Research Article

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Joint measurements of optical parameters by irradiance scintillation and angle-of-arrival fluctuations

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Abstract: A method for joint measuring the power law exponent and the structure constant of atmospheric turbulence is proposed and examined. The measurements are equivalent to solve the simultaneous equations formed by the irradiance scintillation index and the angle-of-arrival fluctuations variance, where the measured parameters are regarded as the unknowns. The measured error analysis is also presented. Based on our proposed method, the measured results accord with the daily trend of atmospheric turbulence.

Keywords: atmospheric turbulence, laser beam transmission, irradiance scintillation index, angle-of-arrival fluctuations

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1 Introduction

Optical wireless communication (OWC) technology has attracted much attention for its large channel capacity and high transmission rate. It uses unguided optical carrier propagation in the free space channel for data transmission [1, 2]. As one typical kind of the free space, the atmosphere contains numerous turbulent eddies. To improve the performance of OWC systems, it is necessary to characterize the atmospheric turbulent eddies. It is the turbulent power law exponent \( \alpha \) and the generalized atmospheric structure constant \( \tilde{C}_n^2 \) that are critical optical parameters for theoretical modeling of the turbulence [3, 4]. \( \alpha \) is dimensionless, and has influence on the shape of turbulent spectrum. For Kolmogorov turbulence, \( \alpha = \frac{5}{3} \), whereas for non-Kolmogorov cases, \( \alpha \in (3, 4) \). \( \tilde{C}_n^2 \) is the quantization of the strength of the refractive index fluctuations, whose typical values near the ground is \( 10^{-15} \sim 10^{-13} \text{m}^{-\alpha} \) [5–8].

In the past few decades, several methods have been proposed and applied to measure various optical parameters for actual atmospheric channels. These methods belong to the indirect measurement techniques, and can be generally classified into meteorology-based and optics-based methods [9]. For path-average measurements, the optics-based methods draw more attention than the meteorology-based ones [10, 11]. The optics-based methods prefer to utilize the irradiance scintillation (IS) or the angle-of-arrival (AOA) fluctuations for the indirect measurements of \( \alpha \) and \( \tilde{C}_n^2 \). IS is connected to the stochastic redistribution of the wavefront amplitude at the receiver, whereas AOA associates with the phase shift in the focal plane [1]. \( \alpha \) is firstly determined by the slope of the temporal spectrum of IS or AOA, and \( \tilde{C}_n^2 \) is subsequently determined by the corresponding mathematical relationship of IS index \( \sigma_{IS}^2 \) or AOA variance \( \sigma_{AOA}^2 \) [12, 13]. However, there are always differences between the results of IS-based and AOA-based measurements, and the unequal measured values may cause confusion for atmospheric calibration [14]. An alternative to avoid the inequality is the joint measurements of \( \alpha \) and \( \tilde{C}_n^2 \) based on \( \sigma_{IS}^2 \) and \( \sigma_{AOA}^2 \) synchronously. Although the joint measurements of multiple optical parameters from multiple physical quantities have been investigated before, their error analysis is still less well established because of the complexity [15].

This paper investigates the path-average measurements of \( \alpha \) and \( \tilde{C}_n^2 \). Our work can be regarded as incremental to the optics-based measurements. Compared with the previous work, our proposed method adopts more than one turbulent effects for joint measurements, and can ensure the uniqueness of the measurements. The rest of the paper is organized as follows. Section 2 not only presents detailed procedures for how to determine \( \alpha \) and \( \tilde{C}_n^2 \), but also gives their measurement errors by the mathematical
models of $\sigma_{\text{IS}}^2$ and $\sigma_{\text{AOA}}^2$, respectively. In Section 3, our proposed method is examined based on both the simulated data and the actual gathered dataset, followed by conclusions in Section 4.

## 2 Theoretical model

Based on the Rytov perturbation approximation and the standard non-Kolmogorov spectrum, $\sigma_{\text{IS}}^2$ and $\sigma_{\text{AOA}}^2$ for a plane wave take the form as [16, 17]

$$
\begin{align*}
\sigma_{\text{IS}}^2 (\alpha, \tilde{C}_n^2) &= -8C_n^2 L^4 k^{3/2} \pi^2 \alpha^{-1} \\
\times A(\alpha) \Gamma \left(1 - \frac{a}{2}\right) \sin \frac{\alpha}{\pi}, \\
\sigma_{\text{AOA}}^2 (\alpha, \tilde{C}_n^2) &= 2^{3-a} \pi^2 d^{a-4} L \tilde{C}_n^2 \beta^{a-4} \\
\times A(\alpha) \Gamma \left(2 - \frac{a}{2}\right),
\end{align*}
$$

(1)

where $L$ is the length along the propagation path, $k = \frac{2\pi}{\lambda}$ is the angular wavenumber of the wave length $\lambda$, $d$ is the diameter of the aperture at the receiver, and $\beta = 0.4832$ is the coefficient. $A(\alpha) = \frac{\Gamma(a-1)}{\Gamma(a)} \cos \frac{\alpha}{\pi}$ is a function of $\alpha$, and $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$ is the gamma function. It must be pointed that $\sigma_{\text{AOA}}^2 (\alpha, \tilde{C}_n^2)$ is valid for $d > \sqrt{k^{-1}L}$.

### 2.1 Solutions of Eq. (1)

It is apparent that both $\alpha$ and $\tilde{C}_n^2$ can be regarded as independent variables of $\sigma_{\text{IS}}^2$ and $\sigma_{\text{AOA}}^2$. Thus, the measurements of $\alpha$ and $\tilde{C}_n^2$ is equivalent to the solutions of the following nonlinear bivariate equations problem

$$
\begin{align*}
\bar{\sigma}_{\text{IS}}^2 &= \sigma_{\text{IS}}^2 (\alpha, \tilde{C}_n^2), \\
\bar{\sigma}_{\text{AOA}}^2 &= \sigma_{\text{AOA}}^2 (\alpha, \tilde{C}_n^2),
\end{align*}
$$

(2)

where $\bar{\sigma}_{\text{IS}}^2$ and $\bar{\sigma}_{\text{AOA}}^2$ are measured values of $\sigma_{\text{IS}}^2$ and $\sigma_{\text{AOA}}^2$, respectively.

To solve Eq. (2), $\alpha$ and $\tilde{C}_n^2$ should be separated. Based on Eq. (1), it is readily shown

$$
\begin{align*}
f(\alpha) &= \frac{\partial}{\partial a} \bar{\sigma}_{\text{IS}}^2 \\
&= 2^a d^{a-4} L^{\frac{3}{2}} \alpha^{-1} k^{3/2} \alpha^{-1} \beta^{a-4} \\
\times \left(\alpha - 2\right)^{-1} \sin \frac{\alpha}{\pi} \\
&= d \left(\frac{\alpha}{\sin k L}\right)^a \times g(\alpha), \\
g(\alpha) &= 2^a \alpha^{-1} \beta^{a-4} (\alpha - 2)^{-1} \sin \frac{\alpha}{\pi}.
\end{align*}
$$

(3)

It is obvious that $f(\alpha)$ is continuous when $\alpha \in (3, 4)$, and $\lim_{\alpha \to 3} f(\alpha) = \frac{3}{2} \sqrt{2d \beta L^4 k^2}$ while $\lim_{\alpha \to 4} f(\alpha) = 0$.

Figure 1 plots the $g(\alpha)$, which strictly monotonically decreases for $\alpha \in (3, 4)$. Because $2^{-\beta} < 1$, it can be found that $f(\alpha)$ strictly monotonically decreases when $\alpha \in (3, 4)$. Based on the extreme value theorem for a real-valued continuous function and the monotonicity, there must be one and only one $\alpha \in (3, 4)$ satisfying

$$
f(\alpha) - \bar{\sigma}_{\text{IS}}^2 = 0
$$

(4)

if $\bar{\sigma}_{\text{IS}}^2 \in \left(0, \frac{3}{2} \sqrt{2d \beta L^4 k^2}\right)$ [18]. Thus, to measure $\alpha$ is equivalent to find the root of Eq. (4) in the interval $(3, 4)$, which can be easily computed by numerical methods.

Simultaneously, to measure $\tilde{C}_n^2$ is equivalent to calculate either of the following expressions

$$
\tilde{C}_n^2 = \frac{-\bar{\sigma}_{\text{IS}}^2}{8\pi^2 L^{3/2} \alpha^{-1/2} A(\alpha) \Gamma \left(1 - \frac{a}{2}\right) \sin \frac{\alpha}{\pi}} \\
\times \frac{2^a \pi^{2a-4} \beta^{a-4} \alpha \Gamma \left(2 - \frac{a}{2}\right)}{L^{a-4} \left(\frac{\alpha}{\sin k L}\right)^a}.
$$

(5)

They can lead to the same solution.

It has been pointed out that the unit of $\tilde{C}_n^2$ is related to $\alpha$. An optional technique to avoid the dependence on the selection of unit is to make the turbulence structure function $D_n(\tau) = C_n^2 r^{a-3}$ constant at $r = \sqrt{k^{-1}L}$ with changing $\alpha$, which means the Kolmogorov structure constant

$$
C_n^2 = \tilde{C}_n^2 \times \left(k^{-1}L\right)^{\frac{a-3}{2}}
$$

(6)

with the unit of $m^{-\frac{a}{2}}$ [19, 20].

### 2.2 Error analysis of the measurements

The observation of $\alpha$ and $\tilde{C}_n^2$ is indirect, so their measurement errors $\delta(\alpha)$ and $\delta\left(\tilde{C}_n^2\right)$ should rely on the measurement errors $\delta \left(\bar{\sigma}_{\text{IS}}^2\right)$ and $\delta \left(\bar{\sigma}_{\text{AOA}}^2\right)$.
For mathematical convenience, define
\[
\begin{align*}
F_1 &= 8\pi^2 L^2 k^{-3/2} a^{-1} A(\alpha)(1 - \frac{a}{2}) \sin \frac{\alpha a}{4} + \frac{2\pi a^2}{C_2}, \\
F_2 &= 2^{4-a} \pi^2 d^{-4} L^3 \beta^{-3} A(\alpha)(2 - \frac{a}{2}),
\end{align*}
\]  
(7)

where \( F_1 = 0 \) and \( F_2 = 0 \) \( \Leftrightarrow \) Eq. (2). \( a \) and \( \tilde{C}_n \) can be regarded as an implicit function of \( \tilde{\sigma}_{IS}^2 \) and \( \tilde{\sigma}_{AOA}^2 \). Based on the implicit function theorem, the Jacobian matrix is defined by [18]

\[
\begin{pmatrix}
\frac{\partial \alpha}{\partial \tilde{\sigma}_{IS}^2} & \frac{\partial \alpha}{\partial \tilde{\sigma}_{AOA}^2} \\
\frac{\partial \tilde{C}_n}{\partial \tilde{\sigma}_{IS}^2} & \frac{\partial \tilde{C}_n}{\partial \tilde{\sigma}_{AOA}^2}
\end{pmatrix} = -\left\langle \frac{\partial F_1}{\partial \alpha} \frac{\partial F_1}{\partial \tilde{C}_n}, \frac{\partial F_2}{\partial \alpha} \frac{\partial F_2}{\partial \tilde{C}_n} \right\rangle^{-1}
\]  
(8)

Based on Eq. (7), it is readily shown that \( \frac{\partial F_1}{\partial \alpha} = \frac{d}{d\alpha} \left( 8\pi^2 L^2 k^{-3/2} \alpha^{-1} A(\alpha)(1 - \frac{a}{2}) \sin \frac{\alpha a}{4} \right) \), \( \frac{\partial F_2}{\partial \alpha} = \frac{d}{d\alpha} \left( 2^{4-a} \pi^2 d^{-4} L^3 \beta^{-3} A(\alpha)(2 - \frac{a}{2}) \right) \), \( \frac{\partial F_1}{\partial \tilde{C}_n} = -\tilde{\sigma}_{IS}^2 \cdot \left( \tilde{C}_n \right)^{-2} \), \( \frac{\partial F_2}{\partial \tilde{C}_n} = \tilde{\sigma}_{AOA}^2 \cdot \left( \tilde{C}_n \right)^{-2} \), and the inverse matrix takes the form as [18]

\[
\begin{pmatrix}
\frac{\partial F_1}{\partial \alpha} & \frac{\partial F_1}{\partial \tilde{C}_n} \\
\frac{\partial F_2}{\partial \alpha} & \frac{\partial F_2}{\partial \tilde{C}_n}
\end{pmatrix}^{-1} = \left( \frac{\partial F_1}{\partial \alpha} \frac{\partial F_1}{\partial \tilde{C}_n} - \frac{\partial F_1}{\partial \alpha} \frac{\partial F_2}{\partial \tilde{C}_n} \right)^{-1} \times \left( \frac{\partial F_2}{\partial \tilde{C}_n} \frac{\partial F_1}{\partial \tilde{C}_n} - \frac{\partial F_1}{\partial \alpha} \frac{\partial F_2}{\partial \tilde{C}_n} \right),
\]  
(9)

On account of

\[
\begin{pmatrix}
\frac{\partial F_1}{\partial \alpha} & \frac{\partial F_1}{\partial \tilde{C}_n} \\
\frac{\partial F_2}{\partial \alpha} & \frac{\partial F_2}{\partial \tilde{C}_n}
\end{pmatrix} = \begin{pmatrix} \frac{1}{\tilde{C}_n^2} & 0 \\ 0 & -\frac{1}{\tilde{C}_n^2} \end{pmatrix},
\]  
(10)

the Jacobian matrix deduces to

\[
\begin{pmatrix}
\frac{\partial \alpha}{\partial \tilde{\sigma}_{IS}^2} & \frac{\partial \alpha}{\partial \tilde{\sigma}_{AOA}^2} \\
\frac{\partial \tilde{C}_n}{\partial \tilde{\sigma}_{IS}^2} & \frac{\partial \tilde{C}_n}{\partial \tilde{\sigma}_{AOA}^2}
\end{pmatrix} = \begin{pmatrix}
\frac{1}{\tilde{C}_n^2} & \frac{\partial F_1}{\partial \tilde{C}_n} \cdot \frac{\partial F_2}{\partial \tilde{C}_n} - \frac{\partial F_1}{\partial \alpha} \frac{\partial F_2}{\partial \alpha} \\
\frac{\partial F_2}{\partial \alpha} \frac{\partial F_1}{\partial \tilde{C}_n} - \frac{\partial F_1}{\partial \alpha} \frac{\partial F_2}{\partial \alpha} & \frac{\partial F_1}{\partial \tilde{C}_n} \cdot \frac{\partial F_2}{\partial \alpha} - \frac{\partial F_2}{\partial \tilde{C}_n} \frac{\partial F_1}{\partial \alpha}
\end{pmatrix}^{-1}
\]  
(11)

Based on the law of error’s propagation, it follows that [21]

\[
\begin{align*}
\delta(\alpha) &= \left| \frac{\partial \alpha}{\partial \tilde{\sigma}_{IS}^2} \right| \delta(\tilde{\sigma}_{IS}^2) + \left| \frac{\partial \alpha}{\partial \tilde{\sigma}_{AOA}^2} \right| \delta(\tilde{\sigma}_{AOA}^2), \\
\delta(\tilde{C}_n) &= \left| \frac{\partial \tilde{C}_n}{\partial \tilde{\sigma}_{IS}^2} \right| \delta(\tilde{\sigma}_{IS}^2) + \left| \frac{\partial \tilde{C}_n}{\partial \tilde{\sigma}_{AOA}^2} \right| \delta(\tilde{\sigma}_{AOA}^2).
\end{align*}
\]  
(12)

In practical situations, the relative error \( \rho \) is widely used to quantify the uncertainty of measurements, where \( \delta(\tilde{\sigma}_{IS}^2) \leq \rho \cdot \tilde{\sigma}_{IS}^2 \) and \( \delta(\tilde{\sigma}_{AOA}^2) \leq \rho \cdot \tilde{\sigma}_{AOA}^2 \).

### 3 Results and discussions

This section mainly examines our proposed method by simulated data and actual gathered dataset, respectively.

#### 3.1 Validation of Eq. (12)

To validate Eq. (12), the simulations follow the procedures below.

1) set the values of parameters in Eq. (1), and denote \( \langle \alpha \rangle \) and \( \langle \tilde{C}_n \rangle \) as the actual values of \( a \) and \( \tilde{C}_n \).
2) calculate \( \tilde{\sigma}_{IS}^2 \) and \( \tilde{\sigma}_{AOA}^2 \) based on Eq. (1).
3) calculate \( \tilde{\sigma}_{IS}^2 = \tilde{\sigma}_{IS}^2 (1 + \rho_{IS}) \) and \( \tilde{\sigma}_{AOA}^2 = \sigma_{AOA}^2 (1 + \rho_{AOA}) \).
4) calculate \( a \) and \( \tilde{C}_n \) based on Eq. (4) and Eq. (5).
5) calculate \( \delta(\alpha) \) and \( \delta(\tilde{C}_n) \) based on Eq. (12).
6) judge whether \( |\alpha - \langle \alpha \rangle| \leq \delta(\alpha) \) and \( |\tilde{C}_n - \langle \tilde{C}_n \rangle| \leq \delta(\tilde{C}_n) \) are true or not.

The simulations are conducted with the default settings: \( \lambda = 1.55 \times 10^{-6} \text{m} \), \( k = 4.0537 \times 10^6 \text{rad/m} \), \( L = 1000 \text{m} \), \( d = 0.1 \text{m} \), \( \rho = 5\% \). Other values can be chosen for the same simulations.

The simulations are repeated several times. In each simulation, \( \rho_{IS} \) and \( \rho_{AOA} \), satisfying max \( \{|\rho_{IS}|, |\rho_{AOA}|\} \leq \rho \), are stochastic. Figure 2 and Figure 3 depict the simulation results under different situations. It can be found that both \( |\alpha - \langle \alpha \rangle| \leq \delta(\alpha) \) and \( |\tilde{C}_n - \langle \tilde{C}_n \rangle| \leq \delta(\tilde{C}_n) \) are true, which means that the measurement errors can bound the deviation between the actual and the measured values.

#### 3.2 Examination by gathered dataset

The actual gathered dataset is provided by Changchun University of Science and Technology (CUST) [22]. The measurement system includes a transmitter and a receiver. The OWC link \( (L = 893 \text{m}) \) between the transmitter and the receiver is in the line-of-sight. The transmitter, including a laser \( (\lambda = 808 \text{nm}) \), a collimator and a Cassegrain...
**Results**

\[
\langle \alpha \rangle = 3.3
\]

**Figure 2:** Simulation results with different \( \langle \alpha \rangle \) and constant \( \langle \hat{C}_2^2 \rangle = 10^{-15} \)

\[
\langle \hat{C}_2^2 \rangle = 10^{-16}
\]

**Figure 3:** Simulation results with different \( \langle \hat{C}_2^2 \rangle \) and constant \( \langle \alpha \rangle = 3.7 \)

\[
\langle \hat{C}_2^2 \rangle = 10^{-15}
\]

\[
\langle \hat{C}_2^2 \rangle = 10^{-14}
\]

**Figure 4:** Daily trend of \( \sigma_{IS}^2 \) and \( \sigma_{AOA}^2 \)
antenna, is located in 13th floor, science and technology building, south campus, CUST. The receiver, including a Cassegrain antenna \((d = 200mm > \sqrt{k}L)\), a camera, a computer and other necessary components, is located in 9th floor, teaching building, east campus, CUST. During the measurements, there are nearly 15000 images captured by the camera in every 10 minutes. \(\sigma_{IS}^2\) and \(\sigma_{AOA}^2\) are synchronously acquired by processing these images. Figure 4 depicts the daily trend of \(\sigma_{IS}^2\) and \(\sigma_{AOA}^2\) from 8:00 to 20:00 on August 20, 2014.

Figure 5 depicts the measured values of \(\alpha, \hat{C}_n^2\) and \(C_n^2\). On one hand, it is shown that \(\alpha\) fluctuates around their mean value, which is slightly larger than \(\frac{11}{3}\). The deviation may be caused by the measurement errors or the altitude of the OWC link. On the other hand, both \(\hat{C}_n^2\) and \(C_n^2\) present similar pattern. Unlike the almost stable curve of \(\alpha\) ver-
sus time, they increase and decrease during the forenoon and after the midafternoon, respectively. Their maximum appears on the meridiem, and the turbulence after nightfall is much weaker than the turbulence in daytime, which coincides with other optics-based measurements \([10–13]\).

Besides, at 20:00, \(C_n^2\) and \(\sigma_{\text{AOA}}^2\) seems to be outlier. The reason is the misalignment of the antennas. During the measurements, the direction of the laser beam will keep drifting with time, especially in the evening. So the light is no longer central aligned, and \(\sigma_{\text{AOA}}^2\) will be larger than regular value. Same phenomenon has been observed \([12]\).

Figure 6 depicts \(\delta(\alpha)\) and \(\delta(C_n^2)\) under different \(\rho \geq \max \{|\rho_{\text{IS}}|, |\rho_{\text{AOA}}|\}\), respectively. It is obvious that both \(\delta(\alpha)\) and \(\delta(C_n^2)\) are tolerable, and the confidence levels of \(\alpha\) and \(C_n^2\) are controllable. Undoubtedly, a smaller \(\rho\) ensures smaller measurement errors. Thus, with the decrease in the measured errors of \(\sigma_{\text{IS}}^2\) and \(\sigma_{\text{AOA}}^2\), the precisions of \(\alpha\) and \(C_n^2\) will enhance. It can also be found that a smaller \(\alpha\) leads to a larger \(\delta(\alpha)\), whereas a smaller \(C_n^2\) results in a smaller \(\delta(C_n^2)\).

### 4 Conclusion

This paper investigates the joint measurements of \(\alpha\) and \(C_n^2\) from \(\sigma_{\text{IS}}^2\) and \(\sigma_{\text{AOA}}^2\). Traditionally, \(\alpha\) and \(C_n^2\) can be measured by either IS or AOA. However, there are always obvious differences among the measured values based on different physical quantities. To avoid the inequality, this paper regards \(\alpha\) and \(C_n^2\) as independent variables of \(\sigma_{\text{IS}}^2\) and \(\sigma_{\text{AOA}}^2\), and their measurements are equivalent to solve the corresponding simultaneous equations. The necessary and sufficient condition for the existence and uniqueness of the solutions is established by the extreme value theorem for a real-valued continuous function and the monotonicity. The measurement errors can be deduced from the Jacobian matrix of implicit function. The proposed method is examined by simulated data and actual gathered dataset. The simulation results indicate that the measurement errors can bound the deviation between the actual and the measured values. Based on our proposed method, the measured results accord with the daily trend of atmospheric turbulence.

We ignore the measurement errors induced by other factors such as \(L\), \(\lambda\), and \(d\). The reasons can be explained. The high-resolution total station theodolite for topography has been marketed long time ago. The commercial laser with high stability ensures the wavelength to hardly drift.

The development of advanced 3D printing makes it possible to manufacture satisfied antenna.

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