} \ y > 0. \]

So the stiffness contribution factor and damping contribution factor are all positive. By substituting \( \mu \rightarrow \tan \frac{\pi \gamma}{2} \), we have
\[ \hat{k} - k = 2GC(\tan \frac{\pi \gamma}{2}) \frac{\pi \omega \tan \frac{\pi \gamma}{2} - 2 \ln(\omega \tan \frac{\pi \gamma}{2})}{4 \ln^2(\omega \tan \frac{\pi \gamma}{2}) + \pi^2 \omega}, \]
\[ \hat{c} - c = 2GC(\tan \frac{\pi \gamma}{2}) \frac{2 \omega \tan \frac{\pi \gamma}{2} \ln(\omega \tan \frac{\pi \gamma}{2}) + \pi}{4 \omega \ln^2(\omega \tan \frac{\pi \gamma}{2}) + \pi^2 \omega}. \] (40)

In Figures 10 and 11, the curves of the stiffness contribution factor \( \hat{k} - k \) and damping contribution factor \( \hat{c} - c \) versus \( \omega \) are shown for \( G = 1 \) and \( \gamma = 0.1, 0.25, 0.5, 0.75, 0.9 \), respectively.

Figures 12 and 13 show, respectively, the stiffness contribution factor \( \hat{k} - k \) and damping contribution factor \( \hat{c} - c \) versus \( \gamma \) for \( G = 1 \) and \( \omega = 0.3, 0.6, 1, 3, 6 \).
3.2 Amplitude and phase difference

From Eq. (34), the amplitude of the response is

\[ |x(t)| = \frac{F}{\sqrt{(k - \omega^2)^2 + (\omega \dot{c})^2}}. \]  (41)

In Figure 14 and 15, the curves of the amplitudes $|x(t)|$ versus $\omega$ and versus $\gamma$ are shown for $k = 1.2, c = 0.1, F = G = 1$ and different values of $\gamma$ and $\omega$, respectively. The resonance peaks descend as the parameter $\gamma$ increases. This

In contrast to Figures 3 and 4, the curves in Figure 12 are monotonic decreasing apart from a small part on the left in the case of $\omega = 6$, and the curves in Figure 13 are monotonic increasing.
implies that increasing of $\gamma$ enhances the system damping. The increasing and decreasing of the amplitudes $|x(t)|$ with the parameter $\gamma$ rely on the driving frequency $\omega$. In the low-frequency region, the amplitudes $|x(t)|$ is an increasing function of $\gamma$, while in the high-frequency region, the amplitudes $|x(t)|$ is a decreasing function of $\gamma$. In Figures 16 and 17, the curves of phase difference $\varphi$ in Eq. (35) versus $\omega$ and versus $\gamma$ are shown for $k = 1.2$, $c = 0.1$, $G = 1$ and different values of $\gamma$ and $\omega$. Similar behaviors to Figures 7 and 8 are displayed.

4 A comparison of single half order and uniform distributed-order

We consider two special cases, one with the single half order derivative,

$$\ddot{x} + k \dot{x} + c \dot{x} + G \cdot \int_{-\infty}^{1/2} x = F e^{i\omega t},$$

(42)

and another with the uniform distributed-order derivative,

$$\ddot{x} + k \dot{x} + c \dot{x} + \int_{0}^{1} D_{t}^{1/2} x \, d\lambda = F e^{i\omega t}. \quad (43)$$

For the system (42), the equivalent stiffness and damping factors are

$$\hat{k} = k + G \sqrt{\frac{\omega}{2}}, \quad \hat{c} = c + G \frac{1}{\sqrt{2\omega}}. \quad (44)$$

We denote the response amplitude and the phase difference as $A(\omega)$ and $\varphi(\omega)$.

For the system (43), the equivalent stiffness and damping factors are

$$\hat{k} = k + 2G \frac{\pi \omega - 2 \ln(\omega)}{4\pi^{2}(\omega) + \pi^{2}}, \quad \hat{c} = c + 2G \frac{2\omega \ln(\omega) + \pi}{4\omega \pi^{2}(\omega) + \pi^{2}}. \quad (45)$$

We denote the response amplitude and the phase difference as $B(\omega)$ and $\theta(\omega)$.

We take $F = G = k = 1$, $c = 0.1$ and 1, respectively, the amplitudes $A(\omega)$ and $B(\omega)$ are plotted in Figure 18, and the phase difference $\varphi(\omega)$ and $\theta(\omega)$ are plotted in Figure 19. The two cases display similar variation trend and dependence on the driving frequency. When $c = 0.1$ the case of uniform distributed-order shows a higher resonance peak than the case of single half order.

5 Conclusions

We considered the steady-state response to harmonic driving for two types of fractional one-degree-of-freedom sys-
tems, one with a single fractional-order derivative and another with a distributed-order derivative. The first case was examined in Section 2, where the stiffness contribution factor and damping contribution factor of the fractional derivatives, the response amplitude and phase difference were explored, especially for their relevance to the driving frequency and the order \( \alpha \) of the fractional derivative. In Section 3, such problems were investigated for the system with a distributed-order derivative, where the weight function is parameterized by \( \mu \) or \( \gamma \). In Section 4, we displayed the results of systems with single half order derivative and uniform distributed-order derivative in the same figures.

The two types of systems display similar response behaviors, but the stiffness contribution factor and damping contribution factor of the distributed-order derivative are almost monotonic varying with the parameter \( \gamma \) on the interval \((0, 1)\) in the weight function \( p(A) \), not exactly like the case of single fractional-order derivative with the order \( \alpha \) on the interval \([0, 1]\). The case of the distributed-order derivative provides us more options for the weight function and involved parameters.

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