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Holonomicity analysis of electromechanical systems

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Abstract: Electromechanical systems are described using state variables that contain electrical and mechanical components. The equations of motion, both electrical and mechanical, describe the relationships between these components. These equations are obtained using Lagrange functions. On the basis of the function and Lagrange - d'Alembert equation the methodology of obtaining equations for electromechanical systems was presented, together with a discussion of the nonholonomicity of these systems. The electromechanical system in the form of a single-phase reluctance motor was used to verify the presented method. Mechanical system was built as a system, which can oscillate as the element of physical pendulum. On the base of the pendulum oscillation, parameters of the electromechanical system were defined. The identification of the motor electric parameters as a function of the rotation angle was carried out. In this paper the characteristics and motion equations parameters of the motor are presented. The parameters of the motion equations obtained from the experiment and from the second order Lagrange equations are compared.

Keywords: Lagrange function, nonholonomic system, electromechanical system, motion equation

PACS: 45.05.+x, 45.20.Jj, 45.20.dg, 45.20.dh

1 Introduction

The foundations of mechanics were created by generations of mathematicians and astronomers trying to describe the motion of such complex mechanical system as the planet system. Lagrange occupies a special place among them. Benefiting from his achievements W. R. Hamilton in 1834 stated that [1] "Among the successors of those illustrious men (autor's remark Galileo, Newton), Lagrange has perhaps done more than any other analyst to give extent and harmony to such deductive researches, by showing that the most varied consequences respecting the motions of systems of bodies may be derived from one radical formula; the beauty of the methods so suiting the dignity of the results as to make of his great work a kind of scientific poem".

Euler also played a great role in the creation of the mechanics of many bodies [2]. However, it was Lagrange who formulated the function called his name, which is a difference between kinetic energy and potential energy.

$$fL = E_k - E_p \tag{1}$$

where: E_k - kinetic energy, E_p - potential energy.

The kinetic energy is the sum of the kinetic energy of all components and, similarly, potential energy is the sum of the potential energy of all components. It is worth noting that Lagrange was considering a conservative arrangement. This means that there are no energy dissipation components in the Lagrange function. For the electromechanical system, a distinction between kinetic electrical and mechanical energy, as well as potential electric and mechanical energy, can be made. For electromechanical rotating devices, the independent (input) mechanical coordinate, variable of the model, is the rotor rotation angle, and the electric input coordinate is the electric charge of the electrical circuit. A characteristic feature of such a system is the lack of potential energy, both electrical and mechanical. But in the Lagrange function, there is a mechanical coordinate in the form of the dependence of the inductance of the electrical circuit from the angle of rotation. In addition, there are the electrical and mechanical coordinate time derivatives, velocities occurring in the expressions of kinetic energy. After the distinction of the generalised coordinates and their velocities, the Lagrange equation of the first kind (Lagrange - d'Alembert equation) is derived:

$$\left(\frac{d}{dt}\frac{\partial fL}{\partial \dot{q}_e}\right)\delta q_e + \left(\frac{d}{dt}\frac{\partial fL}{\partial \dot{q}_m}\right)\delta q_m = 0 \tag{2}$$

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Above equation is the sum of products derived from Euler's variation products of Lagrange function and virtual displacements of individual generalised variables. When these displacements are independent, the system is holonomic and the Euler variation derivatives of the Lagrange function describing the virtual forces balances of the elements of the considered system are called Euler - Lagrange equations:

$$\frac{d}{dt}\frac{\partial fL}{\partial \dot{q}_e} = 0 \tag{3}$$

$$\frac{d}{dt}\frac{\partial fL}{\partial \dot{q}_m} - \frac{\partial fL}{\partial q_m} = 0 \tag{4}$$

The equations above are called the equations of motion and they fully describe the dynamics of the system. But if the virtual displacements are not independent, then the system is a nonholonomic one. The Lagrange - d'Alembert equation should be used in the description of the system and on this basis, the character of interactions between mechanical and electrical system should be determined, "there is no doubt that the correct equations of motion for nonholonomic mechanical systems are given by the Lagrange - d'Alembert principle" [3] (page 236).

From Ref. [3] it also follows that experimental studies should be considered for the analysis of interactions between elements of a complex dynamic system, "the question of applicability of the nonholonomic model ... cannot, in any specific situation, be solved within the framework of an axiomatic scheme without recourse to experimental results" [3] (page 244).

The Lagrange function is most commonly used to describe motion equations of electromechanical systems [4]. It is described by generalized coordinates and velocities as the difference between kinematic and potential energy (1).

In general, potential energy is often not present in electromechanical systems. The Lagrange function does not describe energy flow. It describes relation between particular generalized coordinates and their derivatives. Hence, the holonomicity of the system should be discussed on the basis of Lagrange function of homogeneous and conservative systems. If, electric coordinates (electric charges) q_e and mechanical coordinates (angles) q_m are discriminated among generalized coordinates the Lagrange function of the system may be described as:

$$fL = \frac{1}{2}L(q_m)\cdot\left(\frac{dq_e}{dt}\right)^2 + \frac{1}{2}J\cdot\left(\frac{dq_m}{dt}\right)^2 \tag{5}$$

Based on the Lagrange function, Lagrange - d'Alembert equation may be formulated [5]. For the electromechanical system, it is in form presented as Equation (2).

According to Ref. [5] (page 90), the variations δq_e and δq_m from Lagrange - d'Alembert equation are independent only for the holonomic system, and only for that system, we obtain the Lagrange equations of the second kind (Euler - Lagrange equations) in form (3) and (4).

In order to analyze the motion equation of the electromechanical system, the reluctance motor with a single pair of stator and rotor poles was chosen – Figure 1.

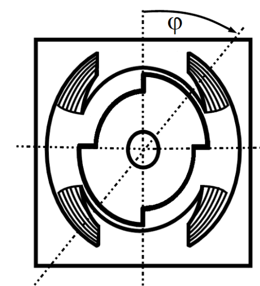

Figure 1: A general diagram of the reluctance motor

A construction of a reluctance motor (RM) is very simple because it has no brushes, magnets and rotor windings. Consequently, a reluctance motor is highly reliable and fault-tolerant. The second advantage of reluctance motors is that the friction torque of the motor depends mainly on friction of bearing, it is rather small and can easily be regulated.

The major drawback of the motor is that it starts to operate only for particular angles of the rotor versus stator. However, the electric circuit is described only by means of one equation. Therefore, it facilitates the analysis and experimental verification of the mathematical model.

The mathematical model of an electromechanical device consists of electrical and mechanical parts, described only by two equations. The analysis of systems is based on the measurement of the electric circuit and the mechanical system parameters. The mechanical and electric equations are most often formulated as follows [6, 7]:

$$J\frac{d\omega}{dt} + k(\omega) + T_L = Te \tag{6}$$

$$L(i,\varphi)\frac{di}{dt} + \frac{dL(i,\varphi)}{d\varphi}i\omega + R_s i = U_s \tag{7}$$

where: J - moment of inertia, $k(\omega)$ - torque of friction, T_L - torque of load, $\partial L\varphi/\partial\varphi$ - derivative of inductance versus the rotor rotation angle, U_s - terminal voltage, i - phase current, R_s - winding resistance and T_e, the electromagnetic torque, is defined as [8]:

$$T_e = \frac{1}{2}\frac{\partial L(\varphi)}{\partial\varphi}i^2 \tag{8}$$

Multiplying the equation (6) by the angular velocity ω, gives a power balance equation for the mechanical part (9). Multiplying the equation (7) by the current i, yields the power equation of electric equation (10):

$$J\frac{d\omega}{dt}\omega + k(\omega)\omega + T_L\omega = \frac{1}{2}\frac{\partial L(\varphi)}{\partial\varphi}\omega i^2 \tag{9}$$

$$L(\varphi)\frac{di}{dt}i + \frac{\partial L\varphi}{\partial\varphi}\omega i^2 + R_s i^2 = U_s i \tag{10}$$

A term on the right side of the equation (9) represents the power which is transferred from the electrical part of motor. Terms on the left side of the equation (9) represent the power related to the changes of rotor kinetic energy, friction power, and load power.

The first term on the left side of the equation (10) defines the rate of change of magnetic energy which is cumulated in the motor inductance. The second term describes the intensity of energy which is transformed to the mechanical system. The last term represents the thermal loss energy of stator winding. A term on the right side describes the electric power supplied to the motor. It should be emphasized, that the power transferred to the mechanical system from the electric system in the equation (10) is different than the intensity of energy transferred to the mechanical system in the equation (9).

Some explanation of the power difference may be found in [9–11]. In these papers, the power transformed from electric circuit is split into two parts:

$$\frac{1}{2}L\frac{di^2}{dt} + \frac{1}{2}i^2\frac{dL}{d\varphi}\omega + \frac{1}{2}i^2\frac{dL}{d\varphi}\omega + R_s i^2 = U_s i \tag{11}$$

and then it is written as:

$$\frac{d}{dt}(\frac{1}{2}Li^2) + \frac{1}{2}i^2\frac{dL}{d\varphi}\omega + R_s i^2 = U_s i \tag{12}$$

The first term is interpreted as the rate of magnetic energy accumulation, the second one is described as power transformed to the mechanical system. It should be noted, that in a steady state, the energy stored in the coil has two terms: the first one - $\frac{1}{2}L\frac{di^2}{dt}$, in the steady state equals zero, and the second one - $\frac{1}{2}i^2\omega\frac{dL}{d\varphi}$, has the same form as the power transferred to the mechanical system. In a steady state the transfer of electric power to mechanical output power is continuous. It means that accumulation rate of magnetic energy is also constant and the magnetic energy of motor increases continuously. Is it possible?

2 Identification of parameters

In order to verify the above relations, the test bench with the reluctance motor was constructed. The stator of the motor had one pair of poles, and a rotor had only one pair of teeth – see Figure 1.

The measurement of the parameters during the rotation process is quite complex, because the moment of load has to be set and measured. Therefore, in order to analyze the coefficients of the equations (6) and (7) , the mechanical system of the motor was modified. A steel rod of 6mm x 6mm x 354mm dimensions and of 100g weight was attached to the rotor axis. The rod and the rotor together formed a physical pendulum – see Figure 2 [12].

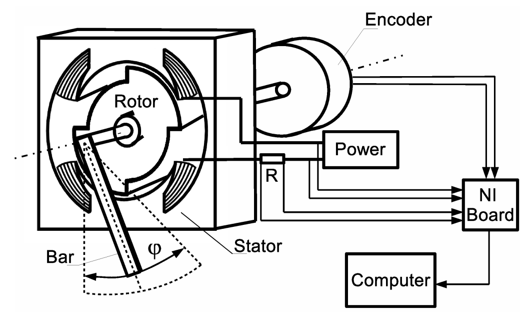

Figure 2: The measurement diagram of the reluctance motor parameters

A resistor R is connected in series with the coil of stator, and it is used to measure current in the stator winding. The values of voltage, current and rotation angle are measured simultaneously using NI 9225 and NI 6216 boards. The boards are serviced by a LabVIEW program installed on the computer where the data are recorded in a text data file *.txt. The data files were loaded into MATLAB and processed using Golay - Savitzky filter to eliminate noise and calculate derivatives. The parameters of the motion equations were calculated using least squares method [13].

On a motor shaft there is also a rotary encoder, which is used to measure the inclination angle of the rotor in relation to the stator.

In order to specify the relation between inductance and the rotation angle of the rotor, the motor winding was supplied by AC 50 Hz voltage source. The rotation angle was set up with a 5 degree step. Due to the symmetry of the motor, the research was carried out for the angle range of 180 degrees. The instantaneous values of current and the voltage of the motor winding were measured simultaneously. The measurements were also conducted using the National Instruments measurement boards, as well as LabVIEW and MATLAB – Simulink programs. The inductance profile, as a function of rotation angle, was established using the measured values - see Figure 3.

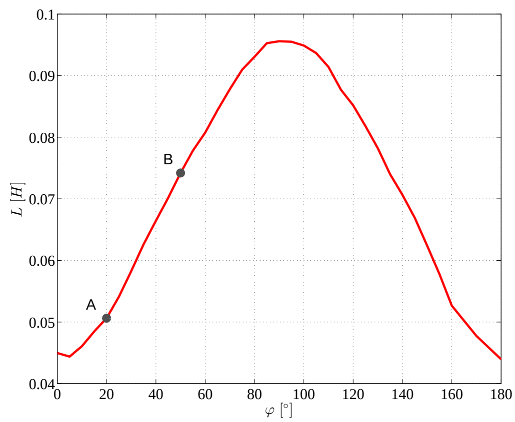

Figure 3: The inductance profile for the angle of rotation

The intermediate value of a derivative of inductance versus rotation angle for the rising quasilinear section of the characteristic in Figure 3 was calculated. Its value between a 20 degree and a 75 degree rotation angle amounts to:

$$\frac{\partial L\varphi}{\partial\varphi} \cong 0.0395\ \frac{H}{rad} \tag{13}$$

In order to check the coefficients of the motion equations (6) and (7), the identification of the equations parameters was carried out. The motor parameters were identified for stabilized current supply, thus derivative of the current in the equation (14) was omitted. The general form of the motion equations of the system was assumed as follows:

$$w_1 i + w_2 \frac{d\varphi}{dt} i = U_s \tag{14}$$

$$w_3 \frac{d^2\varphi}{dt^2} + w_4 sign \frac{d\varphi}{dt} + w_5 i^2 = \frac{mglsin(\varphi)}{2} \tag{15}$$

where: w_1 - stator winding resistance (R_s), w_2, w_5 - derivative of inductance versus the rotor rotation angle in relation to stator $\partial L\varphi/\partial\varphi$, w_3 - moment of inertia (J), w_4 - coefficient of dry friction, m - weight rod, g - acceleration of gravity, l - length of rod.

The rotation angle measurement without current was related to vertical position of the bar and the direction of earth gravity point A in Figure 3. When the stator circuit is powered by a DC voltage source with a stabilized current value the pendulum moves up by about 50 degree. – point B in Figure 3. The value of current is selected to match the inclination angle of the pendulum placed in the middle of positive slope of curve in Figure 3. Then the pendulum position was deflected manually. After release, the oscillations of the steel bar position angle were observed. In the test the changes of inductance are placed on the rising part of the inductance profile – Figure 3.

The approximate values of coefficients in the equations (14) and (15) were calculated. Based on measurement of current i, angle φ, and time derivatives of angle the parameters $w_1, \ldots, w_5$ were identified. Both the voltage induced in the winding, and its approximation calculation from the equation (14) are shown in Figure 4. Similarly, time characteristics of load torque from the equation (15) are displayed in Figure 5.

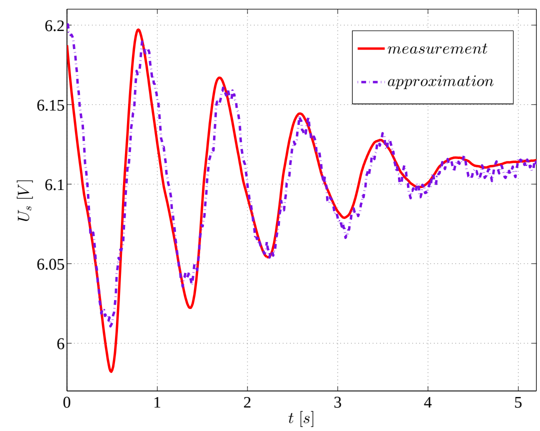

Figure 4: The waveform of voltage oscillation of the stator windings

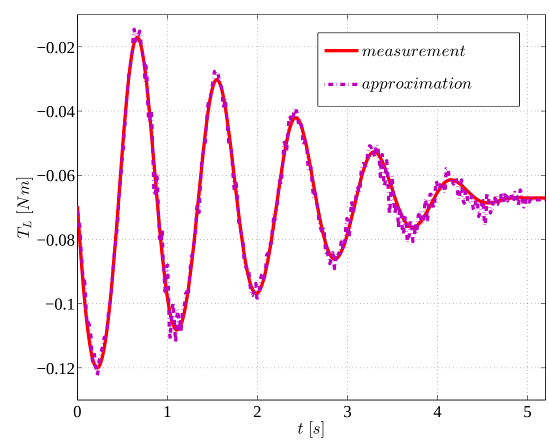

Figure 5: The oscillation of load torque in the mechanical equation

On the basis of the measurement, the coefficients of the equation (14) and the coefficients of the equation (15) are:

$$w_1 = 3.28\ \Omega \tag{16}$$

$$w_2 = 0.0199\ \frac{H}{rad} \tag{17}$$

$$w_3 = 0.003\ kg \cdot m^2 \tag{18}$$

$$w_4 = 0.0028\ \frac{kg \cdot m^2}{s} \tag{19}$$

$$w_5 = 0.0197\ \frac{H}{rad} \tag{20}$$

It can be seen, that the values of the coefficient w_2 from the equation (17) and of the coefficient w_5 from the equation (20) are very similar. Both values almost equal a half of the value of the derivative of inductance profile versus the rotation angle (13) as presented in Figure 3 with the accuracy being one percent. The correlation coefficient between measured and calculated voltages in Figure 4 is 0.9307. For the mechanical equation the correlation coefficient in Figure 5 is equal to 0.9925. The experiment showed that in the equation (14), the term which contains the derivative of inductance with respect to rotation angle, should be multiplied by coefficient $\frac{1}{2}$. The experiment results shown, that the equation of motion of electric circuit ought to be formulated as follows:

$$U_s = L(\varphi)\frac{di}{dt} + \frac{1}{2}\frac{\partial L\varphi}{\partial \varphi}\omega i + R_s i \tag{21}$$

The equation, which resulted from the experiment (21), is different from the widely known electric equation of electric motor [6–8] due the presence of the $\frac{1}{2}$ coefficient. Therefore, the relation between power balance and the holonomicity of the reluctance motor should be analyzed.

3 Holonomicity analysis of electromechanical system

If it is assumed that the virtual displacements are substituted by virtual velocities, the Lagrange - d'Alembert equation (2) becomes the equation of power balance. By substituting in the equation (2) with Lagrange function (5), and by assuming that $\delta q_e = i = const$ and $\delta q_m = \omega = const$, the equation of power balance is not fulfilled in steady state:

$$\frac{\partial L(\varphi)}{\partial \varphi}\omega i^2 - \frac{1}{2}\frac{\partial L(\varphi)}{\partial \varphi}\omega i^2 \neq 0 \tag{22}$$

The power balance (22) is different than zero, hence Lagrange - d'Alembert equation of the reluctance motor is not satisfied. Consequently, it is apparent, that the analyzed electromechanical system is nonholonomic.

If a dynamic system is described by n generalized coordinates, but its equations contain only $m < n$ of these coordinates and all generalized velocities, then the system is nonholonomic [5] (pages 107-108). The same rule is applicable to the equations of electromechanical systems. There is no generalized electric coordinate (electric charge) in electric equations. Therefore, it may be concluded that the electric equation determines whether the analyzed system is holonomic or nonholonomic.

From Ref. [14] it follows that the same procedure may be applied to both nonholonomic and holonomic systems. But, some correction in the equations is necessary. "The modifications that have to be made to the Lagrange equations may be found by looking at the problem in a slightly different way, and regarding the constrained system as the unconstrained system acted on by certain external forces, namely those forces which have to be exerted in order to compel the system to obey the constraint. This formulation has the advantage that in it the coordinates $q_1, q_2, \ldots, q_n$ may be regarded as independent; the constraints now appear as the effect of additional forces, and not as relations between the coordinates. Because the coordinates are independent, Lagrange equations may be used, and the equations of motion of the constrained systems may be obtained by including the effects of the additional forces in the Lagrange equations." [14] (page 145-146). According to above, in the analyzed example in electric equation, a force Q_N was introduced.

$$\left(\frac{d}{dt}\frac{\partial fL}{\partial \dot{q}_e} + Q_N\right)\delta q_e + \left(\frac{d}{dt}\frac{\partial fL}{\partial \dot{q}_m} - \frac{\partial fL}{\partial q_m}\right)\delta q_m = 0 \tag{23}$$

A similar method is described in Ref. [5]. If the system is nonholonomic, the vector of reactions of nonholonomic constraints, as an additional expression ($Q_N \neq 0$) should be added. Then d'Alembert – Lagrange equation may be represented by Equation (23). After the changes, the fragment from [5] (page 92) is described as follows: "if Q_N is the generalized force of reaction in the process of movement of the non-holonomic system, then the equation describes also the process of movement of some holonomic system, along with kinetic energy E_k and the generalized force of reaction Q_N". It allows us to analyze the equations of electric and mechanical systems independently.

On the basis of the equation (23) and the kinetic energy of the conservative system, we may determine a correction Q_N. Assuming that the variations of generalized coordinates are equal to the derivatives of these coordinates, and taking into account the correction in steady-state, we get a following equation:

$$\frac{\partial L(\varphi)}{\partial \varphi}\omega i^2 - \frac{1}{2}\frac{\partial L(\varphi)}{\partial \varphi}\omega i^2 + Q_N i = 0 \tag{24}$$

Hence, an additional force Q_N equals:

$$Q_N = -\frac{1}{2}\frac{\partial L(\varphi)}{\partial \varphi}\omega i \tag{25}$$

If the additional force is used in an electric equation and are added dissipative and potential forces, the following relation is obtained:

$$L(\varphi)\frac{di}{dt} + \frac{1}{2}\frac{\partial L(\varphi)}{\partial \varphi}\omega i + R_s i = U_s \tag{26}$$

It can be seen that the equation (26) is the same as the equation (21), established on the basis of measurements. The multiplication of the equation (26) by the current i yields:

$$L(\varphi)\frac{di}{dt}i + \frac{1}{2}\frac{\partial L(\varphi)}{\partial \varphi}\omega i^2 + R_s i^2 = U_s i \tag{27}$$

The above equation is physically interpretable and along with the mechanical equation (9), describes the power balance of an electromechanical system.

4 Conclusions

It may be concluded that the power balance in the reluctance motor equations is achieved only after using the additional voltage, which plays a role of "nonholonomic force" in the electric equation of the reluctance motor. Only then, the powers in the given motor model are balanced. The power transferred to the mechanical system is equal to the power transferred from the electric equation. It proves that the analyzed motor is a nonholonomic system.

The equation describing an electric circuit for a nonholonomic system should be formulated as in (21). The correction presented in this paper may also be applied in more complex electromagnetic systems e.g. multi phase switching reluctance motor.

It ought to be noted that for system parts of different kind, e.g. electrical and mechanical, the generalized forces are measured in different units and that's why the action and reaction forces between the parts could not be compared or equipared. The action and the reaction between the part should consider only the form of power.

The discussion presented in this article allows to present proposals for the procedure of formulating motion equations based on Lagrange's function. The steps of this procedure are as follows:

1. Formulation of the Lagrange function,
2. The determination of Euler - Lagrange equations and the introduction of nonholomicity correction forces to equations, in which there is no generalized coordinate for which the equation was formulated,
3. Substituting the generalized velocities (generalized variables time derivatives) for virtual displacements (variations of generalized variables), formulation of the Lagrange - d'Alembert equation in steady state and determination of the nonholonomicity correctors. For the holonomic system, the correctors shall be equal to zero,
4. Insertion of nonholonomicity correctors into the Euler-Lagrange equations and addition of generalized external input and output forces such as supply, load, friction, heat dissipation of energy, etc.

As a result, the motion equations are obtained for both the holonomic and nonholonomic system.

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