Research Article

José Luis Morales Guerrero, Manuel Cánovas Vidal*, José Andrés Moreno Nicolás, and Francisco Alhama López

A note on the uniqueness of 2D elastostatic problems formulated by different types of potential functions

https://doi.org/10.1515/phys-2018-0029
Received October 23, 2017; accepted February 6, 2018

Abstract: New additional conditions required for the uniqueness of the 2D elastostatic problems formulated in terms of potential functions for the derived Papkovich-Neuber representations, are studied. Two cases are considered, each of them formulated by the scalar potential function plus one of the rectangular non-zero components of the vector potential function. For these formulations, in addition to the original (physical) boundary conditions, two new additional conditions are required. In addition, for the complete Papkovich-Neuber formulation, expressed by the scalar potential plus two components of the vector potential, the additional conditions established previously for the three-dimensional case in z-convex domain can be applied. To show the usefulness of these new conditions in a numerical scheme two applications are numerically solved by the network method for the three cases of potential formulations.

Keywords: Papkovich-Neuber representation; uniqueness solution; network simulation method; linear elasticity; classical linear elasticity

PACS: 02.30.Em, 02.60.Lj, 02.70.-c, 04.20.Ex, 46.25.-y

1 Introduction

In linear elasticity, the equilibrium equation in terms of displacement, in absence of body forces, is that of Navier equation, whose analytical solution can be obtained by means of potential formulations which simplify the elastic problem by yielding uncoupled governing equations in terms of some potential unknowns. These alternative formulations have allowed to solve a great variety of interesting problems such as Boussinesq and Kelvin [1], among others. As regards numerical solutions, classical methods, such as finite-difference, derive solutions from practical problems based, for example, on the potential formulations of the Airy’s stress function – for 2D elasticity problems – or the Prandtl’s function – for torsion in prismatic bars. Wang et. al. [2] resume the set of potential formulations related to linear elasticity field and their applications. Recently, Weisz-Patrault et al. [3] have presented a new and compact representation of the elastic problem using only two quaternionic-valued monogenic potentials that can be applied to 3D problems. Among them, Papkovich-Neuber representation \([4, 5]\) is extensively used. This work deals with singular aspects of this formulation referred to their uniqueness solutions, a complicated subject partially investigated in the last years \([6–8]\). The study of uniqueness of a potential solution can be useful from the point of view of a numerical analysis or if we want to connect analytical solutions applied to different parts of a domain.

For the case of Papkovich-Neuber (PN) representation, under certain conditions, the number of potential functions (one scalar \(\varphi_0\), and three components of the vector \(\varphi_x, \varphi_y, \varphi_z\)) of the general PN solution can be reduced to a derivate solution with no lack of completeness. Eu-banks and Sternberg [6] studied the necessary conditions to delete either the scalar function or one of the rectangular components of the vector potential function. In addition, they considered the derived PN solution with the
scalar and z-component of the vector potential for the axisymmetric case – named the Boussinesq solution [9].

The completeness of a derived PN solution ensures the suitability of the formulation for the problem under study but new additional conditions can be required to ensure the uniqueness in terms of potential functions as primary unknown. The terms completeness (i.e. the set of displacement functions in the derivative formulation is able to represent an arbitrary deformation of the elastic body) and uniqueness (i.e. the derived displacement potentials are unique for the given deformation) are used here in the sense given by Tran-Cong [8].

Stippes [7] was the first author to deal with the subject of uniqueness but always referring to the derived PN solution that results from deleting the scalar potential. This author noted that Eubanks and Sternberg [6] did not make any reference to this subject, even though the uniqueness condition was implicit in their demonstration of completeness. In a footnote, Stippes [7] wrote on the importance of the uniqueness conditions in reference to a numerical solution, but with arguments only valid for the case of deleting the scalar component of the potential representation. Twenty five years later, Tran-Cong [8], who did not apply his conclusions to particular problems as we do here, came back to the question and, based on the study of the null displacement field problem, proved the necessity of additional conditions for derived PN solutions other than those studied by Stippes [7]; more particularly, the case of deleting the rectangular z-component of the vector function when the problem is z-convex, as well as the case of Boussinesq solution. The uniqueness approach of Tran-Cong [8] completes the work of Eubanks and Sternberg [6] and Stippes [7]. Recently, Morales et al. [10] have successfully deduced results for axisymmetric problems in the Boussinesq solution, applying uniqueness conditions different from those deduced by Tran-Cong [8].

For the 2D elasticity case of plane strain in rectangular coordinates (the extension to plane stress is immediate), the complete PN solution is directly reduced to three potential functions. Choosing the xy-plane as the domain, and thus making the z-component of the vector potential zero, the complete PN solution is defined by three potential functions: φ₀, φₓ, φᵧ. The only possible derived PN functions (maintaining the scalar potential), are two: φ₀, φₓ and φ₀, φᵧ. Golecki [11] suggested that in the case of multiply connected domains, the derived solutions can be multivalued, while Wang and Wang [12] demonstrated that the complete PN solution always remains single valued.

In this work, we investigate the uniqueness conditions for the 2D elasticity problems in rectangular coordinates, using as the main unknowns the potential functions of the complete and derived PN solutions. For the derived formulations, formed by two potentials, it is demonstrated that the solution is unique fixing two additional conditions while for the completed formulation, uniqueness additional conditions can be derived from the tridimensional z-convex case studied by Tran-Cong [8].

To demonstrate the suitability of the proposed conditions for the derived formulations, two applications are solved: a rectangular domain in which the displacement field is zero anywhere, and a domain whose boundary supports an arbitrary load distribution that gives rise to a non-zero displacement field. In addition, for the complete formulation, a particular solution for the above applications that provides the same elastic solutions is also shown. Applications are numerically solved by using the network method [13], a technique already successfully applied to elasticity by Morales et al. [14] and other engineering problems [15–18]. A detailed description of the application of the network model for the numerical solution of the elasticity problem formulated by potential functions can be found in Morales et al. [19]. Solutions are compared with those from the Navier formulation. The network models are generated by using the EPSNET_10 © software [20] and run in the code PSpice © [21].

2 Governing equations and boundary conditions

In term of displacements (u) Navier equation is written as:

\[ \mu \nabla^2 u + (\lambda + \mu) \nabla (\nabla \cdot u) = 0 \]  

(1)

where λ and μ are the Lamé’s constants [1]. Using PN formulation, Eq. (1) is represented by four potential functions, named displacement potentials, in the form

\[ 2\mu \mathbf{u} = \phi - \nabla \left( \phi_0 + \frac{\mathbf{R} \cdot \phi}{4(1 - \nu)} \right) , \]  

(2a)

\[ \nabla^2 \phi = 0 , \]  

(2b)

\[ \nabla^2 \phi_0 = 0 \]  

(2c)

where \( \phi \) is a harmonic vector potential, \( \phi_0 \) a harmonic scalar potential, \( \mathbf{R} \) - the position vector and \( \nu \) - the Poisson ratio. Eqs. (2a, 2b, 2c) are a general solution of Eq. (1) and have proved to be complete for the general case [22].

In rectangular coordinates, the completed PN solutions for the 2D-elasticity plane strain case reduce the dis-
The expressions that relate stress and potential functions, which are required for implementing the boundary condition (5), are [1]:

\[
\begin{align*}
\sigma_{xx} &= -\frac{\partial^2 \phi_0}{\partial x^2} - \left( x \frac{\partial^2 \phi_x}{\partial x^2} + y \frac{\partial^2 \phi_y}{\partial x^2} \right) \frac{1}{4(1-v)} \\
&\quad + \frac{\partial \phi_x}{\partial x} \frac{1}{2} + \frac{\partial \phi_y}{\partial y} \frac{v}{2(1-v)} \\
\sigma_{yy} &= -\frac{\partial^2 \phi_0}{\partial y^2} - \left( x \frac{\partial^2 \phi_x}{\partial y^2} + y \frac{\partial^2 \phi_y}{\partial y^2} \right) \frac{1}{4(1-v)} \\
&\quad + \frac{\partial \phi_x}{\partial y} \frac{1}{2} + \frac{\partial \phi_y}{\partial x} \frac{v}{2(1-v)} \\
\sigma_{xy} &= -\frac{\partial^2 \phi_0}{\partial x \partial y} - \left( x \frac{\partial^2 \phi_x}{\partial x \partial y} + y \frac{\partial^2 \phi_y}{\partial x \partial y} \right) \frac{1}{2(1-v)} \\
&\quad + \frac{\partial \phi_x}{\partial y} \frac{1}{2} + \frac{\partial \phi_y}{\partial x} \frac{1-2v}{4(1-v)}
\end{align*}
\]

For each of the derived solutions, the resulting governing equations and boundary conditions are obtained by omitting one of the rectangular components, \(\phi_x\) or \(\phi_y\), in Eqs. (3-6). Note that, since the physical boundary conditions for a given problem are unique, the problem can be formulated by different sets of PDEs, two governing equations for each derived solution or three governing equations for the complete PN solution.

### 3 Uniqueness of the potential PN solutions in 2D elasticity problems

For the following study, it is convenient to express the general PN solution (2a) as

\[
u = a^2 \Psi - \nabla (\varphi + \mathbf{R} \cdot \Psi)
\]

with \(a \equiv 4(1-v)\). The new harmonic vector and scalar potentials, \(\Psi\) and \(\varphi\), respectively, are related with \(\varphi\) and \(\varphi_0\) by the equations \(\varphi = 2a^2 \Psi\) and \(\varphi_0 = 2 \mu \varphi\).

According to Stippes [7] and Tran-Cong [8], the potential representation (7) is unique only with some additional conditions. One way to find them is to study the form of the potential functions corresponding to a null displacement field.

#### 3.1 Uniqueness of derived PN solutions

Let us consider the derived solution of the completed PN representation reduced to the scalar potential and the...
y-component of the vector potential \((\varphi, \Psi_y)\). Using the nomenclature of Eq. (7), the vector form of the displacement field is

\[
\mathbf{u} = \alpha \Psi_y \mathbf{e}_y - \nabla (\varphi + y\Psi_y) \tag{8}
\]

For a zero displacement field \((\mathbf{u} = \mathbf{0})\), representation (8) satisfies the equation

\[
\alpha \Psi_y \mathbf{e}_y - \nabla (\varphi + y\Psi_y) = \mathbf{0} \tag{9}
\]

Applying the curl to this equation and taking into account the properties of this operator gives: \(\partial \Psi_y / \partial z = \partial \Psi_y / \partial x = 0\), so that \(\Psi_y = \Psi_y(y)\). Also, applying the divergence operator and using the above result and the expressions \(\nabla^2 \varphi = 0\) and \(\nabla^2 \Psi = 0\), it is deduced that \(\partial \Psi_y / \partial y = 0\). In short, for solution (8) to represent a zero displacement field, the non-zero component of the vector potential must be an arbitrary constant,

\[
\Psi_y = k_1 \tag{10}
\]

Introducing this result in Eq. (9) yields

\[
-\frac{\partial \varphi}{\partial x} \mathbf{e}_x + \left[ k_1 (\alpha - 1) - \frac{\partial \varphi}{\partial y} \right] \mathbf{e}_y - \frac{\partial \varphi}{\partial z} \mathbf{e}_z = \mathbf{0} \tag{11}
\]

From the first and last addends of Eq. (11) we obtain \(\partial \varphi / \partial x = \partial \varphi / \partial z = 0\); this implies that \(\varphi = g(y)\). Substituting this result in the second addend, the necessary condition (referring to the scalar potential \(\varphi\) for the derived PN solution (8) to represent a zero displacement field is that \(\varphi\) is a plane parallel to the \(x\)-axe:

\[
\varphi = k_1 (\alpha - 1) y + k_2 \tag{12}
\]

Fixing the arbitrary constants \(k_1\) and \(k_2\), the potential solution (8) is unique for a zero displacement field. By extension, it is also unique for any other displacement field represented by the derived PN solution (8). Figure 2 represents equations (10) and (12): two planes defined only by two constants, \(k_1\) and \(k_2\), since the slope of the \(\varphi\) plane is determined by the expression \(k_1 (\alpha - 1) = \tan \beta\).

The uniqueness conditions can be expressed in the form of a Dirichlet condition

\[
\begin{align*}
\varphi(a) &= c_1 \\
\Psi_y(b) &= c_2
\end{align*} \tag{13}
\]

with \(a\) and \(b\) two arbitrary points of the domain, and two constants: \(c_1\) and \(c_2\), also arbitrarily chosen. Equation (13) is a suitable way of fixing constants \(k_1\) and \(k_2\) and makes solutions (10) and (12) unique: \(k_1 = c_2\) and \(k_2 = c_1 c_2 (\alpha - 1)\). Also, note that the notation of the potential functions introduced in (7), together with the arbitrability of choice constants in (13), let us use \(\varphi_0(a) = c_1\) and \(\Psi_y(b) = c_2\) as additional conditions.

Other possibilities for establishing the uniqueness conditions are: i) to fix values of the scalar potential \(\varphi\) at two different points of the domain, \(\varphi(a) = c_1\) and \(\varphi(b) = c_2\), and ii) to fix a value of the scalar potential \(\varphi\) at one point and its derivative at the same point, \(\varphi(a) = c_1\) (Dirichlet condition) and \(\partial \varphi / \partial y|_a = c_2\) (Neumann condition).

As regards the other derived solution \((\varphi, \Psi_x)\), following similar reasoning, the general form of the potential solution for the null displacement is \(\Psi_x = k_1\) and \(\varphi = k_1 (\alpha - 1) x + k_2\). The resulting uniqueness condition can be similarly expressed as: \(\varphi_0(a) = c_1\) and \(\Psi_x(b) = c_2\), with \(a, b, c_1\) and \(c_2\) have the same meaning as in condition (13).

### 3.2 The complete PN solution

The complete PN solution for 2D elasticity, using the nomenclature of (7) is

\[
\mathbf{u} = \alpha (\Psi_x \mathbf{e}_x + \Psi_y \mathbf{e}_y) - \nabla (\varphi + x \Psi_x + y \Psi_y) \tag{14}
\]

Following the former subsection, the starting point for the uniqueness study is the zero displacement field:

\[
\alpha (\Psi_x \mathbf{e}_x + \Psi_y \mathbf{e}_y) - \nabla (\varphi + x \Psi_x + y \Psi_y) = \mathbf{0} \tag{15}
\]

This equation is equivalent to the null displacement field condition studied by Tran-Cong [8] for the three-dimensional case of \(z\)-convex domains. As a consequence, the uniqueness conditions are also the same. For the case of plane deformation, and by extension of plane tension, the uniqueness conditions according to Tran-Cong can be set in several ways, being the most suitable: i) fixing the value of \(\Psi_x\) at the 2D boundary, Figure 1, by means of a continuous arbitrary function, together with \(\varphi(a) = c_1\) and \(\Psi_y(b) = c_2\); or ii) fixing the value of \(\Psi_y\) at the 2D boundary by means of a continuous arbitrary function, together with \(\varphi(a) = c_1\) and \(\Psi_x(b) = c_2\). As in the former section, \(a\) and \(b\) are arbitrary points of the domain and \(c_1\) and \(c_2\) two constants, also arbitrarily chosen.
4 Applications

4.1 Rectangular plate with zero displacement field

Figure 3 shows the physical model (a) and the 10 × 10 grid (b) for a 2D domain under plane strain. Rigid body movement is restricted by setting zero the perpendicular displacements over the whole boundary $u^0_b = 0$, Figure 3a; in addition, the tangential traction is zero in the whole boundary $t^0_b = 0$. The choice of this grid size is not relevant since we hope a null displacement field. Values of the geometry parameters are $L = 1000$ mm, $H = 1000$ mm, while elastic constants are $E$ (Young modulus) = 210 GPa and $ν$ (Poisson ratio) = 0.3. Additional constants for the two derived solutions, section 3.1, are detailed in Table 1 using the grid reference of Figure 3b. In this figure additional conditions are plotted using the following code: circles, squares and diamonds in red for the potentials $φ_0$, $φ_x$ and $φ_y$, respectively. Note that the values of these additional conditions are not zero because we wish to show a general form of the potential surfaces – a zero value for these constants is a particular value that provides a null (trivial) solution for the potentials. In addition, Table 1 shows additional conditions related to the complete formulation needed to find a particular solution by means of the software EPSNET_10 © [20].

Figure 3: Physical model (a) and grid (b) for a rectangular plate under simple boundary conditions for a null displacement field

Simulation of the three models, one for each solution type, in the code PSpice © [21] using the software EPSNET_10© for preprocessing and postprocessing, gives the following results: Figure 4 shows the displacement field using a large scale to test the null displacement results while Figure 5 shows the potential solutions. Potentials of the derived solutions, Figures 5(a) and (b), agree with the theoretical shape of Figure 2 for the null displacement field. Potential results of the complete PN solution (see Figure 5c) are equivalent to those of derived solution ($φ_0$, φy) due to the extra condition $φ_x|_{AS} = 0$. Changing this condition by other arbitrary continuous linear function, as indicated in the Table 2, new potential results are obtained (see Figure 6) being the displacement solution the same.

This application demonstrates that, in accordance with the uniqueness conditions mentioned in Section 3.1 and the results given in Figure 2, the solution for the potential functions using the formulations $φ_0$, $φ_x$ and $φ_0$, $φ_y$ has the form of constant planes for the vector component and inclined planes for the scalar components. As regards the formulation for $φ_0$, $φ_x$ and $φ_y$, a change at the boundary conditions for one of components of the vector potential function, retaining the fixed values of the other component as well as the scalar potential, provides solutions qualitatively different (with appreciable curvature in some regions) to the former, giving rise to the same solution for the displacements (Tables 1 and 2 and Figures 5c and 6).

4.2 Rectangular plate under arbitrary loads

The new boundary conditions referring to load ($ρ = 100$ MPa) and displacements are shown in Figure 7a. These are: zero normal displacement and zero tangential traction on the bottom and left boundaries, zero tangential and normal traction on the top boundary and zero tangential traction and specified normal traction on the right boundary. Geometry and elastic parameters of the material are the same as in the above problem. In this case, a more refined grid of $20 × 20$ was used for a better resolution. In addition, all the additional conditions were set up to zero and placed as indicated in Figure 7b and Table 3. Figure 8 shows the displacement field of the domain for all the potential solutions as well as the Navier solution [14], while Figure 9 depicts the potential solutions whose forms depend on the physical boundary conditions (loads and displacements) and the arbitrary values of the additional conditions (for an only numerical solution of the potential functions). For each potential formulation, the harmonic functions are smooth surfaces which curve slightly at their boundaries; although these are very different potential surfaces, the displacement solutions for each one are the same.

To see the influence of the grid size in the numerical results of the potential formulation solved by network method and to compare them with a common formulation of the elastic problem solved by a standard numerical code such as FEM, finite element method [23], Table 4 is presented. For the potential formulation ($φ_0$, $φ_x$), this table
Table 1: Additional conditions for derived and completed PN solutions referring to cells in a $10 \times 10$ grid, Figure 3b

<table>
<thead>
<tr>
<th>Solution type</th>
<th>Additional conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varphi_0, \varphi_x$</td>
<td>$\varphi_0(1,1) = 10$</td>
</tr>
<tr>
<td>$\varphi_0, \varphi_y$</td>
<td>$\varphi_0(1,1) = 10$</td>
</tr>
<tr>
<td>$\varphi_0, \varphi_x, \varphi_y$</td>
<td>$\varphi_0(1,1) = 10$</td>
</tr>
</tbody>
</table>

Table 2: Additional conditions for completed PN solutions referring to cells in a $10 \times 10$ grid (Figure 3b)

<table>
<thead>
<tr>
<th>Solution type</th>
<th>Additional conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varphi_0, \varphi_x, \varphi_y$</td>
<td>$\varphi_0(1,1) = 10$</td>
</tr>
</tbody>
</table>

Figure 4: Solution of the displacement field by PSpice © and ESPNET_10 ©: a) Derived PN solution ($\varphi_0, \varphi_x$), b) derived PN solution ($\varphi_0, \varphi_y$) and c) completed PN solution ($\varphi_0, \varphi_x, \varphi_y$) shows the maximum displacement located at the point of maximum load, right-bottom corner, as well as the comparisons with FEM solution for a same grid. As expected, deviations diminish as grid size increases and are always less than 1%. 
Figure 5: Potential solutions: a) Derived PN solution ($\phi_0$, $\phi_x$), b) derived PN solution ($\phi_0$, $\phi_y$) and c) completed PN solution ($\phi_0$, $\phi_x$, $\phi_y$)
Table 3: Additional conditions for derived and completed PN solutions referring to cells in a 20 × 20 grid, Figure 7b

<table>
<thead>
<tr>
<th>Solution type</th>
<th>Additional conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_0, \phi_x$</td>
<td>$\phi_{0}(1,1) = 0$ $\phi_x(1,20) = 0$</td>
</tr>
<tr>
<td>$\phi_0, \phi_y$</td>
<td>$\phi_{0}(1,1) = 0$ $\phi_y(20,1) = 0$</td>
</tr>
<tr>
<td>$\phi_0, \phi_x, \phi_y$</td>
<td>$\phi_{0}(1,1) = 0$ $\phi_{x}</td>
</tr>
</tbody>
</table>

Table 4: Displacements at the point with maximum load and comparison with FEM code solution for different grid size. Deviations are shown between brackets

<table>
<thead>
<tr>
<th>Grid size</th>
<th>10 × 10</th>
<th>20 × 20</th>
<th>40 × 40</th>
<th>80 × 80</th>
</tr>
</thead>
<tbody>
<tr>
<td>FEM</td>
<td>0,3425</td>
<td>0,3430</td>
<td>0,3432</td>
<td>0,3433</td>
</tr>
<tr>
<td>$\phi_0, \phi_x$ solution (deviation)</td>
<td>0,3475 (1,46%)</td>
<td>0,3458 (0,82%)</td>
<td>0,3441 (0,26%)</td>
<td>0,3434 (0,03%)</td>
</tr>
</tbody>
</table>

Figure 6: Potential solutions for the completed PN solution: a) $\phi_0$, b) $\phi_x$, c) $\phi_y$. The uniqueness conditions are shown in Table 2

Figure 7: Physical model (a) and grid (b) for a rectangular plate under arbitrary boundary conditions

Figure 8: Solutions of the non-zero displacement field obtained by ESPNET_10 ©: a) Derived PN solution ($\phi_0, \phi_x$), b) derived PN solution ($\phi_0, \phi_y$) and c) completed PN solution ($\phi_0, \phi_x, \phi_y$) and d) Navier solution

5 Conclusions

Based on the study of the null displacement field, additional conditions must be imposed to obtain a unique so-
A note on the uniqueness of 2D elastostatic problems

Figure 9: Potential solutions for a non-zero displacement field: a) Derived PN solution \((\phi_0, \phi_x)\), b) derived PN solution \((\phi_0, \phi_y)\) and c) completed PN solution \((\phi_0, \phi_x, \phi_y)\)

solution for the 2D planar elasticity problem formulated by the two potential solutions derived from the Papkovich-Neuber representation. These are defined by a scalar potential and one of the components of the vector potential. For each solution, two additional conditions are required. One way for implementing these conditions is to fix an arbitrary value of scalar potential at one arbitrary point of the domain and the same for the component of the vector potential. Other possibilities for implementing these conditions (Dirichlet or Neumann types) are also proposed. For the complete Papkovich-Neuber formulation, formed by three unknown potential functions, it has been demonstrated that the additional conditions proposed by Tran-Cong [8] for the \(z\)-convex domains can also be applied. Again, there are many different forms for implementing these additional conditions. To demonstrate the reliability of these results two applications for each type of potential solutions were numerically solved by network method.

References


