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M-polynomials and topological indices of hex-derived networks

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Abstract: Hex-derived network has a variety of useful applications in pharmacy, electronics, and networking. In this paper, we give general form of the M-polynomial of the hex-derived networks $HDN_1[n]$ and $HDN_2[n]$, which came out of $n$-dimensional hexagonal mesh. We also give closed forms of several degree-based topological indices associated to these networks.

Keywords: M-polynomial, topological index, hex-derived network

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1 Introduction

In the field of mathematical chemistry, several useful structural and chemical properties of a chemical compound can be determined using simple mathematical tools of polynomials (as Hosoya polynomial and M-polynomial) and numbers (as Randic index and Zagreb index) instead of complicated techniques of quantum mechanics; for details, see Ref. [1–12].

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The hexagonal mesh was introduced by Chen et al. in 1990 [13]. The 2-dimensional hexagonal mesh $HX(2)$, which is composed of six triangles, is given in Figure 1:

Figure 1: 2-dimensional hexagonal mesh

If we add a layer of triangles around this 2-dimensional mesh, we obtain the 3-dimensional hexagonal mesh $HX(3)$; see Figure 2:

Figure 2: 3-dimensional hexagonal mesh
The n-dimensional hexagonal mesh $HX(n)$ is obtained by attaching n-2 layers of triangles around $HX(2)$.

A connected planar graph $G$ divides the plane into disjoint regions; each region is called the face of $G$; two faces are said to be adjacent if they have a common edge; the unbounded region lying outside the graph forms the outer face. The 2-dimensional mesh $HX(2)$ has seven faces, as in Figure 3:

If corresponding to every closed face $f$ of $HX(n)$ we mark a vertex $F$ and then join it with all the vertices of the face $f$ through edges we receive the hex-derived network $HDN_1 [n]$; you can see $HDN_1 [4]$ in Figure 4:

If in $HDN_1 [n]$ the faces $f_1, f_2, \ldots, f_k$ are adjacent to a face $f$ and if the vertex $F$ that represents the face $f$ is joined to the vertices $F_1, F_2, \ldots, F_k$ representing the faces $f_1, f_2, \ldots, f_k$ through edges, we receive the hex-derived network $HDN_2 [n]$; one may have a look at $HDN_2 [4]$ in Figure 5:

In this report, we give the closed forms of the M-polynomial of $HDN_1 [n]$ and $HDN_2 [n]$ and recovered several degree-based topological indices from these polynomials.

Throughout this paper, $G$ will represent a connected graph, $V$ its vertex set, $E$ its edge set, and $d_v$ the degree of its vertex $v$.

**Definition 1.** The M-polynomial of $G$ is defined as

$$M(G, x, y) = \sum_{\delta \leq i \leq j \leq \Delta} m_{ij}(G) x^i y^j,$$

where $\delta = \min \{d_v | v \in V(G)\}$, $\Delta = \max \{d_v | v \in V(G)\}$, and $m_{ij}(G)$ is the edge $vu \in E(G)$ such that $\{d_v, d_u\} = \{i, j\}$.

Topological indices are graph invariants and presently play an important role in the field of mathematical chemistry, biology, physics, electronics and other applied areas. So for many useful topological indices have been introduced. In 1975, Milan Randic introduced the Randic index [18], which is defined as

$$R_{-1/2}(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u d_v}}.$$
For a reasonable information about the development and applications of the Randić index, we refer to [28–31].


The general Randić index is defined as

\[ R_a(G) = \sum_{uv \in E(G)} (d_u d_v)^a, \]

and the inverse Randić index is defined as \( RR_a(G) = \sum_{uv \in E(G)} \frac{1}{d_u d_v}. \)

Gutman and Trinajstić introduced first Zagreb index and second Zagreb index, which are respectively \( M_1(G) = \sum_{uv \in E(G)} (d_u + d_v) \) and \( M_2(G) = \sum_{uv \in E(G)} (d_u \times d_v). \)

The second modified Zagreb index is defined as

\[ m M_2(G) = \sum_{uv \in E(G)} \frac{1}{d(u)d(v)}. \]

For detailed information, we refer the reader to [32–36].

The symmetric division index is

\[ SDD(G) = \sum_{uv \in E(G)} \left\{ \min(d_u, d_v) + \max(d_u, d_v) \right\}. \]

Another variant of Randić index is the harmonic index, which is defined as

\[ H(G) = \sum_{uv \in E(G)} \frac{2}{d_u + d_v}. \]

The inverse-sum index is

\[ I(G) = \sum_{uv \in E(G)} \frac{d_u d_v}{d_u + d_v}. \]

The augmented Zagreb index is

\[ A(G) = \sum_{uv \in E(G)} \left\{ \frac{d_u d_v}{d_u + d_v - 2} \right\}^3, \]

which is found useful for computing the heat of formation of alkanes [37, 38].

Some well-known degree-based topological indices are closely related to the M-polynomial [7]; in Table 1 you can see such relations.

Here

\[ D_x = x \frac{\partial f(x, y)}{\partial x}, \quad D_y = y \frac{\partial f(x, y)}{\partial y}, \]

\[ S_x = \int_0^x f(t, y) dt, \quad S_y = \int_0^y f(x, t) dt \]

\( f(x, y) = f(x, x), \quad Q_a(f(x, y)) = x^a f(x, y). \)

Table 1: Derivation of some degree-based topological indices from M-polynomial

<table>
<thead>
<tr>
<th>Topological Index</th>
<th>Derivation from ( M(G; x, y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Zagreb</td>
<td>( (D_x + D_y) (M(G; x, y))_{x=y=1} )</td>
</tr>
<tr>
<td>Second Zagreb</td>
<td>( (D_x D_y) (M(G; x, y))_{x=y=1} )</td>
</tr>
<tr>
<td>Second Modified</td>
<td>( (S_x S_y) (M(G; x, y))_{x=y=1} )</td>
</tr>
<tr>
<td>Inverse Randić</td>
<td>( (D_x^T D_y^T) (M(G; x, y))_{x=y=1} )</td>
</tr>
<tr>
<td>General Randić</td>
<td>( (S_x T y + S_y T x) (M(G; x, y))_{x=y=1} )</td>
</tr>
<tr>
<td>Symmetric Division</td>
<td>( 2 S_x (M(G; x, y))_{x=1} )</td>
</tr>
<tr>
<td>Index</td>
<td>( S_x^3 D_x D_y (M(G; x, y))_{x=1} )</td>
</tr>
<tr>
<td>Augmented Zagreb</td>
<td>( S_y^3 Q_a - J D_x^3 D_y^3 (M(G; x, y))_{x=1} )</td>
</tr>
</tbody>
</table>

2 Main results

This section contains the general closed forms of the M-polynomial and related indices of the hex-derived networks \( HDN_1[n] \) and \( HDN_2[n] \).

**Theorem 1.** The M-polynomial of \( HDN_1[n] \), \( n > 3 \), is

\[ M(HDN_1[n]; x, y) = 12 x^3 y^5 + (18 n - 36) x^3 y^7 \]

\[ + (18 n^2 - 54 n + 42) x^2 y^7 + 12 n x^2 y^7 + 6 x^2 y^{12} \]

\[ + (6 n - 18) x^2 y^{12} + (12 n - 24) x y^{12} \]

\[ + (9 n^2 - 33 n + 30) x^{12} y^{12}. \]

**Proof.** Depending on degrees, the vertex set \( E \) of \( HDN_1[n] \) can be divided into four disjoint subsets, \( V_1, V_2, V_3, \) and \( V_4 \), containing vertices of degrees 3, 5, 7, and 12, respectively.

Moreover, \( |V_1(HDN_1[n])| = 6 n^2 - 12 n + 6, \)

\( |V_2(HDN_1[n])| = 6, \quad |V_3(HDN_1[n])| = 6 n - 12, \)

\( |V_4(HDN_1[n])| = 3 n^2 - 9 n + 7. \)

Similarly, the edge set \( E \) of \( HDN_1[n] \) can be divided into eight disjoint subsets:

\[ E_1(HDN_1[n]) = \{ e = uv \in E(HDN_1[n]) : d_u = 3, d_v = 5 \}, \]

\[ E_2(HDN_1[n]) = \{ e = uv \in E(HDN_1[n]) : d_u = 3, d_v = 7 \}, \]

\[ E_3(HDN_1[n]) = \{ e = uv \in E(HDN_1[n]) : d_u = 3, d_v = 12 \}, \]

\[ E_4(HDN_1[n]) = \{ e = uv \in E(HDN_1[n]) : d_u = 5, d_v = 7 \}, \]

\[ E_5(HDN_1[n]) = \{ e = uv \in E(HDN_1[n]) : d_u = 5, d_v = 12 \}, \]
Also, we have
\[ E_1(HDN_1[n]) = \{ e = uv \in E(HDN_1[n]) : d_u = 7, d_v = 7 \}, \]
\[ E_2(HDN_1[n]) = \{ e = uv \in E(HDN_1[n]) : d_u = 7, d_v = 12 \}, \]
\[ E_3(HDN_1[n]) = \{ e = uv \in E(HDN_1[n]) : d_u = 7, d_v = 6 \}, \]
\[ E_4(HDN_1[n]) = \{ e = uv \in E(HDN_1[n]) : d_u = d_v = 12 \}. \]

Also,
\[ |E_1(HDN_1[n])| = 12, \]
\[ |E_2(HDN_1[n])| = 18n - 36, \]
\[ |E_3(HDN_1[n])| = 18n^2 - 54n + 42, \]
\[ |E_4(HDN_1[n])| = 12, \]
\[ |E_5(HDN_1[n])| = 6, \]
\[ |E_6(HDN_1[n])| = 6n - 18, \]
\[ |E_7(HDN_1[n])| = 12n - 24, \]
\[ |E_8(HDN_1[n])| = 9n^2 - 33n + 30. \]

Now, we have
\[
M(HDN_1[n] ; x, y) = \sum_{i,j} m_{i,j} x^i y^j = \sum_{3 \leq i \leq 5} m_{3,5} x^3 y^5
\]
\[ + \sum_{3 \leq i \leq 12} m_{3,12} x^3 y^{12} + \sum_{5 \leq i \leq 7} m_{5,7} x^5 y^7
\]
\[ + \sum_{5 \leq i \leq 12} m_{5,12} x^5 y^{12} + \sum_{7 \leq i \leq 7} m_{7,7} x^7 y^7 + \sum_{7 \leq i \leq 12} m_{7,12} x^7 y^{12}
\]
\[ + \sum_{12 \leq i \leq 12} m_{12,12} x^{12} y^{12} = \sum_{uv \in E(HDN_1[n])} m_{3,5} x^3 y^5
\]
\[ + \sum_{uv \in E(HDN_1[n])} m_{3,12} x^3 y^{12} + \sum_{uv \in E(HDN_1[n])} m_{5,7} x^5 y^7
\]
\[ + \sum_{uv \in E(HDN_1[n])} m_{5,12} x^5 y^{12} + \sum_{uv \in E(HDN_1[n])} m_{7,7} x^7 y^7 + \sum_{uv \in E(HDN_1[n])} m_{7,12} x^7 y^{12}
\]
\[ + \sum_{uv \in E(HDN_1[n])} m_{12,12} x^{12} y^{12} = |E_1(HDN_1[n])| x^3 y^5
\]
\[ + |E_2(HDN_1[n])| x^3 y^{12} + |E_3(HDN_1[n])| x^5 y^7 + |E_4(HDN_1[n])| x^5 y^{12}
\]
\[ + |E_5(HDN_1[n])| x^7 y^7 + |E_6(HDN_1[n])| x^7 y^{12} + |E_7(HDN_1[n])| x^{12} y^7
\]
\[ + |E_8(HDN_1[n])| x^{12} y^{12} = 12x^3 y^5 + (18n - 36)x^3 y^7
\]
\[ + (18n^2 - 54n + 42)x^3 y^{12} + 12nx^7 y^7 + 6x^7 y^{12} + (6n - 18)x^7 y^{12} + (12n - 24)x^7 y^{12} + (9n^2 - 33n + 30)x^{12} y^{12}.
\]

Some degree-based topological indices of HDN_1[n] are given in the following proposition.

**Proposition 1.** For the hex-derived network HDN_1[n], we have

1. \( M_1(HDN_1[n]) = 486n^2 - 966n + 480. \)
2. \( M_2(HDN_1[n]) = 1944n^3 - 4596n + 2718. \)
3. \( m^2 M_2(HDN_1[n]) = -\frac{181}{1960} \cdot \frac{1}{11760} n^9 + \frac{9}{16} n^2. \)
4. \( RR_a(HDN_1[n]) = 12 \times 15^a + (18n - 36)21^a. \)
5. \( Kk + (18n^2 - 54n + 42)36^a + 12n35^a + 6 \times 60^a \)
\[ + (6n - 18)49^a + (12n - 24)54^a \]
\[ + (9n^2 - 33n + 30)144^a. \]
6. \( R_a(HDN_1[n]) = 12 \times 15^a + 18n - 36 + \frac{9n^2 - 54n + 42}{36^a} \)
\[ + \frac{12n}{35^a} + 6 \times 60^a + \frac{12n - 24}{49^a} + \frac{9n^2 - 33n + 30}{144^a}. \]
7. \( SSD(HDN_1[n]) = 3221 \times 3^a - \frac{12659}{70} n + 189 n^2. \)
8. \( S_1(HDN_1[n]) = \frac{11121}{6520} - \frac{5931}{5320} n + 63 \times 40^a. \)
9. \( I(HDN_1[n]) = \frac{257661}{3230} - \frac{17171}{95} n + 486 \times 5^a. \)
10. \( A(HDN_1[n]) = \frac{2324977571262767}{551678532544} - \frac{1974148336790655}{3009155745024} n + 7382820168 \times 2924207 n^2. \)

**Proof.** Consider the M-polynomial of HDN_1[n], which is
\[ M(HDN_1[n]; x, y) = f(x, y) = 12x^3 y^5 + (18n - 36)x^3 y^7
\]
\[ + (18n^2 - 54n + 42)x^3 y^{12} + 12nx^7 y^7
\]
\[ + 6x^7 y^{12} + (6n - 18)x^7 y^7 + (12n - 24)x^7 y^{12}
\]
\[ + (9n^2 - 33n + 30)x^{12} y^{12}.
\]
Then

\[ D_x f(x, y) = 36x^3 y^5 + 3(18n - 36)x^3 y^7 \\
+ 3(18n^2 - 54n + 42)x^3 y^{12} + 60nx^5 y^7 + 30x^5 y^{12} \\
+ 7(6n - 18)x^7 y^7 + 7(12n - 24)x^7 y^{12} \\
+ 12(9n^2 - 33n + 30)x^{12} y^{12}, \]

\[ D_y f(x, y) = 60x^3 y^5 + 7(18n - 36)x^3 y^7 \\
+ 12(18n^2 - 54n + 42)x^3 y^{12} + 84nx^5 y^7 + 72x^5 y^{12} \\
+ 7(6n - 18)x^7 y^7 + 12(12n - 24)x^7 y^{12} \\
+ 12(9n^2 - 33n + 30)x^{12} y^{12}, \]

\[ D_x D_y f(x, y) = 180x^3 y^5 + 21(18n - 36)x^3 y^7 \\
+ 36(18n^2 - 54n + 42)x^3 y^{12} + 420nx^5 y^7 + 360x^5 y^{12} \\
+ 49(6n - 18)x^7 y^7 + 84(12n - 24)x^7 y^{12} \\
+ 144(9n^2 - 33n + 30)x^{12} y^{12}, \]

\[ S_y (f(x, y)) = \frac{12}{5}x^3 y^5 + \frac{7}{5}(18n - 36)x^3 y^7 \\
+ \frac{1}{12}(18n^2 - 54n + 42)x^3 y^{12} + \frac{12}{7}nx^5 y^7 + \frac{1}{2}x^5 y^{12} \\
+ \frac{1}{7}(6n - 18)x^7 y^7 + \frac{1}{12}(12n - 24)x^7 y^{12} \\
+ \frac{1}{12}(9n^2 - 33n + 30)x^{12} y^{12}, \]

\[ S_x S_y (f(x, y)) = \frac{12}{15}x^3 y^5 + \frac{21}{2}(18n - 36)x^3 y^7 \\
+ \frac{1}{36}(18n^2 - 54n + 42)x^3 y^{12} + \frac{12}{35}nx^5 y^7 + \frac{1}{10}x^5 y^{12} \\
+ \frac{1}{49}(6n - 18)x^7 y^7 + \frac{1}{84}(12n - 24)x^7 y^{12} \\
+ \frac{1}{144}(9n^2 - 33n + 30)x^{12} y^{12}, \]

\[ D_x^a D_y^a (f(x, y)) = 12 \times 3^a 5^a 9^a 13^a (18n - 36)x^3 y^7 \\
+ 3^a 12^a (18n^2 - 54n + 42)x^3 y^{12} + 12 \times 5^a 7^a 11^a nx^5 y^7 \\
+ 6 \times 5^a 12^a 17^a 19^a (6n - 18)x^7 y^7 \\
+ 7^a 12^a (12n - 24)x^7 y^{12} + 12^a (9n^2 - 33n + 30)x^{12} y^{12}, \]

\[ S_y D_x (f(x, y)) = \frac{36}{5}x^3 y^5 + \frac{3}{7}(18n - 36)x^3 y^7 \\
+ \frac{1}{4}(18n^2 - 54n + 42)x^3 y^{12} + \frac{60}{7}nx^5 y^7 + \frac{30}{12}x^5 y^{12} \\
+ (6n - 18)x^7 y^7 + \frac{7}{12}(12n - 24)x^7 y^{12} \\
+ (9n^2 - 33n + 30)x^{12} y^{12}, \]

\[ S_x D_y (f(x, y)) = 20x^3 y^5 + \frac{7}{3}(18n - 36)x^3 y^7 \\
+ 4(18n^2 - 54n + 42)x^3 y^{12} + \frac{84}{5}nx^5 y^7 + \frac{72}{5}x^5 y^{12} \\
+ (6n - 18)x^7 y^7 + \frac{12}{11}(12n - 24)x^7 y^{12} \\
+ (9n^2 - 33n + 30)x^{12} y^{12}, \]

\[ S_x S_y f(x, y) = \frac{3}{2}x^8 + \frac{1}{10}(18n - 36)x^{10} \\
+ \frac{1}{15}(18n^2 - 54n + 42)x^{15} + nx^{12} + \frac{6}{17}x^{17} \\
+ \frac{1}{14}(6n - 18)x^{14} + \frac{1}{19}(12n - 24)x^{19} \\
+ \frac{1}{24}(9n^2 - 33n + 30)x^{26}, \]

\[ S_x^3 Q - JD_x^3 D_y^3 f(x, y) = \frac{12 \times 3^3 5^3}{6^3} x^6 + \frac{3^3 7^3}{8^3}(18n - 36)x^8 \\
+ \frac{3^3 12^3}{13^3} (18n^2 - 54n + 42)x^{13} + \frac{12 \times 5^3 7^3}{10^3} nx^{10} \\
+ \frac{6 \times 5^3 12^3}{15^3} x^{15} + \frac{7^6 (6n - 18)}{12^3} x^{12} + \frac{7^3 12^3}{17^3} (12n - 24)x^{17} \\
+ \frac{12^6}{22^3} (9n^2 - 33n + 30)x^{22}, \]

Now, we go for indices.

1. \[ M_1 (HDN_1[n]) = D_x + D_y (f(x, y)) \bigg|_{x=y=1} = 486n^2 - 966n + 480. \]
2. \[ M_2 (HDN_1[n]) = D_x D_y (f(x, y)) \bigg|_{x=y=1} = 1944n^2 - 4596n + 2718. \]
3. \[ m M_2 (HDN_1[n]) = S_x S_y (f(x, y)) \bigg|_{x=y=1} = -\frac{181}{1960}. \]
4. \[ R_a (HDN_1[n]) = D_x^a D_y^a (f(x, y)) \bigg|_{x=y=1} = 12 \times 15^a + (18n - 36)21^a + (18n^2 - 54n + 42)36^a + 12n 35^a \\
+ 6 \times 60^a + (6n - 18)49^a + (12n - 24)84^a \\
+ (9n^2 - 33n + 30)144^a. \]
5. \[ R R_a (HDN_1[n]) = S_x^a S_y^a (f(x, y)) \bigg|_{x=y=1} = \frac{12^a}{15^a}. \]
6. SSD(HDN1[n]) = (S_yD_x + S_xD_y) \left( f(x, y) \right) |_{x=y=1}
   = \frac{3221}{35} - \frac{12659}{70} n + \frac{189}{2} n^2.

7. H(HDN1[n]) = 2S_xJ \left( f(x, y) \right) |_{x=1} = \frac{11121}{45220}
   \frac{5931}{5320} n + \frac{63}{40} n^2.

8. I(HDN1[n]) = S_xJD_xD_y \left( f(x, y) \right) |_{x=1} = \frac{257661}{3230}
   \frac{17111}{95} n + \frac{486}{5} n^2.

9. A(HDN1[n]) = S_x^3Q_zJD_x^3D_y^3 \left( f(x, y) \right) |_{x=1}
   = \frac{23249577512672767}{551678553254} - \frac{19741483367980655}{3009155745024} n
   + \frac{7382820168}{2924207} n^2.

**Theorem 2.** The M-polynomial of HDN2[n] is

\[
M(HDN2[n] ; x, y) = 18x^5y^5 + (12n - 24)x^5y^6
+ (12n - 12)x^5y^7 + 6nx^5y^{12} + (9n^2 - 33x + 30)x^6y^6
+ (6n - 12)x^6y^7 + (18n^2 - 60n + 48)x^6y^{12}
+ (6n - 18)x^7y^7 + (12n - 24)x^7y^{12}
+ (9n^2 - 33n + 30)x^{12}y^{12}.
\]

**Proof.** Consider the hexagonal mesh HDN2[n], where n > 3. The vertex set of HDN2[n] has the following four partitions.

\[
V_1(HDN2[n]) = \{ u \in V(HDN2[n]) : d_u = 5 \}, \quad (48)
\]

\[
V_2(HDN2[n]) = \{ u \in V(HDN2[n]) : d_u = 6 \}, \quad (49)
\]

\[
V_3(HDN2[n]) = \{ u \in V(HDN2[n]) : d_u = 7 \}, \quad (50)
\]

and

\[
V_4(HDN2[n]) = \{ u \in V(HDN2[n]) : d_u = 12 \}. \quad (51)
\]

Also

\[
|V_1(HDN2[n])| = 6n, \quad (52)
\]

\[
|V_2(HDN2[n])| = 6n^2 - 18n + 12, \quad (53)
\]

\[
|V_3(HDN2[n])| = 6n - 12, \quad (54)
\]

\[
|V_4(HDN2[n])| = 3n^2 - 9n + 7. \quad (55)
\]

Moreover we divide the edge set of HDN2[n] into the following ten partitions:

\[
E_1(HDN2[n]) = \{ e = uv \in E(HDN2[n]) : d_u = d_v = 5 \},
\]

\[
E_2(HDN2[n]) = \{ e = uv \in E(HDN2[n]) : d_u = 5, d_v = 6 \},
\]

\[
E_3(HDN2[n]) = \{ e = uv \in E(HDN2[n]) : d_u = 5, d_v = 7 \},
\]

\[
E_4(HDN2[n]) = \{ e = uv \in E(HDN2[n]) : d_u = 5, d_v = 12 \},
\]

\[
E_5(HDN2[n]) = \{ e = uv \in E(HDN2[n]) : d_u = 6, d_v = 6 \},
\]

\[
E_6(HDN2[n]) = \{ e = uv \in E(HDN2[n]) : d_u = 6, d_v = 7 \},
\]

\[
E_7(HDN2[n]) = \{ e = uv \in E(HDN2[n]) : d_u = 6, d_v = 12 \},
\]

\[
E_8(HDN2[n]) = \{ e = uv \in E(HDN2[n]) : d_u = 7, d_v = 7 \},
\]

\[
E_9(HDN2[n]) = \{ e = uv \in E(HDN2[n]) : d_u = 7, d_v = 12 \},
\]

and

\[
E_{10}(HDN2[n]) = \{ e = uv \in E(HDN2[n]) : d_u = 12, d_v = 12 \},
\]

Also, we have

\[
|E_1(HDN2[n])| = 18, \quad (66)
\]

\[
|E_2(HDN2[n])| = 12n - 24, \quad (67)
\]

\[
|E_3(HDN2[n])| = 12n - 12, \quad (68)
\]

\[
|E_4(HDN2[n])| = 6n, \quad (69)
\]

\[
|E_5(HDN2[n])| = 9n^2 - 33n + 30, \quad (70)
\]

\[
|E_6(HDN2[n])| = 6n - 12, \quad (71)
\]
\[ |E_7(HDN_2[n])| = 18n^2 - 60n + 48, \quad (72) \]
\[ |E_6(HDN_2[n])| = 6n - 18, \quad (73) \]
\[ |E_9(HDN_2[n])| = 12n - 24, \quad (74) \]

and
\[ |E_{10}(HDN_2[n])| = 9n^2 - 33n + 30. \quad (75) \]

Now, by the definition of the $M$-polynomial, we have
\[
M(HDN_2[n]; x, y) = \sum_{i\leq j} m_{ij}x^i y^j = \sum_{5 \leq i \leq 5} m_{5,5}x^5 y^5 \quad (76)
\]
\[
+ \sum_{5 \leq i \leq 5} m_{5,5}x^5 y^5 + \sum_{6 \leq i \leq 6} m_{5,6}x^6 y^6 + \sum_{7 \leq i \leq 7} m_{7,7}x^7 y^7 + \sum_{12 \leq i \leq 12} m_{12,12}x^{12} y^{12}
\]
\[
= \left[ E_1(HDN_2[n]) \right] x^5 y^5 + \left[ E_2(HDN_2[n]) \right] x^5 y^6 + \left[ E_3(HDN_2[n]) \right] x^7 y^7 + \left[ E_4(HDN_2[n]) \right] x^7 y^{12} + \left[ E_5(HDN_2[n]) \right] x^5 y^6 + \left[ E_6(HDN_2[n]) \right] x^5 y^7 + \left[ E_7(HDN_2[n]) \right] x^6 y^{12} + \left[ E_8(HDN_2[n]) \right] x^{12} y^{12} + \left[ E_9(HDN_2[n]) \right] x^{12} y^{12} + \left[ E_{10}(HDN_2[n]) \right] x^{12} y^{12}
\]
\[
= 18x^5 y^5 + (12n - 24)x^5 y^6 + (12n - 24)x^5 y^7 + 6nx^5 y^{12} + (9n^2 - 33n + 30)x^5 y^7 + (18n^2 - 60n + 48)x^6 y^{12} + (6n - 18)x^6 y^{12} + (12n - 24)x^7 y^{12} + (9n^2 - 33n + 30)x^{12} y^{12}.
\]

Now we compute some degree-based topological indices of the hexagonal mesh.

**Proposition 2.** For we have

1. $M_1(HDN_2[n]) = 648n^2 - 1500n + 852$.
2. $M_2(HDN_2[n]) = 2916n^2 - 7566n + 4764$.

![Figure 7: The plot for the $M$-polynomial of $HDN_2[1]$](image)

3. $M_2(HDN_2[n]) = \frac{10193}{29400} - \frac{8586}{11760} n + \frac{9}{16}n^2$.

4. $R_a(HDN_2[n]) = 18 \times 25^{a} + (12n - 24)30^{a} + (12n - 24)35^{a} + (6n - 12)42^{a} + (18n^2 - 60n + 48)72^{a} + (6n - 18)49^{a} + (12n - 24)84^{a} + (9n^2 - 33n + 30)144^{a}$.

5. $R_{R_a}(HDN_2[n]) = \frac{18}{25^{a}} + \frac{12n - 24}{30^{a}} + \frac{12n - 24}{35^{a}} + \frac{6n - 12}{36^{a}} + \frac{6n - 12}{42^{a}} + \frac{18n^2 - 60n + 48}{72^{a}} + \frac{6n - 18}{49^{a}} + \frac{12n - 24}{84^{a}} + \frac{9n^2 - 33n + 30}{144^{a}}$.

6. $SSD(HDN_2[n]) = -\frac{11453}{70} n + \frac{432}{5} + 81n^2$.

7. $H_1(HDN_2[n]) = \frac{1783487}{1141140} - \frac{27103217}{7759752} n + \frac{17}{8}n^2$.

8. $I(HDN_2[n]) = \frac{539789}{2717} - \frac{16381345}{46189} n + 153n^2$.

9. $A(HDN_2[n]) = \frac{3218008599887179}{376658092800} - \frac{998781364704403}{78470436000} n + \frac{30506488487}{665500} n^2$.

**Proof.** Let
\[
f(x, y) = 18x^5 y^5 + (12n - 24)x^5 y^6 + (12n - 24)x^5 y^7 + (9n^2 - 33n + 30)x^5 y^{12} + (6n - 12)x^6 y^7 + (18n^2 - 60n + 48)x^6 y^{12}.
\]
\[ + (6n - 18)x^7y^7 + (12n - 24)x^7y^{12} + (9n^2 - 33n + 30)x^{12}y^{12}. \]

Then

\[ D_f(x, y) = 90x^5y^5 + 6(12n - 24)x^5y^6 + 7(12n - 12)x^5y^7 + 72nx^5y^{12} + 6(9n^2 - 33n + 30)x^6y^6 + 7(6n - 12)x^6y^7 + 12(18n^2 - 60n + 48)x^6y^{12} + 7(6n - 18)x^7y^7 + 12(12n - 24)x^7y^{12} + 12(9n^2 - 33n + 30)x^{12}y^{12}, \]

\[ D_f(x, y) = 450x^5y^5 + 6(12n - 24)x^5y^6 + 35(12n - 12)x^5y^7 + 360nx^5y^{12} + 36(9n^2 - 33n + 30)x^6y^6 + 42(6n - 12)x^6y^7 + 72(18n^2 - 60n + 48)x^6y^{12} + 49(6n - 18)x^7y^7 + 84(12n - 24)x^7y^{12} + 144(9n^2 - 33n + 30)x^{12}y^{12}, \]

\[ S_xS_y(f(x, y)) = \frac{18}{25}x^5y^5 + \frac{1}{30}(12n - 24)x^5y^6 + \frac{1}{35}(12n - 12)x^5y^7 + \frac{1}{16}nx^5y^{12} + \frac{1}{36}(9n^2 - 33n + 30)x^6y^6 + \frac{1}{42}(6n - 12)x^6y^7 + \frac{1}{72}(18n^2 - 60n + 48)x^6y^{12} + \frac{1}{49}(6n - 18)x^7y^7 + \frac{1}{84}(12n - 24)x^7y^{12} + \frac{1}{144}(9n^2 - 33n + 30)x^{12}y^{12}, \]

\[ D_sD_t(f(x, y)) = 18x^5y^5 + (12n - 24)x^5y^6 + 6x^5(12n - 24)x^5y^7 + 6\times 5^{\frac{2}{3}}n^5x^6y^{12} + 6\times 5^{\frac{7}{9}}(12n - 12)x^5y^7 + 6\times 6^a(9n^2 - 33n + 30)x^6y^6 + 6\times 6^a(6n - 12)x^5y^7 + 6\times 6^{12}(18n^2 - 60n + 48)x^6y^{12} + 7^{2a}(6n - 18)x^7y^7 + 7^{12a}(12n - 24)x^7y^{12} + 12^{2a}(9n^2 - 33n + 30)x^{12}y^{12}, \]

\[ S_x^3Q_xD_x^3D_y^3f(x, y) = \frac{9}{4}x^5y^8 + \frac{5^{\frac{3}{2}}}{9^{\frac{3}{2}}}(12n - 24)x^9 + \frac{5^{\frac{3}{2}}}{10^{\frac{3}{2}}}(12n - 12)x^{10} + \frac{6}{15^3}nx^{12} + \frac{6}{15^3}(9n^2 - 33n + 30)x^{10} + \frac{6^{\frac{7}{3}}}{11^3}(6n - 12)x^{11} + \frac{6^{\frac{12}{3}}}{16^3}(18n^2 - 60n + 48)x^{16} + \frac{7^{\frac{6}{3}}}{12^3}(6n - 18)x^{12} + \frac{7^{\frac{12}{3}}}{17^3}(12n - 24)x^{17}, \]
In this article, we computed the M-polynomial of and HDN_2(n). The First and the second Zagreb indices, Generalized Randic index, Inverse Randic index, Symmetric division index, Inverse sum index and Augmented Zagreb index of these hex-derived networks have also been computed. These indices are actually functions of chemical graphs and encode many chemical properties as viscosity, strain energy, and heat of formation. Graphical description, given in Figures 6 and 7, also demonstrates the behavior of the M-polynomial of the networks. It is notable that our results about Randic index extend the results given in [39].

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